On the generalized vector F-implicit complementarity problems and vector F-implicit variational inequality problems*

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Abstract. *In this paper, we introduce and analyze some new classes of generalized vector-F implicit complementarity problems and the general mixed vector-F variational inequalities. Under suitable conditions, we prove the equivalences between these new problems. We establish several existence theorems for these classes of vector-F complementarity and general mixed vector-F variational inequalities using a new version of the Fan-KKM theorem in Hausdorff topological vector spaces, and without even using the classical assumptions in this context, like monotonicity or continuity. Results obtained in this paper represent significant improvement and refinement of the previously known results.*

Key words: *topological vector space, vector F-implicit complementarity problems, KKM mapping, vector F-implicit variational inequality problems, positively homogeneous mapping*

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1. Introduction

Complementarity problems theory, which was introduced by Lemke [7], has emerged as a powerful tool to investigate and study a wide class of unrelated problems arising in industry, finance, management, economics, regional, ecology, pure and applied sciences in a unified and flexible framework. Complementarity problems have been generalized and extended in several directions using novel and innovative techniques. Closely related to the complementarity problems is the theory of variational inequalities. It is well known that the variational inequalities and complementarity

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problems are equivalent if the underlying set is a convex cone. This equivalence has played a crucial part in the existence results and numerical methods for solving these problems.

Itoh, Takahashi and Yanagi [9] considered the F-complementarity problems and proved that the F-complementarity problems are equivalent to the mixed variational inequalities. This class of F-complementarity problems has bee considered and investigated in [9]. Giannessi (see [1]) considered the vector variational inequalities and discussed its applications. For the recent developments and applications of the vector variational inequalities and vector complementarity problems, see [1-18] and the references therein. Li and Huang [10] considered the vector F-implicit complementarity problems and proved the equivalence between vector F-implicit complementarity problems and vector F-implicit variational inequalities along with some existence results for the their solution.

It is worth mentioning that the variational inequality theory developed so far is applicable for studying even order and symmetric problems. In 1988, Noor [8] introduced and studied a new class of variational inequalities involving two operators, now known as general(Noor) variational inequalities. It has been shown that a large class of odd-order and nonsymmetric problems arising in various branches of pure and applied science can be studied in the unified and general framework of the general variational inequalities, see [11,12,14] and the references therein. It has been shown that the general variational inequalities are equivalent to the general complementarity problems and the Wiener-Hopf equations, see Noor [14].

Motivated and inspired by the research work going in these fascinating and interesting fields, we introduce some new classes of general vector F-implicit complementarity problems and general vector F-implicit variational inequalities in Hausdorff topological vector spaces. We establish the equivalence between these new problems under certain suitable assumptions. Furthermore, we also obtain some new existence theorems for solutions of generalized vector F-implicit complementarity problems and the generalized vector F-implicit variational inequality problems by using a new version of the Fan-KKM Theorem [3] in Hausdorff topological vector spaces without monotonicity and even without continuity assumption. Since the general vector F-implicit complementarity problems and general vector F-implicit (quasi) variational inequalities include complementarity problems, variational inequalities and other related optimization problems as special cases, our results continue to hold for these problems. In this respect, our new results generalize and improve the recent results in several direction.

In the rest of this section we recall some definitions and preliminaries results which are used in the next section.

We shall denote by 2^A the family of all subsets of A and by $\mathcal{F}(A)$ the family of all nonempty finite subsets of A. Let X be a real Hausdorff topological vector space (in short, t.v.s.). A nonempty subset P of X is called convex cone if (i) $P+P=P$, (ii) $\lambda P \subset P$, for all $\lambda \geq 0$. Cone P is said to be pointed whenever $P \cap -P = \{0\}$. Let Y be a t.v.s. and $P \subset Y$ be a cone. The cone P induces an ordering on Y (in this case the pair (Y,P) is called an ordered t.v.s.) which is defined as follows:

$$x \leq y \Leftrightarrow y - x \in P. \tag{1}$$

This ordering is anti-symmetric if P is pointed. Let K be a nonempty convex subset of a t.v.s. X and let K_0 be a subset of K. A multi-valued map $\Gamma : K_0 \to 2^K$ is said to be a KKM map if

$$coA \subseteq \cup_{x\in A}\Gamma(x), \quad \forall A \in \mathcal{F}(K_0),$$

where co denotes the convex hull.

