Tilings of Annular Regions

What is Tiling?

In our context a tile is a collection of unit cells in the plane. Let $\mathcal{T}$ be a set of tiles, then we say a region Γ is tileable by $\mathcal{T}$ if we can cover every cell of Γ once and only once by tiles from $\mathcal{T}$. We can use as many copies of a tile from $\mathcal{T}$ in a tiling as we wish. Also, the tiles cannot be rotated or flipped, but if we would like to allow for rotations and flips we can include them in the tile set.

Common tiling questions for a family of regions $\mathcal{R}$ and a tile set $\mathcal{T}$ are:

1. Which regions in $\mathcal{R}$ can be tiled by $\mathcal{T}$?

2. How many ways can we tile a region $\Gamma \in \mathcal{R}$ by $\mathcal{T}$?

3. How are two different tilings of a given region related?

- Is there a local move property? That is, is there a set of local moves that allow one to transform any tiling of a region to any other tiling of that region?
- Are there certain relations among the tiles in $\mathcal{T}$ that must persist in any tiling of a region $\Gamma \in \mathcal{R}$? In other words, what are the tile invariants associated to $\mathcal{T}$ and $\mathcal{R}$?
- What is the tile counting group associated to $\mathcal{T}$ and $\mathcal{R}$ (the tile counting group records the various tile invariants associated to $\mathcal{T}$ and $\mathcal{R}$)?

Goals of Our Research

In this project we investigate tilings of rectangular annular regions which we denote $A_n(a, b)$ using the set of T and skew tetrominoes (from this point on $\mathcal{T}$ will refer to this tile set). Figure 5 below depicts our tile set.

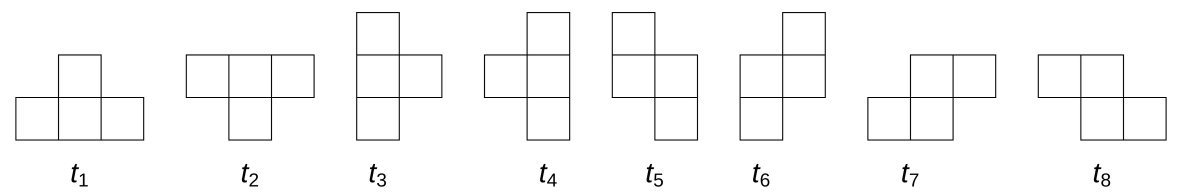

Figure 1: Our tile set $\mathcal{T}$ of T and skew tetrominoes.

The region $A_n(a, b)$ is defined as $a \times b$ rectangle with n rows of cells around it. The $a \times b$ rectangle is not included in the region. Thus these regions are not simply connected (they have a hole in the middle). Figure 2 gives an example of two of these regions.

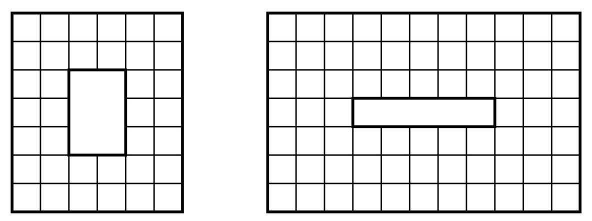

Figure 2: The annular regions $A_2(3, 2)$ and $A_3(1, 5)$

It can be quite difficult to show that a region is untileable by a given tile set. Thankfully many techniques exist to help decide if a region is tileable. However, many of these techniques require that the region is simply connected (no holes). We hope to develop new techniques that work for non-simply connected regions. This is why we decided to look at the $A_n(a, b)$ regions, as they seemed like easy examples of non-simply connected regions.

We also sought to answer some of the common tiling questions listed above for our tile set and family of annular regions.

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Main Result

Theorem 1. *$A_n(a, b)$ is tileable by $\mathcal{T}$ if and only if*

- *n is even,*
- *$n = 3$ and $a \equiv b \pmod 2$ and it is not the case that $a, b \equiv 0 \pmod 4$,*
- *$n \geq 5$ is odd and $a \equiv b \pmod 2$.*

The theorem above completely classifies which of our $A_n(a, b)$ regions are tileable by $\mathcal{T}$. Below we present a proof of the first case when n is even. To do this we first develop a few definitions and lemmas.

