Multi-step iterations with errors for common fixed points of a finite family of nonself asymptotically nonexpansive mappings*

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Abstract. *In this paper we established strong and weak convergence theorems for a multi-step iterative scheme with errors for nonself asymptotically nonexpansive mappings in the real uniformly convex Banach space. Our results extend and improve the ones announced by Lin Wang [Lin Wang, Strong and weak convergence theorems for common fixed points of nonself asymptotically nonexpansive mappings, J. Math. Anal. Appl.(2005)].*

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1. Introduction

Let K be a nonempty closed convex subset of a real normed linear space E. A self-mapping $T : K \to K$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for $\forall x, y \in K$. A self-mapping $T : K \to K$ is called asymptotically nonexpansive if there exist sequences $\{k_n\} \subset [1, \infty), k_n \to 1$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \tag{1}$$

for $\forall x, y \in K$ and each $n \geq 1$.

Being an important generalization of the class of nonexpansive self-mappings, the class of asymptotically nonexpansive self-mappings was introduced by Geobel and Kirk [3] in 1972, who proved that if K is a nonempty closed convex subset

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of a real uniformly convex Banach space and T is an asymptotically nonexpansive self-mapping on K, then T has a fixed point.

Iterative techniques for approximating fixed points of nonexpansive self-mappings have been studied by various authors (see, e.g., [3,4]), using the Mann iteration process or the Ishikawa iteration process. For nonself nonexpansive mappings, some authors (see, e.g., [5,6]) have studied the strong and weak convergence theorems in Hilbert space or uniformly convex Banach spaces, using one or two-step iteration. The concept of nonself asymptotically nonexpansive mappings was introduced by Chidume [1] in 2003 as the generalization of asymptotically nonexpansive self-mappings. The nonself asymptotically nonexpansive mapping is defined as follows:

Definition 1.1[1]. *Let K be a nonempty subset of real normed linear space E. Let $P : E \to K$ be the nonexpansive retraction of E onto K. A nonself mapping $T : K \to E$ is called asymptotically nonexpansive if there exist sequences$\{k_n\} \subset [1, \infty), k_n \to 1$ as $n \to \infty$ such that*

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq k_n\|x - y\| \tag{2}$$

for $\forall x, y \in K$ and each $n \geq 1$.

By studying the following iteration process:

$$x_1 \in K, x_{n+1} = P((1 - \alpha_n)x_n + \alpha_n T(PT)^{n-1}x_n), \tag{3}$$

Chidume [1] got the following strong and weak convergence theorems for nonself asymptotically nonexpansive mappings.

Theorem 1[1]. *Let E be a real uniformly convex Banach space and K a nonempty closed convex subset of E. Let $T : K \to E$ be a completely continuous and asymptotically nonexpansive map with sequence $\{k_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$ and $F(T) \neq \emptyset$. Let $\{\alpha_n\} \subset (0, 1)$ be such that $\epsilon \leq 1 - \alpha_n \leq 1 - \epsilon, \forall n \geq 1$ and some $\epsilon > 0$. From arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by (3). Then $\{x_n\}$ converges strongly to some fixed point of T.*

Theorem 2[1]. *Let E be a real uniformly convex Banach space which has a Fréchet differentiable norm and K a nonempty closed convex subset of E. Let $T : K \to E$ be an asymptotically nonexpansive map with sequence $\{k_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$ and $F(T) \neq \emptyset$. Let $\{\alpha_n\} \subset (0, 1)$ be such that $\epsilon \leq 1 - \alpha_n \leq 1 - \epsilon, \forall n \geq 1$ and some $\epsilon > 0$. From arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by (3). Then $\{x_n\}$ converges weakly to some fixed point of T.*

Remark 1.1. *If T is a self-mapping, then P becomes the identity mapping so that (2) reduces to (1).*

By studying the following iterative process:

$$\begin{cases} x_1 \in K, \\ x_{n+1} = P((1 - \alpha_n)x_n + \alpha_n T_1(PT_1)^{n-1}y_n), \\ y_n = P((1 - \beta_n)x_n + \beta_n T_2(PT_2)^{n-1}x_n), n \geq 1, \end{cases} \tag{4}$$

Lin Wang [2] constructed an iteration scheme for approximating common fixed points of two nonself asymptotically nonexpansive mappings and got the following strong and weak convergence theorems for such mappings in uniformly convex Banach spaces.