Let X be a nonempty set, Y a topological space, and $\Gamma : X \to 2^Y$ a multivalued map. Then, Γ is called transfer closed-valued if, $y \notin \Gamma(x)$ there exists $x' \in X$ such that $y \notin cl\Gamma(x')$, where $cl\Gamma(x')$ denotes the topological closure of $\Gamma(x')$. It is clear that, $\Gamma : X \to 2^Y$ is transfer closed-valued if and only if

$$\bigcap_{x\in X}\Gamma(x) = \bigcap_{x\in X} cl\Gamma(x). \tag{2}$$

If $A \subseteq X$ and $B \subseteq Y$, then $\Gamma : A \to 2^B$ is called transfer closed-valued if the multivalued mapping $x \to \Gamma(x) \cap B$ is transfer closed-valued. In this case where $X = Y$ and $A = B$, Γ is called transfer closed-valued on A.

We need the following lemma in the next section (see [3]).

Lemma 1.1. *Let K be a nonempty convex subset of X. Suppose that $\Gamma, \widehat{\Gamma} : K \to 2^K$ are two multivalued mappings such that:*

(a) $\widehat{\Gamma}(x) \subseteq \Gamma(x)$, $\forall\, x \in K$;

(b) $\widehat{\Gamma}$ *is a KKM map;*

(c) for each $A \in \mathcal{F}(K)$, Γ *is transfer closed-valued on* coA;

(d) for each $A \in \mathcal{F}(K)$, $cl_K(\bigcap_{x\in coA}\Gamma(x))\bigcap coA = (\bigcap_{x\in coA}\Gamma(x))\bigcap coA$;

(e) there is a nonempty compact convex set $B \subseteq K$ *such that* $cl_K(\bigcap_{x\in B}\Gamma(x))$ *is compact.*

Then, $\bigcap_{x\in K}\Gamma(x) \neq \emptyset$.

2. Main results

Throughout this section, suppose X and Y are real Hausdorff t.v.s. and K a nonempty convex subset of X. Denote by $L(X,Y)$ the space of all continuous linear mappings from X into Y, and $\langle t, x\rangle$ the value of the linear continuous mapping $t \in L(X,Y)$ at x. Let $A, T, g : K \to K, F : K \to Y$, $N : K \times K \to L(X,Y)$, and $C : K \to 2^Y$, with nonempty convex pointed cone values, that is $C(x)$ is nonempty and convex cone for each $x \in K$. We consider the following generalized vector F-implicit complementarity problem (GVF-ICP)in t.v.s. Find $x \in K$ such that

$$\langle N(Ax, Tx),\ g(x)\rangle + F(g(x)) = 0$$

and

$$\langle N(Ax, Tx),\ g(y)\rangle + F(g(y)) \in C(x), \qquad \forall y \in K. \tag{2.1}$$

Special cases

(1) The following vector F-implicit complementarity problem (VF-ICP) which consists of finding $x \in K$ such that

$$\langle f(x), g(x)\rangle + F(x) = 0 \quad \text{and} \quad \langle f(x), y\rangle + F(y) \in C(x), \qquad \forall y \in K.$$

is a particular form of (GVF-ICP) for the identities A and T and a mapping $f : K \to L(X,Y)$ defined by $N(x,y) = f(x)$ for all $x, y \in K$, where $f : K \to L(X,Y)$, which was considered and studied in [10] for a constant cone P, that is, $C(x) = P$, for all $x \in K$.

(2) If g is an identity mapping on K, then (VF-ICP) reduces to the vector F-complementary problem (in short VF-CP) which consists of finding $x \in K$ such that

$$\langle f(x), x\rangle + F(x) = 0 \quad \text{and} \quad \langle f(x), y\rangle + F(y) \in C(x), \qquad \forall y \in K.$$

(3) If $F = 0$, then (VF-CP) reduces to the vector complementary problem (in short VCP) which consists of finding $x \in K$ such that

$$\langle f(x), x\rangle = 0 \quad \text{and} \quad \langle f(x), y\rangle \in C(x), \qquad \forall y \in K,$$

which has been studied by Chen and Yang [2] in particular case $C(x) = P, \forall x \in K$.

