Further results on a filtered multiplicative basis of group algebras*

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Abstract. *Let FG be a group algebra of a finite non-abelian p-group G and F a field of characteristic p. In this paper we give all minimal non-abelian p-groups and minimal non-metacyclic p-groups whose group algebras FG possess a filtered multiplicative F-basis.*

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1. Introduction

Let Λ be a finite-dimensional algebra over a field F. The following concept was introduced by H. Kupisch [11]. An F-basis B of the algebra Λ is called a filtered multiplicative F-basis if B has the following properties:

(i) if $b_1, b_2 \in B$, then either $b_1b_2 = 0$ or $b_1b_2 \in B$;

(ii) $B \cap rad\ (\Lambda)$ is an F-basis for $rad\ (\Lambda)$, where $rad\ (\Lambda)$ denotes the Jacobson radical of Λ.

Denote by $mod\ \Lambda$ the category of finite dimensional right Λ-modules. Λ has a finite representation type (is representation-finite) if there are only finitely many isomorphism classes of indecomposable modules $M \in mod\ \Lambda$. Let F be an algebraically closed field. R. Bautista, P. Gabriel, A. Roiter, and L. Salmeron [3] proved that if Λ has a finite representation type, then Λ has a filtered multiplicative F-basis. The question when a filtered multiplicative F-basis exists in a group algebra FG, where G is a finite p-group and F a field of characteristic p also derives from [3]. Let $C_{p^n} = \langle\, a \mid a^{p^n} = 1\,\rangle$ be the cyclic group of order p^n and F a field of characteristic p. Then the set $B = \left\{(a-1)^i \mid 0 \le i < p^n\right\}$ is a filtered multiplicative F-basis for FC_{p^n}. Note that if FG_1 and FG_2 have a filtered multiplicative F-basis which we

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denote by B_1 and B_2 respectively, then $B_1 \times B_2$ is a filtered multiplicative F-basis for the group algebra $F[G_1 \times G_2]$. Let G be a finite abelian p-group. Since G is a direct product of finite cyclic p-groups from the above line reasoning, it follows that the group algebra FG over a field F of characteristic p admits a filtered multiplicative F-basis.

Higman [9] proved that the group algebra FG over a field of characteristic p is representation-finite if and only if all the Sylow p-subgroups of G are cyclic. In this case FG admits a filtered multiplicative basis. We mention however that finite type is not necessary for existence of filtered multiplicative F-basis. For example, if $G = C_p \times C_p$, then the group algebra FG over a field F of characteristic p possesses a filtered multiplicative F-basis, but the representation type of FG is not finite.

L. Paris in [12] gave examples of non-abelian p-groups G such that the group algebras FG have filtered multiplicative F-basis. In [5] these results were extended to metacyclic p-groups by V. Bovdi. Further results can be found in [6] by V. Bovdi and [2] by Z. Balogh. In [6] negative answer was given for this question when G is either a powerful p-group or a two generated p-group ($p \neq 2$) with central cyclic commutator subgroup. In [2] all groups can be found with order less then p^5 or equal 2^5 whose group algebra possesses a filtered multiplicative F-basis.

In this paper we shall study the existence of filtered multiplicative F-basis of group algebra FG, where G is either a minimal non-abelian p-group or a minimal non-metacyclic p-group. We first remark that a non-abelian p-group G is called minimal non-abelian if all of its proper subgroup are abelian. A finite p-group G is called minimal non-metacyclic if all of whose proper subgroups are metacyclic.

Our main results are as follow.

Theorem 1. *Let G be a finite minimal non-abelian p-group and F a field of characteristic p. Then FG possesses a filtered multiplicative F-basis if and only if $p = 2$ and one of the following conditions holds.*

(i) G is the quaternion group Q_8 of order 8 and F contains a primitive cube root of the unity.

(ii) $G = \langle\, a, b \mid a^4 = b^{2^m} = 1, a^b = a^3 \,\rangle$, where $m \geq 1$.

(iii) $G = \langle\, a, b, c \mid a^{2^n} = b^{2^m} = c^2 = 1, (a, b) = c, (a, c) = (b, c) = 1 \,\rangle$, where $n, m \geq 2$.

As a consequence of *Theorem 1* we have

Corollary 1. *Let FG be a group algebra of finite p-group G over a field F of characteristic p, such that all elements of the unit group of FG of order p are commute. Then FG admits a filtered multiplicative F-basis if and only if $p = 2$ and one of the following conditions holds.*

(i) G is either Q_8 or $Q_8 \times C_{2^n}$ and F contains a primitive cube root of the unity.

(ii) $G = \langle\, a, b \mid a^4 = b^{2^m} = 1, a^b = a^3 \,\rangle$, where $m \geq 2$.

Denote by $G \curlyvee H$ the central product of G and H.