Theorem 3[2]. *Let K be a nonempty closed convex subset of a real uniformly convex Banach space E. Suppose $T_1, T_2 : K \to E$ are two nonself asymptotically nonexpansive mappings with sequences $\{k_n\}, \{l_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n - 1) < \infty$ and $k_n \to 1, l_n \to 1$, as $n \to \infty$, respectively. Let $\{x_n\}$ be defined by (4), where $\{\alpha_n\}, \{\beta_n\}$ are two sequences in $[\epsilon, 1-\epsilon)$ for some $\epsilon > 0$. If one of T_1 and T_2 is completely continuous, and $F(T_1) \bigcap F(T_2) \neq \emptyset$, then $\{x_n\}$ converges strongly to a common fixed point of T_1 and T_2.*

Theorem 4[2]. *Let K be a nonempty closed convex subset of a real uniformly convex Banach space E satisfying Opial's condition. Suppose $T_1, T_2 : K \to E$ are two nonself asymptotically nonexpansive mappings with sequences $\{k_n\}, \{l_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n - 1) < \infty$ and $k_n \to 1, l_n \to 1$, as $n \to \infty$, respectively. Let $\{x_n\}$ be defined by (4), where $\{\alpha_n\}, \{\beta_n\}$ are two sequences in $[\epsilon, 1-\epsilon)$ for some $\epsilon > 0$. If $F(T_1) \bigcap F(T_2) \neq \emptyset$, then $\{x_n\}$ converges weakly to a common fixed point of T_1 and T_2.*

Remark 1.2. *As $T_1 = T_2$ and $\beta_n = 0$ for all $n \geq 1$, the iteration scheme (4) reduces to (3).*

The purpose of this paper is to construct an iteration scheme for approximating common fixed points of m nonself asymptotically nonexpansive mappings and to prove some strong and weak convergence theorems for such mappings in uniformly convex Banach spaces.

2. Preliminaries

Let E be a real uniformly convex Banach space and K a nonempty closed convex subset of E, which is also a nonexpansive retract of E with retraction P. Let $T_i : K \to E$ be nonself asymptotically nonexpansive mappings. For approximating the common fixed points of m nonself asymptotically nonexpansive mappings, we generalize the iteration scheme as follows:

$$
\begin{aligned}
x_n^{(1)} &= P(\alpha_n^{(1)} T_1 (PT_1)^{n-1} x_n + \beta_n^{(1)} x_n + \gamma_n^{(1)} \mu_n^{(1)}) \\
x_n^{(2)} &= P(\alpha_n^{(2)} T_2 (PT_2)^{n-1} x_n^{(1)} + \beta_n^{(2)} x_n + \gamma_n^{(2)} \mu_n^{(2)}) \\
x_n^{(3)} &= P(\alpha_n^{(3)} T_3 (PT_3)^{n-1} x_n^{(2)} + \beta_n^{(3)} x_n + \gamma_n^{(3)} \mu_n^{(3)}) \\
&\vdots \\
x_n^{(m-1)} &= P(\alpha_n^{(m-1)} T_{m-1} (PT_{m-1})^{n-1} x_n^{(m-2)} + \beta_n^{(m-1)} x_n + \gamma_n^{(m-1)} \mu_n^{(m-1)}) \\
x_{n+1} &= x_n^{(m)} = P(\alpha_n^{(m)} T_m (PT_m)^{n-1} x_n^{(m-1)} + \beta_n^{(m)} x_n + \gamma_n^{(m)} \mu_n^{(m)}) \qquad (5)
\end{aligned}
$$

where $\{\mu_n^1\}, \{\mu_n^2\}, \ldots, \{\mu_n^m\}$ are bounded sequences in K, and $\{\alpha_n^{(i)}\}, \{\beta_n^{(i)}\}, \{\gamma_n^{(i)}\}$ are appropriate real sequences in $[0, 1]$ such that $\alpha_n^{(i)} + \beta_n^{(i)} + \gamma_n^{(i)} = 1$ for each $i \in \{1, 2, \ldots, m\}$.

For the sake of convenience, we restate the following concepts results:
Let E be a Banach space with dimension $E \geq 2$. The modulus of E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\epsilon) = \inf\{1 - \|\frac{1}{2}(x + y)\| : \|x\| = 1, \|y\| = 1, \epsilon = \|x - y\|\}.$$

Banach space E is uniformly convex if and only if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0, 2]$.

