A fixed point theorem on asymptotic contractions

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Abstract. *The aim of this paper is to prove a fixed point theorem on asymptotic contractions with hypotheses slightly different from that of Chen [1], Theorem 2.2.*

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1. Introduction

Throughout this paper, (M, d) denotes a complete metric space, R^+ the set of all nonnegative reals, and Φ the class of all mappings $\varphi : R^+ \to R^+$ satisfying (i) φ is continuous, and (ii) $\varphi(t) < t$ for $t > 0$. For $x \in M$, the orbit of x is $\{x, Tx, T^2x, \cdots\}$ and is denoted by $O(x)$.

In 2003, Kirk [2] introduced a new class of mappings namely *asymptotic contractions* and established the existence of fixed points for such mappings by using ultra filter methods.

Definition 1.1 (Kirk [2]). *A mapping $T : M \to M$ is said to be an* asymptotic contraction, *if there exists a sequence $\{\varphi_n\}_{n=1}^{\infty}$, $\varphi_n : R^+ \to R^+$ and $\varphi \in \Phi$ such that for all $n \in N$*

$$d(T^n x, T^n y) \le \varphi_n(d(x, y)) \tag{1}$$

for all $x, y \in M$, where $\varphi_n \to \varphi$ uniformly on the range of d.

Theorem 1.2 (Kirk [2]). *Suppose $T : M \to M$ is an asymptotic contraction for which the mappings φ_n in* (1) *are also continuous. Assume that some orbit of T is bounded. Then T has a unique fixed point $x_* \in M$, and moreover the Picard sequence $\{T^n x\}_{n=1}^{\infty}$ converges to x_* for each $x \in M$.*

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In 2005, Chen [1] proved *Theorem 1.2*, under weaker assumptions without using ultra filter methods.

Theorem 1.3 (Chen [1], Theorem 2.2). *Let $T : M \to M$ be a mapping satisfying*

$$d(T^n x, T^n y) \leq \varphi_n(d(x, y)) \tag{1.3.1}$$

for all $x, y \in M$, where $\varphi_n : R^+ \to R^+$ and $\lim_{n\to\infty} \varphi_n = \varphi$ uniformly on any bounded interval $[0, b]$. Suppose that φ is upper semicontinuous and $\varphi(t) < t$ for $t > 0$,

(1.3.2) *there exists a positive integer n_* such that φ_{n_*} is upper semicontinuous and $\varphi_{n_*}(0) = 0$, and*

(1.3.3) *there exists $x_0 \in M$ such that $O(x_0)$ is bounded.*

Then T has a unique fixed point $x_* \in M$ such that $\lim_{n\to\infty} T^n x = x_*$ for all $x \in M$.

The following example shows that T may not have a fixed point if condition (1.3.2) is dropped in *Theorem 1.3*.

Example 1.4. *Let $M = \{0, 1, 2^{-1}, 2^{-2}, \cdots\}$ with the usual metric. Define $T : M \to M$ by $T0 = 1$; $T(2^{-n}) = 2^{-(n+1)}$ for $n = 0, 1, 2, \cdots$. Define φ_n on R^+ by $\varphi_n(t) = 2^{-1}t + n^{-1}$ and φ on R^+ by $\varphi(t) = 2^{-1}t$, $t \in R^+$. Then $\{\varphi_n\}$ and φ satisfy conditions (1.3.1) and (1.3.3). Here we observe that $\varphi_n(0) \neq 0$ for every n, so that condition (1.3.2) does not hold and T has no fixed points.*

In this paper, we prove that *Theorem 1.3* holds well if (1.3.2) is replaced by

(1.3.2)′: There exists $x_0 \in M$ such that $O(x_0)$ is bounded and there exists $y \in \overline{O(x_0)}$ such that $O(Ty)$ is closed.

