

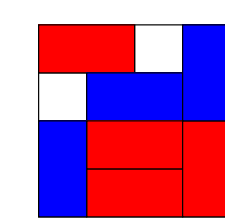
# Competitive Tiling

## The Tiling Game

The  $(T, \Gamma, d)$ -tiling game consists of two players, Alice and Bob, a tile set  $T$ , a region  $\Gamma$ , and a non-negative integer  $d$ . Alice and Bob alternate placing tiles on  $\Gamma$ , with Alice going first. On each turn, players place a tile from  $T$  on untiled squares in  $\Gamma$ . The game ends when no more tiles can be placed on  $\Gamma$ . Alice wins if at most  $d$  squares are untiled at the end of the game and Bob wins if more than  $d$  squares are untiled.

## The $(D, 4 \times 4, 2)$ Tiling Game: An Example

The game played with dominoes on the  $4 \times 4$  square with non-negative integer  $d = 2$ , or the  $(D, 4 \times 4, 2)$  tiling game. (Red and blue corresponding to Alice and Bob respectively.)



2 squares remain untiled at the end of the game, thus Alice wins.

## The Game Tiling Number

For certain regions and tile sets, we are interested in the smallest value  $d$  such that Alice can win. Thus, we define the *game tiling number*.

The *game tiling number* is the smallest  $d$  such that Alice has a winning strategy. We use the following notation,

$$\gamma(T, \Gamma) = d.$$

\*For later purposes let  $\gamma^*(T, \Gamma) = d$  be the game tiling number for a game when Bob plays first.

## The Game Tiling Number: An Example

Consider a game played with dominoes on a  $1 \times 6$  region.



There exists a strategy for Alice such that she can ensure the entire region gets tiled. Thus,  $\gamma(D, 1 \times 6) = 0$ .

For the game when Bob plays first, he can play in such a way that gives a different game tiling number.



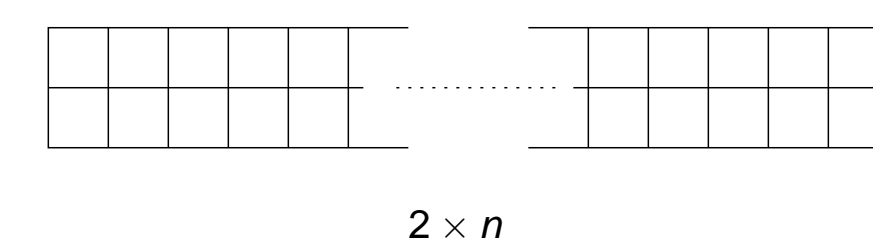
Thus,  $\gamma^*(D, 1 \times 6) = 2$ .

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## Main Results

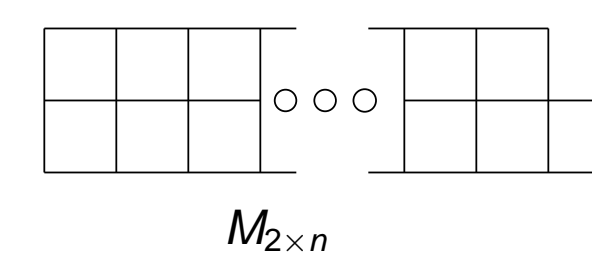
We found the game tiling number for the following generalized regions:

(a)  $2 \times n$ ,



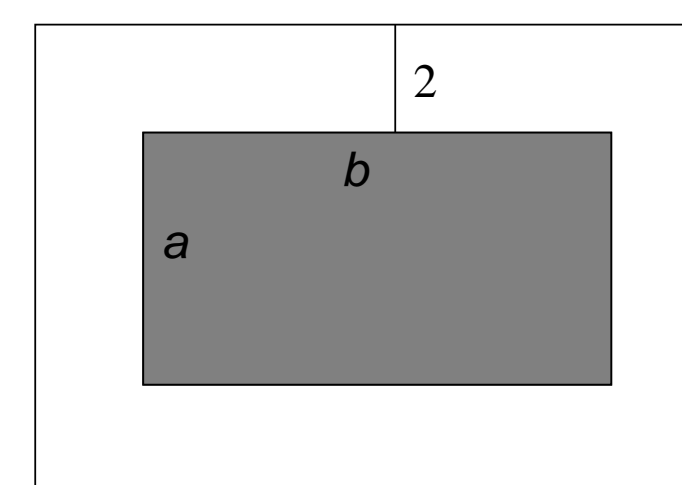
$$\gamma(D, 2 \times n) = \gamma^*(D, 2 \times n) = 0.$$

(b) Modified  $2 \times n$ ,



$$\gamma(D, M_{2 \times n}) = \gamma^*(D, M_{2 \times n}) = 1.$$

(c) Extended 2 annular regions,



$$\begin{aligned} \gamma(D, A_2(1, b)) &= 2 \\ \gamma(D, A_2(a, b)) &= 4 \text{ for } a \geq 2 \end{aligned}$$

## Bibliography

C. Dunn, "Complete multipartite graphs and the relaxed coloring game," Order, 29, Issues 3 (2012), 507-512.

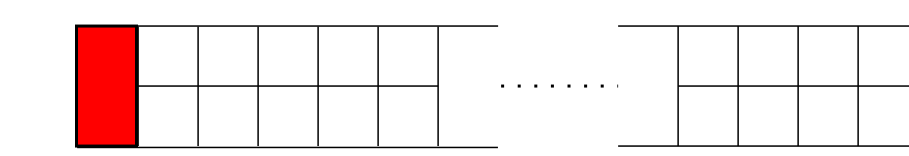
M. Korn, Geometric and algebraic properties of polyomino tilings, Ph.D. thesis, MIT (2004), <http://hdl.handle.net/1721.1/16628>.

## Finding The Game Tiling Number

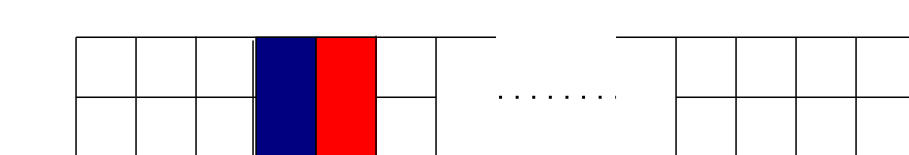
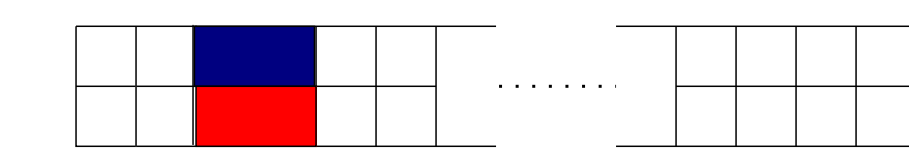
The proof of the  $2 \times n$  result is shown below to display the process of how the game tiling number is found for certain regions.

We begin by implementing a strategy for Alice.

On Alice's first move she will place a domino on the far left.



Now, Alice's moves will be in response to Bob by placing a tile of the same orientation directly next to his last placed tile.



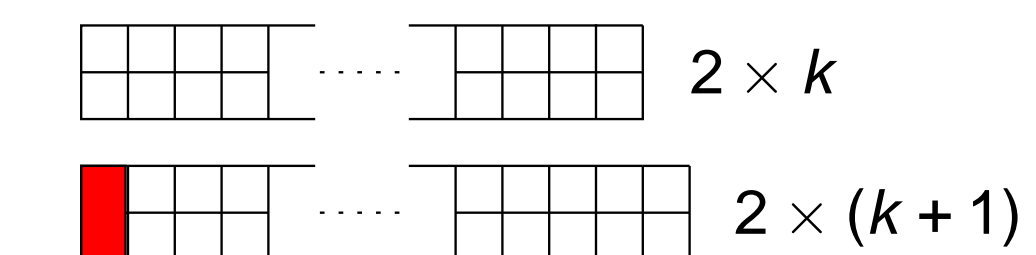
Notice that Alice's strategy isn't dependent on her playing first, but Alice's responses ensure the region is becoming comprised of smaller and smaller  $2 \times n$  regions where either Alice or Bob are playing first. Thus, we use strong induction to prove the game tiling number.