A subset K of E is said to be retract if there exists continuous mapping $P : E \to K$ such that $Px = x$ for all $x \in K$. Every closed convex subset of a uniformly convex Banach space is a retract. A mapping $P : E \to E$ is said to be a retraction if $P^2 = P$.

Remark 2.1. *If a mapping P is a retraction, then $Pz = z$ for every $z \in R(P)$, range of P.*

A Banach space E is said to satisfy Opial's condition if for any sequence $\{x_n\} \subset E, x_n \rightharpoonup x$ implies that $\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$ for all $y \in E$ with $y \neq x$, where $x_n \rightharpoonup x$ denotes that $\{x_n\}$ converges weakly to x.

A mapping $T : K \to E$ is said to be semi-compact if for any sequence $\{x_n\}$ in K such that $\|x_n - Tx_n\| \to 0 (n \to \infty)$, there exists subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to $x^* \in K$.

A mapping T with domain $D(T)$ and $R(T)$ in E is said to be demiclosed at p if whenever $\{x_n\}$ is a sequence in $D(T)$ such that $\{x_n\}$ converges weakly to $x^* \in D(T)$ and $\{Tx_n\}$ converges strongly to p, then $Tx^* = p$.

Lemma 2.1[7]. *Let$\{\alpha_n\}$ and $\{t_n\}$ be two nonnegative sequences satisfying*

$$\alpha_{n+1} \leq \alpha_n + t_n \quad for\, all\, n \geq 1.$$

If $\sum_{n=1}^{\infty} t_n < \infty$, then $\lim_{n\to\infty} \alpha_n$ exists.

Lemma 2.2[8]. *Let E be a real uniformly convex Banach space and $0 \leq p \leq t_n \leq q < 1$ for all positive integers $n \geq 1$. Also suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences of E such that $\limsup_{n\to\infty} \|x_n\| \leq r, \limsup_{n\to\infty} \|y_n\| \leq r$ and $\limsup_{n\to\infty} \|t_n x_n + (1 - t_n) y_n\| = r$ hold for some $r \geq 0$, then $\limsup_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2.3[1]. *Let E be a real uniformly convex Banach space, K a nonempty closed subset of E, and let $T : K \to E$ be nonself asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ and $k_n \to 1$ as $n \to \infty$. Then $I - T$ is demiclosed at zero.*

3. Main results

Lemma 3.1. *Let K be a nonempty closed convex and bounded subset of a real normed linear space E. Let $T_i : K \to E$ be m nonself asymptotically nonexpansive mappings with sequences $k_n^{(i)} \subset [1, \infty)$ such that $\Sigma_{n=1}^{\infty} (k_n^{(i)} - 1) < \infty, k_n^{(i)} \to 1$ as $n \to \infty, i \in \{1, 2, \cdots, m\}$. Suppose that $\{\alpha_n^{(i)}\}$, $\{\beta_n^{(i)}\}$ and $\{\gamma_n^{(i)}\}$ are three real sequences in $[0, 1)$, $\{x_n\}$ is defined by (5) with the following restrictions:*

(i) $\alpha_n^{(i)} + \beta_n^{(i)} + \gamma_n^{(i)} = 1$ for all $i \in \{1, 2, \cdots, m\}, n \geq 1$.

(ii) $\Sigma_{n=1}^{\infty} \gamma_n^{(i)} < \infty$ for all $i \in \{1, 2, \cdots, m\}, n \geq 1$.

If $\bigcap_{i=1}^{m} F(T_i) \neq \emptyset$, then $\lim_{n\to\infty} \|x_n - q\|$ exists for each $q \in \bigcap_{i=1}^{m} F(T_i)$.

Proof. Setting $k_n^{(i)} = 1 + l_n^{(i)}$. Since $\Sigma_{n=1}^{\infty}(k_n^{(i)} - 1) < \infty$, so $\Sigma_{n=1}^{\infty} l_n^{(i)} < \infty$. For any $q \in \bigcap_{i=1}^{m} F(T_i)$, by (5), we have