Theorem 2. *Let G be a finite non-abelian minimal non-metacyclic p-group and F a field of characteristic p. Then FG has a filtered multiplicative F-basis if and only if $p = 2$ and one of the following conditions holds.*

(i) G is $Q_8 \times C_2$ and F contains a primitive cube root of the unity.

(ii) G is the central product $D_8 \curlyvee C_4$ of the dihedral group of order 8 and the cyclic group of order 4.

We remark that *Theorems 1, 2* and *Corollary 1* confirm the conjecture that if G is a finite p-group and p is odd, then the group algebra FG does not possess any filtered multiplicative F-basis.

2. Preliminaries and proof of the main results

Let FG be a group algebra of a finite p-group G over a field F of characteristic p. Assume that B is a filtered multiplicative F-basis for FG. Since FG is a local ring with maximal ideal $rad\ (FG)$ and $dim_F\big(FG/rad\ (FG)\big) = 1$, we can assume that $1 \in B$ and $B \cap rad\ (FG)$ is a basis for $rad\ (FG)$.

In this paper we use freely the following simple properties of B (see [5]):

(i) $B \cap rad\ (A)^n$ is an F-basis of $rad(A)^n$ for all $n \geq 1$.

(ii) If $u, v \in B \setminus rad(A)^k$ and $u \equiv v \pmod{rad(A)^k}$ then $u = v$.

Denote the augmentation ideal of FG by $\omega(FG)$. It is well-known that if G is a finite p-group and the characteristic of F is equal to p, then $\omega(FG)$ coincides with $rad\ (FG)$. By [8] the Frattini subalgebra of $rad\ (FG)$ coincides with $rad\ (FG)^2$. Therefore $B \setminus \big(\{1\} \cup \omega(FG)^2\big)$ is a generating set of $\omega(FG)$ as an F-algebra.

The M-series $\mathfrak{D}_i$ of G is due to Brauer, Jennings and Zassenhaus. We define this series by recursion as follows $\mathfrak{D}_1 = G$ and $\mathfrak{D}_i(\mathfrak{D}_{i-1}, G)\mathfrak{D}^p_{\lceil i/p \rceil}$, where $\lceil r \rceil$ denotes the upper integral part of the real number r. It is easy to see that $\mathfrak{D}_2$ coincides with the Frattini subgroup $\Phi(G)$ of G. Moreover, the n^{th} dimension subgroup of G over F coincides with the n^{th} term of the Brauer-Jennings-Zassenhaus M-series by [10]. Denote the set $\big\{j \in \mathbb{N} \mid \mathfrak{D}_j(G) \neq \mathfrak{D}_{j+1}(G)\big\}$ by I. Let p^{d_j} $(j \in I)$ be the order of the elementary Abelian p-group $\mathfrak{D}_j(G)/\mathfrak{D}_{j+1}(G)$. Evidently,

$$\mathfrak{D}_j(G)/\mathfrak{D}_{j+1}(G) = \prod_{k=1}^{d_j} \langle\ g_{jk}\mathfrak{D}_{j+1}\ \rangle$$

for some $g_{jk} \in G$.

According to Jennings [10], the elements $\prod_{j\in I}\big(\prod_{k=1}^{d_j}(g_{jk} - 1)^{l_{jk}}\big) \in \omega(FG)$, where $0 \leq l_{jk} < p$ and the indices of the factors are in lexicographic order form a basis for $\omega(FG)$. Moreover, the above mentioned elements with the property $\sum_{j\in I}\big(\sum_{k=1}^{d_j} j \cdot l_{jk}\big) \geq t$ form an F-basis called Jennings basis for $\omega(FG)^t$. Therefore for any $u \in B \setminus \big(\{1\} \cup \omega(FG)^2\big)$ we have

$$u \equiv \sum_{i=1}^{d_1} \alpha_{1i}(g_{1i} - 1) \pmod{\omega(FG)^2} \tag{1}$$

for some $\alpha_{1i} \in F$. Using the identity

$$\begin{aligned}(y-1)(x-1) = [(x-1)(y-1) + (x-1) + (y-1)](z-1)\\ + (x-1)(y-1) + (z-1),\end{aligned} \tag{2}$$

where $x, y \in G$ and $z = (y, x) = y^{-1}x^{-1}yx$ we have that

$$(g_{1j}-1)(g_{1i}-1) \equiv (g_{1i}-1)(g_{1j}-1) + (z_{ji}-1) \pmod{\omega(FG)^3}, \tag{3}$$

where $z_{ji} = (g_{1j}, g_{1i})$.

We are now ready to prove the main results.

Proof. [Proof of Theorem 1] By Rédei [13] a group G is minimal non-abelian if and only if it is one of the following groups:

(i) quaternion group Q_8 of order 8.

(ii) $\langle\, a, b \mid a^{p^n} = b^{p^m} = 1, a^b = a^{1+p^{n-1}} \,\rangle$, where $n \geq 2$, $m \geq 1$.