(4) If $L(X,Y) = X^*$ and $F : K \to \mathbb{R}$, then (VF-ICP) reduces to the $F-$ implicit complementary problems (in short F-ICP) which consists of finding $x \in K$ such that

$$\langle f(x), x\rangle + F(x) = 0 \quad \text{and} \quad \langle f(x), y\rangle + F(y) \in C(x), \;\; \forall y \in K.$$

which were considered by Huang and Li [6], in particular case $C(x) = P, \forall x$.

(5) If g is an identity mapping on K, then (F-ICP) reduces to the F-complementary problem (in short F-CP) which consists of finding $x \in K$ such that

$$\langle f(x), x\rangle + F(x) = 0 \quad \text{and} \quad \langle f(x), y\rangle + F(y) \in C(x), \qquad \forall y \in K,$$

which has been studied by Yin et al. [17], in particular case $C(x) = P, \forall x$.

(6) If $F = 0$, then ($F-$ ICP) reduces to the implicit complementary problem (in short ICP)which consists of finding $x \in K$ such that

$$\langle f(x), g(x)\rangle = 0 \quad \text{and} \quad \langle f(x), y\rangle \in C(x), \qquad \forall y \in K,$$

which has been studied by Noor [8]. See also [17,18] .

We also consider the following generalized vector F-implicit variational inequality problem (GVF-IVIP): find $x \in K$ such that

$$\langle N(Ax,Tx),\; g(y) - g(x)\rangle + F(g(y)) - F(g(x)) \in C(x), \qquad \forall y \in K.$$

The above problem reduces to the vector F-implicit variational inequality problem considered in [10] if, we set A, T the identities mapping and define $N(x,y) = f(x)$

for all $x, y \in K$, where $f : K \to L(x, y), C(x) = P$, where (Y, P) is an ordered Banach space.

The following theorem establishes the equivalence between (GVF-ICP) and (GVF-IVIP).

Theorem 2.1.

(i) If x solves (GVF-ICP), then x solves (GVF-IVIP).

(ii) Let $0 \in K$, $2K \subset K$ (K is not necessarily a cone) and $F : K \to Y$ satisfy $F(2x) = 2F(x)$, for all $x \in K$. If g is onto and x solves (GVF-IVIP), then x solves (GVF-ICP).

Proof. By the definitions of (GVF-ICP) and (GVF-IVIP), (i) trivially holds. Now let $x \in K$ solves (GVF-IVIP). Hence

$$\langle N(Ax, Tx),\ g(y) - g(x)\rangle + F(g(y)) - F(g(x)) \in C(x),\ \forall y \in K. \qquad (I)$$

Since g is onto, there exist $y, y' \in K$ such that $g(y) = 0, g(y') = 2g(x)$. By letting $g(y), g(y')$ in (I) we obtain that

$$\langle N(Ax, Tx), g(x)\rangle + F(g(x)) \in -C(x)$$

and

$$\langle N(Ax, Tx), g(x)\rangle + F(g(x)) \in C(x),$$

and hence $\langle N(Ax, Tx), g(x)\rangle + F(g(x)) \in C(x) \cap -C(x)$.

Since $C(x)$ is a pointed cone

$$\langle N(Ax, Tx), g(x)\rangle + F(g(x)) = 0. \qquad (II)$$

By adding (I) and (II) the result follows, i.e.,

$$\langle N(Ax, Tx),\ g(y)\rangle + F(g(y)) \in C(x) + C(x) \subset C(x), \quad \forall y \in K. \qquad \square$$

Corollary 2.1 ([10])

(i) If x solves (VF-ICP), then x solves (VF-IVIP).

(ii) Let K be a closed convex cone. If $F : K \to Y$ is positively homogeneous and x solves (VF-IVIP), then x solves (VF-ICP).

The following example shows that the assumption g is onto cannot be omitted from *Theorem 2.1.*

Example 2.1. *Let $X = Y = K = \mathbb{R}$ and $Ax = Tx = F(x) = x, g(x) = 1, N(x, y) = xy$ for all $x, y \in K$. The (GVF-ICP) does not have any solution, because of $\langle N(Ax, Tx), g(x)\rangle + F(g(x)) = x^2 + 1 = 0$, does not have solution, while every member of K is a solution of the (GVF-IVIP).*

Remark 2.1. *It is obvious that if K is a closed convex cone, then $0 \in K$ and $2K \subset K$, while $K = Q$, rational numbers, as a subset of $X = \mathbb{R}$ is not a cone and $0 \in K$ and $2K \subset K$. If $F : K \to Y$ is positively homogeneous, then $F(2x) = 2F(x)$. The following example shows that the converse does not hold in general. These facts show that* Theorem 2.1 *improves Theorem 3.1* (Corollary 2.1) in [10].