$$\begin{aligned}
\|x_n^{(1)} - q\| &= \|P(\alpha_n^{(1)} T_1 (PT_1)^{n-1} x_n + \beta_n^{(1)} x_n + \gamma_n^{(1)} \mu_n^{(1)}) - q\| \\
&\le \|\alpha_n^{(1)} T_1 (PT_1)^{n-1} x_n + \beta_n^{(1)} x_n + \gamma_n^{(1)} \mu_n^{(1)} - q\| \\
&\le \alpha_n^{(1)} \|T_1 (PT_1)^{n-1} x_n - q\| + \beta_n^{(1)} \|x_n - q\| + \gamma_n^{(1)} \|\mu_n^{(1)} - q\| \\
&\le \alpha_n^{(1)} (1 + l_n^{(1)}) \|x_n - q\| + \beta_n^{(1)} \|x_n - q\| + \gamma_n^{(1)} \|\mu_n^{(1)} - q\| \\
&= (\alpha_n^{(1)} + \beta_n^{(1)}) \|x_n - q\| + \alpha_n^{(1)} l_n^{(1)} \|x_n - q\| + \gamma_n^{(1)} \|\mu_n^{(1)} - q\| \\
&\le \|x_n - q\| + d_n^{(1)} \qquad (6)
\end{aligned}$$

where $d_n^{(1)} = \alpha_n^{(1)} l_n^{(1)} \|x_n - q\| + \gamma_n^{(1)} \|\mu_n^{(1)} - q\|$, since $\Sigma_{n=1}^{\infty} l_n^{(1)} < \infty$, $\Sigma_{n=1}^{\infty} \gamma_n^{(1)} < \infty$, and $\{\alpha_n^{(1)}\}$, $\{\|x_n - q\|\}$, $\{\|\mu_n^{(1)} - q\|\}$ are bounded, we see that $\Sigma_{n=1}^{\infty} d_n^{(1)} < \infty$. It follows from (6) that

$$\begin{aligned}
\|x_n^{(2)} - q\| &\le \alpha_n^{(2)} (1 + l_n^{(2)}) \|x_n^{(1)} - q\| + \beta_n^{(2)} \|x_n - q\| + \gamma_n^{(2)} \|\mu_n^{(2)} - q\| \\
&\le \alpha_n^{(2)} (1 + l_n^{(2)}) (\|x_n - q\| + d_n^{(1)}) + \beta_n^{(2)} \|x_n - q\| + \gamma_n^{(2)} \|\mu_n^{(2)} - q\| \\
&= (\alpha_n^{(2)} + \beta_n^{(2)}) \|x_n - q\| + \alpha_n^{(2)} l_n^{(2)} \|x_n - q\| + \gamma_n^{(2)} \|\mu_n^{(2)} - q\| \\
&\quad + \alpha_n^{(2)} (1 + l_n^{(2)}) d_n^{(1)} \\
&\le \|x_n - q\| + \alpha_n^{(2)} l_n^{(2)} \|x_n - q\| + \gamma_n^{(2)} \|\mu_n^{(2)} - q\| + \alpha_n^{(2)} (1 + l_n^{(2)}) d_n^{(1)} \\
&= \|x_n - q\| + d_n^{(2)}
\end{aligned}$$

where $d_n^{(2)} = \alpha_n^{(2)} l_n^{(2)} \|x_n - q\| + \gamma_n^{(2)} \|\mu_n^{(2)} - q\| + \alpha_n^{(2)} (1 + l_n^{(2)}) d_n^{(1)}$, since $\Sigma_{n=1}^{\infty} l_n^{(2)} < \infty, \Sigma_{n=1}^{\infty} \gamma_n^{(2)} < \infty, \Sigma_{n=1}^{\infty} d_n^{(1)} < \infty$, and $\{\alpha_n^{(2)}\}$, $\{\|x_n - q\|\}, \{\|\mu_n^{(2)} - q\|\}$ are bounded, we see that $\Sigma_{n=1}^{\infty} d_n^{(2)} < \infty$.

By continuing the above method, there are nonnegative real sequences $\{d_n^{(i)}\}$ such that $\Sigma_{n=1}^{\infty} d_n^{(i)} < \infty$ and

$$\|x_n^{(i)} - q\| \le \|x_n - q\| + d_n^{(i)}, \qquad (7)$$

for all $i \in \{1, 2, \cdots, m\}$. This together with *Lemma 2.1* gives that $\lim_{n\to\infty} \|x_n - q\|$ exists. This completes the proof. □