(iii) $\langle\, a, b, c \mid a^{p^n} = b^{p^m} = c^p = 1, (a, b) = c, (a, c) = (b, c) = 1 \,\rangle$, where $n, m \geq 2$ if $p = 2$.

Paris [12, Proposition 2] proved that FQ_8 has a filtered multiplicative F-basis if and only if F contains a primitive cube root of the unity.

Let $G = \langle\, a, b \mid a^{p^n} = b^{p^m} = 1, a^b = a^{1+p^{n-1}} \,\rangle$, where $n \geq 2$, $m \geq 1$. If either p is odd or $p = 2$ and $n \geq 3$, then G is powerful and FG does not possess any filtered multiplicative F-basis by [6, Theorem 1]. Thus we got that $n = 2$ and F is a field of characteristic 2. It is easy to see that if $m = 1$, then G is isomorphic to the dihedral group of order 8 and FG has a filtered multiplicative F-basis by [12]. The M-series of G in the case of $m > 1$ is as follows

$$\mathfrak{D}_1 = G, \quad \mathfrak{D}_2 = G^2, \quad \mathfrak{D}_3 = G^4, \cdots, \mathfrak{D}_m = G^{2^{m-1}}, \quad \mathfrak{D}_{m+1} = \langle\, 1 \,\rangle.$$

Choose $u, v \in \omega(FG)$ such that

$$\begin{aligned} u &\equiv (1+a) + (1+b) \pmod{\omega(FG)^2},\\ v &\equiv (1+b) \pmod{\omega(FG)^2}.\end{aligned}$$

Evidently, u and v modulo $\omega(FG)^2$ form a basis for $\omega(FG)/\omega(FG)^2$. From the congruence equation (3) we have that

$$(1+b)(1+a) \equiv (1+a)(1+b) + (1+a)^2 \pmod{\omega(FG)^3}, \tag{4}$$

and a simple calculation shows that

$$\begin{aligned} uv &\equiv (1+a)(1+b) + (1+b)^2 \pmod{\omega(FG)^3},\\ vu &\equiv (a+1)(1+b) + (1+a)^2 + (1+b)^2 \pmod{\omega(FG)^3},\\ u^2 = v^2 &\equiv (1+b)^2 \pmod{\omega(FG)^3}\end{aligned} \tag{5}$$

and they are linearly independent elements. The F-dimension of $\omega(FG)^2/\omega(FG)^3$ is equal to 3, so $\{uv, vu, u^2\}$ modulo $\omega(FG)^3$ forms an F-basis for $\omega(FG)^2/\omega(FG)^3$.

Now, we shall prove by induction on i that the elements $uvuv^i, vuv^{i+1}, u^2v^{i+1}, u^3v^i$ modulo $\omega(FG)^{i+4}$ form an F-basis for $\omega(FG)^{i+3}/\omega(FG)^{i+4}$. Using (4) and (5) we have that

$$\begin{aligned} uv\cdot u &\equiv (1+a)^2(1+b)+(1+a)^3+(1+b)^3 && \pmod{\omega(FG)^4},\\ vu\cdot v &\equiv (1+a)(1+b)^2+(1+a)^2(1+b)+(1+b)^3 && \pmod{\omega(FG)^4},\\ u^2\cdot v &\equiv (1+b)^3 && \pmod{\omega(FG)^4},\\ u^2\cdot u &\equiv (1+a)(1+b)^2+(1+b)^3 && \pmod{\omega(FG)^4}. \end{aligned}$$

Since uvu, vuv, u^2v, u^3 are linearly independent elements and the F-dimension of $\omega(FG)^3/\omega(FG)^4$ also equals 4, the statement is true for $i=0$. Using $v \equiv (1+b) \pmod{\omega(FG)^2}$ we get that

$$\begin{aligned} uvuv^i &\equiv (1+a)^2(1+b)^{i+1}+(1+a)^3(1+b)^i+(1+b)^{i+3} && \pmod{\omega(FG)^{i+4}},\\ vuv^{i+1} &\equiv (1+a)(1+b)^{i+2}+(1+a)^2(1+b)^{i+1}+(1+b)^{i+3} && \pmod{\omega(FG)^{i+4}},\\ u^2v^{i+1} &\equiv (1+b)^{i+3} && \pmod{\omega(FG)^{i+4}},\\ u^3v^i &\equiv (1+a)(1+b)^{i+2}+(1+b)^{i+3} && \pmod{\omega(FG)^{i+4}}, \end{aligned}$$

for any $i \geq 0$. Since the above mentioned elements are linearly independent and the F-dimension of $\omega(FG)^{i+3}/\omega(FG)^{i+4}$ is the same than the number of the non-zero elements of the set $\{uvuv^i, vuv^{i+1}, u^2v^{i+1}, u^3v^i\}$, so we have proved that $\big\{1, u, v, uv, vu, u^2, uvuv^i, vuv^{i+1}, u^2v^{i+1}, u^3v^i \mid i \geq 0\big\}$ is a filtered multiplicative F-basis for FG.