Base Step:  $2 \times 1$  case,  $d = 0$ .



(Also note that  $d = 0$  for the  $2 \times 1$  case when Bob goes first.)

Induction Step: Assume  $d = 0$  for the region  $2 \times k$  and then we look at the  $2 \times (k + 1)$  region.



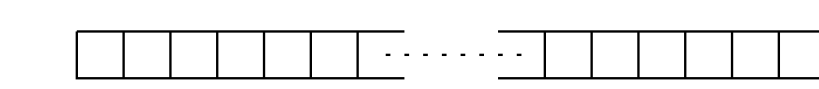
Notice that Alice will place a domino on the far left creating a  $2 \times k$  region which by the induction assumption has  $d = 0$ .

(Also note that in the game where Bob plays first Alice's strategy ensures smaller  $2 \times n$  regions where  $n \leq k$ . Therefore our strong induction holds.)

Thus,  $\gamma(D, 2 \times n) = \gamma^*(D, 2 \times n) = 0$ .

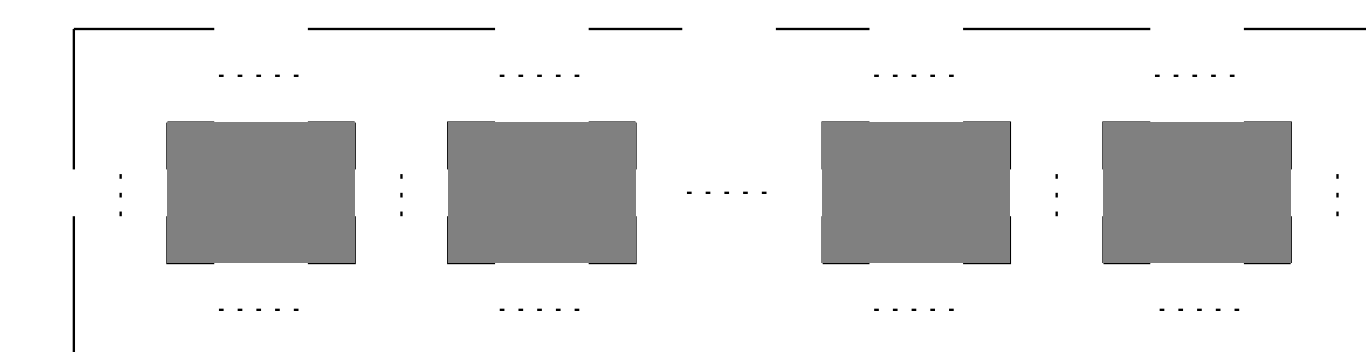
## Other Findings

Conjecture:



For the game played with dominoes on the  $1 \times n$ , when  $n$  is even,  $\gamma(D, 1 \times n) = 2 \lfloor \frac{n+6}{14} \rfloor$ , and when  $n$  is odd,  $\gamma(D, 1 \times n) = 2 \lfloor \frac{n-1}{14} \rfloor + 1$ .

Conjecture:



For the game played with dominoes on the extended annular regions, with  $h$  being the number of holes in the region,  $\gamma(D, A_2(a, b)_h) = 2 \lfloor \frac{h}{2} \rfloor$ , where  $a = 1$ , and  $\gamma(D, A_2(a, b)_h) = 2(h + 1)$  where  $a \geq 2$ .