Definition 1. *An extended T, X_n, is any rotation of a region formed by removing the two corner squares from first row of a $2 \times n$ rectangle for all $n \geq 3$ (see Fig. 3).*

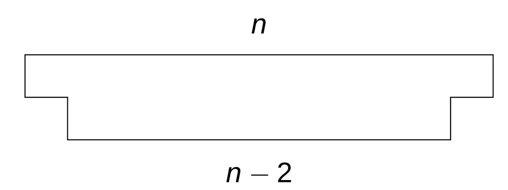

Figure 3: An extended T of length n

Lemma 2. *The region X_n is tileable by T for all odd $n \geq 3$.*

Proof. We proceed by induction. For $n = 3$ we have a T-tile.
Now assume X_n is tileable by T for some odd $n \geq 3$. Now we must show that X_{n+2} is tileable by $\mathcal{T}$. Simply add a skew tetromino to one end of the region. The remaining area is tileable by the induction hypothesis.

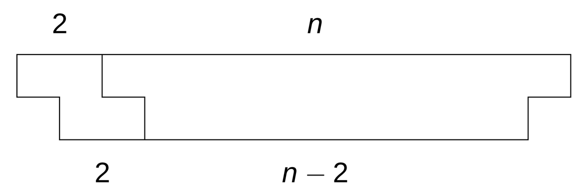

Figure 4: Illustration of the induction step.

□

Lemma 3. *The region $A_2(a, b)$ is tileable by $\mathcal{T}$ for all $a, b \in \mathbb{Z}^+$.*

Proof. Consider the following 3 cases:

$$a, b \equiv 0 \pmod 2,$$
$$a \equiv 0 \pmod 2 \text{ and } b \equiv 1 \pmod 2,$$
$$a, b \equiv 1 \pmod 2$$

(note $a \equiv 1 \pmod 2, b \equiv 0 \pmod 2$) is covered in the second case by symmetry).

In each case the region can be broken down into four extended T's of odd length, which are tileable by lemma 2. Below is a picture of such a decomposition for the first case.

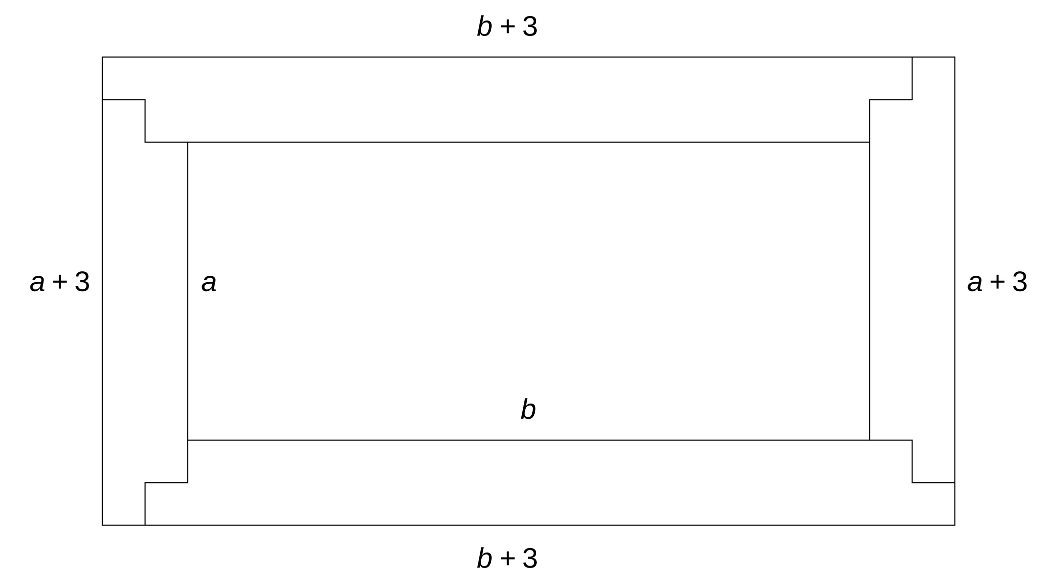

Figure 5: An arrangement of odd length extended T's showing that the region is tileable ($a, b \equiv 0 \mod 2$).

□

From these lemmas, it is not difficult to show that $A_n(a, b)$ is tileable by $\mathcal{T}$ for all even values of n. This can be done by induction on n. The previous lemma provides the base case.

Other Results

- We proved that the tile counting group for the $A_2(a, b)$ regions with respect to $\mathcal{T}$ is isomorphic to $\mathbb{Z}^3 \times \mathbb{Z}_2$. This means that we know what all the tile invariants look like for these regions with our tile set.
- We were able to show that the extended T's have a local move property.
- We also proved that there are 2^{a+b+1} ways to tile $A_2(a, b)$ by $\mathcal{T}$.

References

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