Let $G = \langle\, a, b, c \mid a^{p^n} = b^{p^m} = c^p = 1, (a,b) = c, (a,c) = (b,c) = 1 \,\rangle$ and $p > 2$. The first three term of the M-series of G is the following:

$$\mathfrak{D}_1 = G, \quad \mathfrak{D}_2 = \Phi(G), \quad \mathfrak{D}_3 = G^p.$$

Evidently, $\mathfrak{D}_1/\mathfrak{D}_2$ is generated by $a\mathfrak{D}_2$ and $b\mathfrak{D}_2$. Therefore the F dimension of $\omega(FG)/\omega(FG)^2$ is equal to 2. Assume that $B \setminus \{1\}$ is a filtered multiplicative F-basis of $\omega(FG)$. Let $u, v \in B \setminus (1 \cup \omega(FG)^2)$ be. According to (1) we have

$$\begin{aligned} u &\equiv \alpha_1(a-1)+\alpha_2(b-1) && \pmod{\omega(FG)^2},\\ v &\equiv \beta_1(a-1)+\beta_2(b-1) && \pmod{\omega(FG)^2} \end{aligned}$$

for some $\alpha_i, \beta_j \in F$. Being u and v linearly independent over F we have that $\Delta = \alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$.

We shall compute the all $b_ib_jb_k$ modulo $\omega(FG)^4$, where $b_i, b_j, b_k \in \{u, v\}$. For the sake of convenience our result will be summarized in a table, consisting of the coefficients of the decomposition $b_ib_jb_k$ with respect to the Jennings basis of $\omega(FG)^3/\omega(FG)^4$.

	$(a-1)^3$	$(a-1)^2(b-1)$	$(a-1)(c-1)$	$(a-1)(b-1)^2$	$(b-1)(c-1)$	$(b-1)^3$
uvu	$\alpha_1^2\beta_1$	$\alpha_1^2\beta_2+2\alpha_1\alpha_2\beta_1$	$-\alpha_1^2\beta_2$	$2\alpha_1\alpha_2\beta_2+\alpha_2^2\beta_1$	$-2\alpha_1\alpha_2\beta_2+\alpha_2^2\beta_1$	$\alpha_2^2\beta_2$
vu^2	$\alpha_1^2\beta_1$	$\alpha_1^2\beta_2+2\alpha_1\alpha_2\beta_1$	$-\alpha_1\alpha_2\beta_1$	$2\alpha_1\alpha_2\beta_2+\alpha_2^2\beta_1$	$-\alpha_1\alpha_2\beta_2$	$\alpha_2^2\beta_2$
u^3	α_1^3	$3\alpha_1^2\alpha_2$	$-\alpha_1^2\alpha_2$	$3\alpha_1\alpha_2^2$	$-\alpha_1\alpha_2^2$	α_2^3
u^2u	$\alpha_1\beta_1^2$	$2\alpha_1\beta_1\beta_2+\alpha_2\beta_1^2$	$-\alpha_1\beta_1\beta_2$	$2\alpha_2\beta_1\beta_2+\alpha_1\beta_2^2$	$\alpha_2\beta_1\beta_2-2\alpha_1\beta_2^2$	$\alpha_2\beta_2^2$
uv^2	$\alpha_1\beta_1^2$	$2\alpha_1\beta_1\beta_2+\alpha_2\beta_1^2$	$-\alpha_1\beta_1\beta_2$	$2\alpha_2\beta_1\beta_2+\alpha_1\beta_2^2$	$\alpha_2\beta_1\beta_2$	$\alpha_2\beta_2^2$
vuv	$\alpha_1\beta_1^2$	$2\alpha_1\beta_1\beta_2+\alpha_2\beta_1^2$	$-\alpha_2\beta_1^2$	$2\alpha_2\beta_1\beta_2+\alpha_1\beta_2^2$	$-2\alpha_2\beta_1\beta_2+\alpha_1\beta_2^2$	$\alpha_2\beta_2^2$
u^2v	$\alpha_1^2\beta_1$	$2\alpha_1\alpha_2\beta_1+\alpha_1^2\beta_2$	$-\alpha_1\alpha_2\beta_1$	$2\alpha_1\alpha_2\beta_2+\alpha_2^2\beta_1$	$\alpha_1\alpha_2\beta_2-2\alpha_2^2\beta_1$	$\alpha_2^2\beta_2$
v^3	β_1^3	$3\beta_1^2\beta_2$	$-\beta_1^2\beta_2$	$3\beta_1\beta_2^2$	$-\beta_1\beta_2^2$	β_2^3

Assume that $uvu \equiv 0 \pmod{\omega(FG)^4}$. Since the coefficients of $(a-1)^3, (b-1)^3, (a-1)^2(b-1), (a-1)(b-1)^2$ are equal to zero, we have either $u \equiv 0 \pmod{\omega(FG)^2}$ or $v \equiv 0 \pmod{\omega(FG)^2}$ which is impossible. By similar arguments we can see that the above mentioned eight elements are not congruent to zero modulo $\omega(FG)^4$.