Example 2.2. *Let* $X = Y = \mathbb{R},\ K = \mathbb{R}, A(x) = T(x) = 0, g(x) = x,\ N(x, y) = 0$, *for all* $x, y \in K$ *and* $F : K \to Y$ *defined by*

$$F(x) = \begin{cases} x & \text{if} \quad x \in Q \\ 0 & \text{if} \quad x \notin Q. \end{cases} \tag{3}$$

The following theorem improves and extends Theorem 3.2 in [10].

Theorem 2.2. *Assume that*

(a) for all $A \in \mathcal{F}(K)$ *the multivalued map* $\Gamma_A : coA \to 2^K$ *defined by* $\Gamma_A(x) = \{y \in K : \langle N(Ay, Ty), g(x) - g(y)\rangle + F(g(x)) - F(g(y)) \in C(y)\}$ *is transfer-closed valued mapping;*

(b) there exist a nonempty compact subset B *and a nonempty convex compact subset* D *of* K *such that, for each* $x \in K \backslash B$, *there exists* $y \in D$ *such that* $\langle N(Ax, Tx), g(y) - g(x)\rangle + F(g(y)) - F(g(x)) \notin C(x)$.

(c) there exists a mapping $h : K \times K \to Y$ *such that*

- *(i)* $h(x, x) \in C(x)$, *for all* $x \in K$;
- *(ii)* $\langle N(Ay, Ty), g((x) - g(y)\rangle + F(g(x)) - F(g(y)) - h(y, x) \in C(y), \forall x, y \in K$;
- *(iii) the set* $\{y \in K : h(x, y) \notin C(x)\}$ *is convex, for all* $x \in K$.

Then, the solution set of (GVF-IVIP) is nonempty and compact.

Proof. Define $\Gamma, \hat{\Gamma} : K \to 2^K$ as follows
$\hat{\Gamma}(x) = \{y \in K : h(y, x) \in C(y)\}$,
$\Gamma(x) = \{y \in K : \langle N(Ay, Ty), g((x) - g(y)\rangle + F(g(x)) - F(g(y)) \in C(y)\}$.
We show that $\Gamma, \widehat{\Gamma}$ satisfy conditions of *Lemma 1.1*. By (c)(ii), $\widehat{\Gamma}(y) \subseteq \Gamma(y)$, for all $y \in K$. If $A = \{x_1, x_2, ..., x_n\} \subseteq K$, $z \in coA$ and $z \notin \bigcup_{i \in \{1,2,...,n\}} \widehat{\Gamma}(x_i)$, then $h(z, x_i) \notin C(z)$ for $i = 1, 2, 3, ..., n$. It follows by (c) and (iii) that $h(z, z) \notin C(z)$ which contradicts (i)of (c). By (a),Γ satisfies the assumptions of (b) and (c) of Lemma (1). Hence, by *Lemma 1.1*, we have,

$$\bigcap_{x \in K} \Gamma(x) \neq \emptyset, \tag{4}$$

which shows that the problem (GVF-IVIP) has a solution. □

Remark 2.2. *In the previous theorem, if we let the mappings* N, A, T, g *and* F *be continuous and let* $C : K \to 2^K$ *have closed graph, then condition (a) trivially holds. Moreover, if for all* $x \in K$ *the set* $\{y \in K : \langle N(Ay, Ty), g((x) - g(y)\rangle + F(g(x)) - F(g(y)) \in C(y)\}$ *is closed in K, for all* $x \in K$, *then condition (a) holds and in this case the solution set of (GVF-IVIP) is a compact subset of* K.

In view of the proof of *Theorem 2.2* and *Remark 2.2*, we can obtain the following existence result of the solution set of (GVF-IVIP) .

Theorem 2.3. *Suppose that*

(i) $h(x, x) \in C(x), \quad \forall x \in K$;

(ii) the set $\{y \in K : h(y, x) \in C(y)\}$ *is closed in* K, *for all* $x \in K$;

(iii) *the set* $\{y \in K : h(x, y) \notin C(x)\}$ *is convex,* $\forall x \in K$;

(v) *there exist a nonempty compact subset* B *and a nonempty convex compact subset* D *of* K *such that, for each* $x \in K \backslash B$, *there exists* $y \in D$ *such that* $h(x, y) \notin C(x)$.