2. Main result

Theorem 2.1. *Let $T : M \to M$ be a mapping satisfying (1.3.1) and (1.3.2)′. Then T has a unique fixed point $x_* \in M$ and $\lim_{n\to\infty} T^n x_0 = x_*$. In fact $\lim_{n\to\infty} T^n x = x_*$ for all $x \in M$.*

Proof. For $x_0 \in M$, the sequence $\{T^n x_0\}$ is Cauchy, which follows from the proof of *Theorem 1.3* (see [1], Theorem 2.2) and hence converges, say, to x_*. Thus

$$\overline{O(x_0)} = \{x_*\} \cup O(x_0). \tag{2.1.1}$$

Since $O(x_0)$ is bounded, there exists $b > 0$ such that $d(T^n x_0, T^m x_0) \leq b$ for all n, m. Now, by using (1.3.1), we have

$$d(T^n x_0, T^n x_*) \leq \varphi_n(d(x_0, x_*)).$$

Since $\varphi_n \to \varphi$ as $n \to \infty$, there exists a natural number N such that

$$\varphi_n(d(x_0, x_*)) < \varphi(d(x_0, x_*)) + 1 \qquad \text{for all } n \geq N.$$

Take

$$a = \max_{1\leq k\leq N-1} \{d(T^k x_0, T^k x_*), \varphi(d(x_0, x_*)) + 1\},$$

so that $d(T^n x_0, T^n x_*) \le a$ for all n. Now

$$\begin{aligned} d(T^n x_*, T^m x_*) &\le d(T^n x_*, T^n x_0) + d(T^n x_0, T^m x_0) + d(T^m x_0, T^m x_*) \\ &\le a + b + a = 2a + b \qquad \text{for all } n, m. \end{aligned}$$

Hence $O(x_*)$ is bounded. Thus $\{T^n x_*\}$ converges, say, to y_*.

We now show that $y_* = x_*$. If possible, suppose that $y_* \ne x_*$.

For a fixed positive integer N, we have

$$d(T^{n+N} x_0, T^{n+N} x_*) \le \varphi_n(d(T^N x_0, T^N x_*)).$$

By letting $n \to \infty$, we get

$$d(x_*, y_*) \le \varphi(d(T^N x_0, T^N x_*)).$$

Now, by letting $N \to \infty$, we have

$$d(x_*, y_*) \le \varphi(d(x_*, y_*)) < d(x_*, y_*),$$

a contradiction. Thus $T^n x_* \to x_*$ as $n \to \infty$.

By condition $(1.3.2)'$ and (2.1.1) the following two cases arise.

Case (i): $y = x_*$

Since $T^n x_* \to x_*$, $x_* \in O(Tx_*)$. Therefore there exists a positive integer N such that $x_* = T^N x_*$. Hence $T^{nN+1} x_* = T x_*$ for $n = 1, 2, \ldots$. Consequently, the subsequence $\{T^{nN+1} x_*\}$ of $\{T^n x_*\}$ converges to Tx_*. Thus $Tx_* = x_*$.

Case (ii): $y \ne x_*$

In this case $y \in O(x_0)$ by (2.1.1) and $O(Ty)$ is closed by $(1.3.2)'$. Also observe that $O(Ty)$ is bounded. Hence the conclusion follows by case (i).

Now, for $x \in M$ we have

$$d(T^n x, x_*) = d(T^n x, T^n x_*) \le \varphi_n(d(x, x_*)) < \varphi(d(x, x_*)) + 1 \qquad \text{for a large } n.$$

Hence $\{T^n x\}$ is bounded. Thus $T^n x \to x_*$ as $n \to \infty$. This completes the proof of the theorem. □

The following is an example which satisfies $(1.3.2)'$ but not (1.3.2).

Example 2.2. *Let* $M = \{0, 1, 2^{-1}, 3^{-1}, \cdots\}$ *with the usual metric. Define* $T : M \to M$ *by* $T0 = 0$; $T(n^{-1}) = (n+1)^{-1}$ *for* $n = 1, 2, \ldots$. *For* $t \in R^+$, *define* $\varphi_n(t) = n^{-1}$ *for all* n, *and* $\varphi(t) = 0$ *for all* t. *Clearly* $\varphi(t) < t$ *for* $t > 0$ *and* φ_n *converges to* φ *uniformly on* M. *It is easy to verify the condition (1.3.1). Put* $x_0 = 1$. *Then,* $O(x_0) = \{1, 2^{-1}, 3^{-1}, \cdots\}$ *is bounded. We have* $0 \in \overline{O(x_0)}$ *and* $T^k x_0 \to 0$ *as* $k \to \infty$. *Hence* $0 \in O(Tx_0)$ *so that* $O(Tx_0)$ *is closed. Consequently* $(1.3.2)'$ *holds. Thus* T *satisfies all the conditions of* Theorem 2.1 *and 0 is the only fixed point of* T.

However, we observe that $\varphi_n(0) \ne 0$ *for every* n, *so that condition (1.3.2) fails to hold.*

References

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