Since the F-dimension of $\omega(FG)^3/\omega(FG)^4$ equals six, and we have obtained eight non-zero elements modulo $\omega(FG)^4$, we conclude that some of them coincide (see property (ii)). For example, suppose that $uvu = vu^2$. If it occurs, then the coefficients of $(a-1)(c-1)$ equal each other and so do the coefficients of $(b-1)(c-1)$. Thus $\alpha_1\Delta = -\alpha_2\Delta = 0$. Since $\Delta \neq 0$ we have $u \equiv 0 \pmod{\omega(FG)^2}$ which is a contradiction. In a similar manner we can verify that the above mentioned eight elements are different. The F-dimension of $\omega(FG)^3/\omega(FG)^4$ is equal to six, but we have eight different non-zero elements modulo $\omega(FG)^4$ which is impossible.

Let $G = \langle\, a, b, c \mid a^{2^n} = b^{2^m} = c^2 = 1, (a, b) = c, (a, c) = (b, c) = 1 \,\rangle$ and $n, m \geq 2$. Then FG has a filtered multiplicative F-basis by [2, Theorem 2]. □

Proof. [Proof of Corollary 1] Let FG be a group algebra of a finite p-group G over a field F of characteristic p, such that all elements of the unit group of FG of order p commute. According to [1, Theorem 2] and [7, Theorem 1.1] G is one of the following groups.

(i) Q_8 or $Q_8 \times C_{2^n}$.

(ii) $\langle\, a, b \mid a^{p^n} = b^{p^m} = 1, a^b = a^{1+p^{n-1}} \,\rangle$, where $n \geq 2$, $m \geq 1$, if $p = 2$, then $m \neq 1$.

(iii) $\langle\, a, b, c \mid a^4 = 1, a^2 = b^2 = (a, b), c^{2^n} = 1, (a, c) = c^{2^{n-1}} \,\rangle$, where $n \geq 2$.

(iv) $\langle\, a, b, c \mid a^4 = b^4 = 1, a^2 = (b, a), b^2 = c^2 = (c, a), x^2y^2 = (c, b) \,\rangle$.

Let G be either Q_8 or $Q_8 \times C_{2^n}$. By [12, Proposition 2] FG has a filtered multiplicative F-basis if and only if F contains a primitive cube root of the unity. Let $G = \langle\, a, b \mid a^{p^n} = b^{p^m} = 1, a^b = a^{1+p^{n-1}} \,\rangle$ and n, m, p such as in *(ii)*. From the proof of *Theorem 1* we can see that FG has a filtered multiplicative F-basis if and only if $G = \langle\, a, b \mid a^4 = b^{2^m} = 1, a^b = a^3 \,\rangle$, where $m \geq 2$.

Let $G = \langle\, a, b, c \mid a^4 = 1, a^2 = b^2 = (a, b), c^{2^n} = 1, (a, c) = c^{2^{n-1}} \,\rangle$, where $n \geq 2$. If $n = 2$, then G is isomorphic to the group G_{35} of order 2^5, where 35 is the index of this group in GAP. According to [2, Theorem 4], FG has no filtered multiplicative F-basis. Assume that $n > 2$ and FG has a filtered multiplicative F-basis. Since $G/\mathfrak{D}_2 = \langle\, a\mathfrak{D}_2, b\mathfrak{D}_2, c\mathfrak{D}_2 \,\rangle$ we can write that

$$\begin{aligned} b_1 &\equiv \alpha_1(1+a) + \alpha_2(1+b) + \alpha_3(1+c) \pmod{\omega(FG)^2}, \\ b_2 &\equiv \beta_1(1+a) + \beta_2(1+b) + \beta_3(1+c) \pmod{\omega(FG)^2}, \\ b_3 &\equiv \gamma_1(1+a) + \gamma_2(1+b) + \gamma_3(1+c) \pmod{\omega(FG)^2}, \end{aligned}$$

for some $\alpha_i, \beta_i, \gamma_i \in F$ by (1). Denote by Δ the determinant of the matrix $\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix}$. Evidently, $\Delta \neq 0$ because b_1, b_2, b_3 are linearly independent in $\omega(FG) \setminus \omega(FG)^2$. It is easy to check that $(1 + g^{-1}) = \sum_{i=1}^{|g|-1}(1+g)^i$ holds for any $g \in G$, where $|g|$ denotes the order of g. Using the well known identity

$$(1 + gh) = (1 + g) + (1 + h) + (1 + g)(1 + h),$$

where $g,h\in G$ and the identity above an easy calculation shows that

$$\begin{aligned}(1+b)(1+a)&=\sum_{i=1}^{3}(1+a)^i(1+b)+(1+a)^2+(1+a)^3,\\(1+c)(1+a)&=(1+a)(1+c)+\sum_{i=2^{n-1}}^{2^{n-1}+1}\big[(1+c)^i+(1+a)(1+c)^i\big].\end{aligned}\tag{6}$$

We shall compute b_ib_j modulo $\omega(FG)^3$, where $i,j\in\{1,2,3\}$. The result of our computation will be written in a table as in the proof of *Theorem 1.* The following table consists of the coefficients of the decompositions with respect to the Jennings basis of $\omega(FG)^2/\omega(FG)^3$.