If, for every $x, y \in K$, *the following implication holds*

$$\langle N(Ax, Tx), g(y) - g(x)\rangle + F(g(y)) - F(g(x)) - h(x, y) \in C(x).$$

then the solution set of (GVF−IVIP) is nonempty.

The following Theorem improves Theorem 3.3. in [10].

Theorem 2.4. *If all of the assumptions of* Theorem 2.1 *and* Theorem 2.2 *(resp,* Theorem 2.3*) are satisfied, then the solution set of (GVF−ICP) is nonempty and compact (resp, nonempty).*

Proof. The result follows directly from *Theorems 2.1* and *2.2* (resp, *2.3*). □

3. Applications

In this section, we show that the minimum of a class of differentiable nonconvex functions on the g-convex set K in a real Hilbert space can be characterized by a special case of general variational inequality (I) with $F(.) = 0$. For this purpose, we recall the following well-know concepts, see Noor [18].

Definition 3.1. *Let* K *be any set in* H. *The set* K *is said to be* g*-convex, if there exists a function* $g : H \longrightarrow H$ *such that*

$$g(u) + t(g(v) - g(u)) \in K, \quad \textit{for all } u, v \in K, t \in [0, 1].$$

Note that every convex set is g-convex, but the converse is not true.

Definition 3.2. *The function* $F : K \longrightarrow H$ *is said to be* g*-convex, if*

$$F(g(u) + t(g(v) - g(u))) \leq (1 - t)F(g(u)) + tF(g(v)), \quad \textit{for all } u, v \in K, t \in [0, 1].$$

Clearly every convex function is g-convex, but the converse is not true.

We now show that the minimum of a differentiable g-convex function on K in H can be characterized by the general variational inequality (2.1) and this is the main motivation of our next result.

Lemma 3.3. *Let* $F : K \longrightarrow H$ *be a differentiable* g*-convex function. Then* $u \in K$ *is the minimum of* g*-convex function* F *on* K *if and only if* $u \in K$ *satisfies the inequality*

$$\langle F'(g(u)), g(v) - g(u)\rangle \geq 0, \quad \textit{for all } g(v) \in K, \tag{5}$$

where F' *is the differential of* F *at* $g(u)$.

Proof. Let $u \in K$ be a minimum of g-convex function F on K. Then

$$F(g(u)) \leq F(g(v)), \quad \text{for all } g(v) \in K. \tag{6}$$

Since K is a g-convex set, so for all $u, v \in K, t \in [0,1], g(v_t) = g(u)+t(g(v)-g(u)) \in K$. Setting $g(v) = g(v_t)$ in (2), we have

$$F(g(u)) \leq F(g(u) + t(g(v) - g(u)) \leq F(g(u)) + t(F(g(v) - g(u)).$$

Dividing the above inequality by t and taking $t \longrightarrow 0$, we have

$$\langle F'(g(u)), g(v) - g(u)\rangle \geq 0,$$

which is the required result(1).

Conversely, let $u \in K, g(u) \in K$ satisfy inequality (1). Since F is a g-convex function, for all $u, v \in K, \quad t \in [0,1], \quad g(u) + t(g(v) - g(u)) \in K$ and

$$F(g(u) + t(g(v) - g(u))) \leq (1-t)F(g(u)) + tF(g(v)),$$

which implies that

$$F(g(v)) - F(g(u)) \geq \frac{F(g(u) + t(g(v) - g(u))) - F(g(u))}{t}.$$

Letting $t \longrightarrow 0$, we have

$$F(g(v)) - F(g(u)) \geq \langle F'(g(u)), g(v) - g(u)\rangle \geq 0, \quad \text{using (1)},$$

which implies that

$$F(g(u)) \leq F(g(v)), \quad \text{for all } g(v) \in K,$$

showing that $u \in K$ is the minimum of F on K in H. □

Lemma 3.3 implies that g-convex programming problem can be studied via the general (Noor) variational inequality with $Tu = F'(g(u))$, see Noor [8,13,14]. In a similar way, one can show that the general variational inequality is the Fritz-John condition of the inequality constrained optimization problem.

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