	$(1+a)^2$	$(1+a)(1+b)$	$(1+a)(1+c)$	$(1+b)(1+c)$	$(1+c)^2$
b_1b_2	$\alpha_1\beta_1+\alpha_2\beta_1+\alpha_2\beta_2$	$\alpha_1\beta_2+\alpha_2\beta_1$	$\alpha_1\beta_3+\alpha_3\beta_1$	$\alpha_2\beta_3+\alpha_3\beta_2$	$\alpha_3\beta_3$
b_2b_1	$\alpha_1\beta_1+\alpha_1\beta_2+\alpha_2\beta_2$	$\alpha_1\beta_2+\alpha_2\beta_1$	$\alpha_1\beta_3+\alpha_3\beta_1$	$\alpha_2\beta_3+\alpha_3\beta_2$	$\alpha_3\beta_3$
b_1b_3	$\alpha_1\gamma_1+\alpha_2\gamma_1+\alpha_2\gamma_2$	$\alpha_1\gamma_2+\alpha_2\gamma_1$	$\alpha_1\gamma_3+\alpha_3\gamma_1$	$\alpha_2\gamma_3+\alpha_3\gamma_2$	$\alpha_3\gamma_3$
b_3b_1	$\alpha_1\gamma_1+\alpha_1\gamma_2+\alpha_2\gamma_2$	$\alpha_1\gamma_2+\alpha_2\gamma_1$	$\alpha_1\gamma_3+\alpha_3\gamma_1$	$\alpha_2\gamma_3+\alpha_3\gamma_2$	$\alpha_3\gamma_3$
b_2b_3	$\beta_1\gamma_1+\beta_2\gamma_1+\beta_2\gamma_2$	$\beta_1\gamma_2+\beta_2\gamma_1$	$\beta_1\gamma_3+\beta_3\gamma_1$	$\beta_2\gamma_3+\beta_3\gamma_2$	$\beta_3\gamma_3$
b_3b_2	$\beta_1\gamma_1+\beta_1\gamma_2+\beta_2\gamma_2$	$\beta_1\gamma_2+\beta_2\gamma_1$	$\beta_1\gamma_3+\beta_3\gamma_1$	$\beta_2\gamma_3+\beta_3\gamma_2$	$\beta_3\gamma_3$
b_1^2	$\alpha_1^2+\alpha_1\alpha_2+\alpha_2^2$	0	0	0	α_3^2
b_2^2	$\beta_1^2+\beta_1\beta_2+\beta_2^2$	0	0	0	β_3^2
b_3^2	$\gamma_1^2+\gamma_1\gamma_2+\gamma_2^2$	0	0	0	γ_3^2

If $i\neq j$, then $b_ib_j\not\equiv 0\pmod{\omega(FG)^3}$ because $\Delta\neq 0$. It is easy to see that $b_ib_j\not\equiv b_kb_l\pmod{\omega(FG)^3}$ if either $k\notin\{i,j\}$ or $l\notin\{i,j\}$. Indeed, for example, if $b_1b_2\equiv b_1b_3\pmod{\omega(FG)^3}$, then the coefficients of $(1+a)(1+b)$ are equal to each other and so are the coefficients of $(1+a)(1+c)$ or $(1+b)(1+c)$. Thus $\Delta=det\begin{pmatrix}\alpha_1&\alpha_2&\alpha_3\\\gamma_1&\gamma_2&\gamma_3\\\gamma_1&\gamma_2&\gamma_3\end{pmatrix}=0$ which is a contradiction.

Since the F-dimension of $\omega(FG)^2/\omega(FG)^3$ is equal to five but we get nine elements we conclude that there exist two indices i,j such that $b_ib_j=b_jb_i$ for example $b_1b_3=b_3b_1$. Evidently, If $b_i^2\equiv 0\pmod{\omega(FG)^3}$ for any $i\in\{1,2,3\}$, then $\Delta=0$ which is impossible. Thus we have

$$\begin{aligned}b_1&\equiv\omega(1+a)+(1+b)+\alpha_3(1+c)&\pmod{\omega(FG)^2},\\b_2&\equiv(1+a)+\omega(1+b)+\beta_3(1+c)&\pmod{\omega(FG)^2},\\b_3&\equiv(1+c)&\pmod{\omega(FG)^2},\end{aligned}$$

where ω is a primitive cube root of the unity. The other cases are symmetric to this one.

Clearly, every element of the Jennings basis of FG can be written in the form $(1+a)^{t_1}(1+b)^{t_2}(1+c)^{t_3}$ for some $t_1,t_2,t_3\in\mathbb{N}$. Thus we can write that

$$b_1=\omega(1+a)+(1+b)+\alpha_3(1+c)+(1+a)A_1+(1+b)A_2+\sum_{s=2}^{2^n-1}\delta_s(1+c)^s,$$

where $\delta_s\in F$, $A_1,A_2\in\omega(FG)$ and $(1+a)A_1,(1+b)A_2$ are linear combinations of the elements of Jennings basis over F. Similarly,

$$b_3=(1+c)+(1+a)B_1+(1+b)B_2+\sum_{t=2}^{2^n-1}\epsilon_t(1+c)^t,$$

where $\epsilon_t \in F$, $B_1, B_2 \in \omega(FG)$ and $(1+a)B_1, (1+b)B_2$ are linear combinations of the Jennings basis over F.

By the help of (6) let us calculate $b_1b_3 + b_3b_1$ modulo $\omega(FG)^{2^{n-1}+1}$.

$$b_1b_3 + b_3b_1 \equiv \omega(1+c)^{2^{n-1}} + (1+a)C_1 + (1+b)C_2 \pmod{\omega(FG)^{2^{n-1}+1}},$$

for some $C_1, C_2 \in \omega(FG)^2$. Therefore $b_1b_3 \neq b_3b_1$ which is a contradiction. Thus FG does not possess any filtered multiplicative F-basis.

If $G = \langle\, a, b, c \mid a^4 = b^4 = 1, a^2 = (b,a), b^2 = c^2 = (c,a), x^2y^2 = (c,b)\,\rangle$, then $G \cong G_{32}$, where G_{32} is the group of order 32 with index 32 in GAP. Then, FG has no filtered multiplicative F-basis by [2, Theorem 4].

□

Proof. [proof of Theorem 2] The finite non-metacyclic p-groups all of whose proper subgroup are metacyclic was classified by Blackburn [4, Theorem 3.2].

If $p = 2$, then $G = Q_8 \times C_2$ or $G = Q_8 \curlyvee C_4$ or $G = G_{32}$. If $G = Q_8 \times C_2$ and F contains a primitive cube root of the unity, then FG admits a filtered multiplicative F-basis by [12, Proposition 2]. According to [2, Theorem 1] FG has a filtered multiplicative F-basis for $G = Q_8 \curlyvee C_4$. The group algebra FG_{32} has no multiplicative filtered F-basis by [2, Theorem 4].

If p is odd and G is the group of order p^3 with exponent p, then G is a powerful group and FG has no filtered multiplicative F-basis by [6, Theorem 1].

We have remained only the following group $G = \langle\, a, b, c \mid b^9 = c^3 = 1, a^3 = b^{-3}, (c,b) = 1, (b,a) = c, (c,a) = b^{-3}\,\rangle$. The M-series of G is the following:

$$\mathfrak{D}_1 = G, \quad \mathfrak{D}_2 = \langle\, c, b^3\,\rangle, \quad \mathfrak{D}_3 = \langle\, b^3\,\rangle, \quad \mathfrak{D}_4 = \langle\, 1\,\rangle.$$

Since $G/\mathfrak{D}_2 = \langle\, a\mathfrak{D}_2, b\mathfrak{D}_2\,\rangle$ we have only to apply (1) to see that the elements of an F-basis of $\omega(FG)/\omega(FG)^2$ can be written in the form

$$\begin{aligned} u &\equiv \alpha_1(a-1) + \alpha_2(b-1) \pmod{\omega(FG)^2},\\ v &\equiv \beta_1(a-1) + \beta_2(b-1) \pmod{\omega(FG)^2}, \end{aligned}$$

for some $\alpha_i, \beta_j \in F$. Evidently, $\Delta = \alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$ because u and v are linearly independent over F. Using the identities (2) and (3) we have

$$\begin{aligned} (b-1)(a-1) &\equiv (a-1)(b-1) + (c-1) \pmod{\omega(FG)^3},\\ (c-1)(a-1) &\equiv (a-1)(c-1) - (b-1)^3 \pmod{\omega(FG)^4}. \end{aligned} \tag{7}$$

By the help of congruence equations (7) let us compute $b_ib_jb_k$ modulo $\omega(FG)^4$, where $b_i, b_j, b_k \in \{u, v\}$. The result of our computation will be written in a table as before. We shall divide our table into two parts. The coefficients corresponding to the first three elements of the Jennings basis of $\omega(FG)^3/\omega(FG)^4$ will be in the first part of the table, while the next three will be in the second one.

	$(a-1)^3$	$(a-1)^2(b-1)$	$(a-1)(c-1)$
uvu	$\alpha_1^2\beta_1$	$2\alpha_1\alpha_2\beta_1+\alpha_1^2\beta_2$	$\alpha_1^2\beta_2+2\alpha_1\alpha_2\beta_1$
vu^2	$\alpha_1^2\beta_1$	$2\alpha_1\alpha_2\beta_1+\alpha_1^2\beta_2$	$2\alpha_1^2\beta_2+\alpha_1\alpha_2\beta_1$
u^3	α_1^3	0	0
v^2u	$\alpha_1\beta_1^2$	$\alpha_2\beta_1^2+2\alpha_1\beta_1\beta_2$	0
uv^2	$\alpha_1\beta_1^2$	$2\alpha_1\beta_1\beta_2+\alpha_2\beta_1^2$	$\alpha_1\beta_1\beta_2+2\alpha_2\beta_1^2$
vuv	$\alpha_1\beta_1^2$	$2\alpha_1\beta_1\beta_2+\alpha_2\beta_1^2$	$2\alpha_1\beta_1\beta_2+\alpha_2\beta_1^2$
u^2v	$\alpha_1^2\beta_1$	$2\alpha_1\alpha_2\beta_1+\alpha_1^2\beta_2$	0
v^3	β_1^3	0	0

	$(a-1)(b-1)^2$	$(b-1)(c-1)$	$(b-1)^3$
uvu	$2\alpha_1\alpha_2\beta_2+\alpha_2^2\beta_1$	$\alpha_2^2\beta_1+2\alpha_1\alpha_2\beta_2$	$\alpha_2^2\beta_2-\alpha_1\alpha_2\beta_1$
vu^2	$2\alpha_1\alpha_2\beta_2+\alpha_2^2\beta_1$	0	$\alpha_2^2\beta_2-\alpha_1^2\beta_2$
u^3	0	0	$\alpha_2^3-\alpha_1^2\alpha_2$
v^2u	$2\alpha_2\beta_1\beta_2+\alpha_1\beta_2^2$	$\alpha_2\beta_1\beta_2+2\alpha_1\beta_2^2$	$\alpha_2\beta_2^2-\alpha_1\beta_1\beta_2$
uv^2	$\alpha_1\beta_2^2+2\alpha_2\beta_1\beta_2$	0	$\alpha_2\beta_2^2-\alpha_2\beta_1^2$
vuv	$\alpha_1\beta_2^2+2\alpha_2\beta_1\beta_2$	$\alpha_1\beta_2^2+2\alpha_2\beta_1\beta_2$	$\alpha_2\beta_2^2-\alpha_1\beta_1\beta_2$
u^2v	$2\alpha_1\alpha_2\beta_2+\alpha_2^2\beta_1$	$2\alpha_2^2\beta_1+\alpha_1\alpha_2\beta_2$	$\alpha_2^2\beta_2-\alpha_1\alpha_2\beta_1$
v^3	0	0	$\beta_2^3-\beta_1^2\beta_2$

We have obtained eight elements. It is easy to prove that each of them is in $\omega(FG)^3$. Indeed, for example, if $uvu\notin\omega(FG)^3$, then we have $uvu\equiv 0 \pmod{\omega(FG)^4}$. Thus the coefficient of $(b-1)(c-1)$ are equal to 0, that is $\alpha_2^2\beta_1+2\alpha_1\alpha_2\beta_2=0$. Assume that $\alpha_1=0$. Then $\alpha_2^2\beta_1+2\alpha_1\alpha_2\beta_2=\alpha_2^2\beta_1=0$ so $\Delta=0$ which is impossible. Since $\alpha_1\neq 0$ and $2\alpha_1\alpha_2\beta_1+\alpha_1^2\beta_2=0$ because the coefficient of $(a-1)^2(b-1)$ also equals 0 we have that $\Delta=2\alpha_2\beta_1+\alpha_1\beta_2=0$ which is also impossible. In a similar manner the other cases can be verified.

Since the F-dimension of $\omega(FG)^3/\omega(FG)^4$ is six but we have eight elements we conclude that some of the above mentioned elements are equal to some other elements. For example if $uvu=vu^2$, then we have that $\alpha_2^2\beta_1+2\alpha_1\alpha_2\beta_2=0$ because the coefficients of $(b-1)(c-1)$ are equal to each other. Clearly, $\alpha_1\neq 0$. Indeed, if $\alpha_1=0$, then $\alpha_2^2\beta_1=0$ and $\Delta=0$ which is impossible. From the coefficients of $(a-1)(c-1)$ we get that $\alpha_1\beta_2=\alpha_2\beta_1$ and $\Delta=0$ which is also a contradiction. A similar method can be used to prove that the only possible case is $u^3=v^3$. Thus only seven elements are left. However, the F-dimension of $\omega(FG)^3/\omega(FG)^4$ is six which is impossible. We have proved that FG has no filtered multiplicative F-basis.

□

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