Lemma 3.2. *Let K be a nonempty closed convex subset of a real uniformly convex Banach space E. Let $T_i : K \to E$ be m nonself asymptotically nonexpansive mappings with sequences $k_n^{(i)} \subset [1, \infty)$ such that $\Sigma_{n=1}^{\infty}(k_n^{(i)} - 1) < \infty, k_n^{(i)} \to 1$, as $n \to \infty, i \in \{1, 2, \cdots, m\}$. Suppose that $\{\alpha_n^{(i)}\}$, $\{\beta_n^{(i)}\}$ and $\{\gamma_n^{(i)}\}$ are three real sequences in $[0, 1)$, $\{x_n\}$ is defined by (5) with the following restrictions:*

(i) $\alpha_n^{(i)} + \beta_n^{(i)} + \gamma_n^{(i)} = 1$ *for all* $i \in \{1, 2, \cdots, m\}, n \ge 1$.

(ii) $\Sigma_{n=1}^{\infty} \gamma_n^{(i)} < \infty$ *for all* $i \in \{1, 2, \cdots, m\}, n \ge 1$.

If $\bigcap_{i=1}^{m} F(T_i) \ne \emptyset$, *then* $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ *for all* $i \in \{1, 2, \cdots, m\}$.

Proof. Setting $k_n^{(i)} = 1 + l_n^{(i)}$. Taking $q \in \bigcap_{i=1}^{m} F(T_i), i \in \{1, 2, \cdots, m\}$, by *Lemma 3.1*, we see that $\lim_{n\to\infty} \|x_n - q\|$ exists. Assume $\lim_{n\to\infty} \|x_n - q\| = a, a \ge 0$. From (7), we note that

$$\|x_n^{(m-1)} - q\| \le \|x_n - q\| + d_n^{(m-1)}, \forall n \ge 1$$

where $\{d_n^{(m-1)}\}$ is a nonnegative real sequence such that $\Sigma_{n=1}^{\infty} d_n^{(m-1)} < \infty$. It follows that

$$\limsup_{n\to\infty} \|x_n^{(m-1)} - q\| \le \limsup_{n\to\infty} \|x_n - q\| = \lim_{n\to\infty} \|x_n - q\| = a, \tag{8}$$

from where we have

$$\limsup_{n\to\infty} \|T_m(PT_m)^{n-1}x_n^{(m-1)} - q\| \le \limsup_{n\to\infty}(1 + l_n^{(m)})\|x_n^{(m-1)} - q\| \le a.$$

Next, we deserve that

$$\begin{aligned}\|T_m(PT_m)^{n-1}x_n^{(m-1)} - q + \gamma_n^{(m)}(\mu_n^{(m)} - x_n)\| \le{}& \|T_m(PT_m)^{n-1}x_n^{(m-1)} - q\| \\ &+\|\gamma_n^{(m)}(\mu_n^{(m)} - x_n)\|.\end{aligned}$$

Thus we have

$$\limsup_{n\to\infty} \|T_m(PT_m)^{n-1}x_n^{(m-1)} - q + \gamma_n^{(m)}(\mu_n^{(m)} - x_n)\| \le a. \tag{9}$$

Also,

$$\|x_n^{(m-1)} - q + \gamma_n^{(m)}(\mu_n^{(m)} - x_n)\| \le \|x_n^{(m-1)} - q\| + \|\gamma_n^{(m)}(\mu_n^{(m)} - x_n)\|$$

gives that

$$\limsup_{n\to\infty} \|x_n^{(m-1)} - q + \gamma_n^{(m)}(\mu_n^{(m)} - x_n)\| \le a \tag{10}$$

and note that

$$\begin{aligned} a &= \lim_{n\to\infty} \|x_n^{(m)} - q\| \\ &= \lim_{n\to\infty} \|\alpha_n^{(m)}T_m(PT_m)^{n-1}x_n^{(m-1)} + (1-\alpha_n^{(m)})x_n - \gamma_n^{(m)}x_n + \gamma_n^{(m)}\mu_n^{(m)} \\ &\quad -(1-\alpha_n^{(m)})q - \alpha_n^{(m)}q\| \\ &= \lim_{n\to\infty} \|\alpha_n^{(m)}T_m(PT_m)^{n-1}x_n^{(m-1)} - \alpha_n^{(m)}q + \alpha_n^{(m)}\gamma_n^{(m)}\mu_n^{(m)} - \alpha_n^{(m)}\gamma_n^{(m)}x_n \\ &\quad +(1 - \alpha_n^{(m)})x_n - (1 - \alpha_n^{(m)})q - \gamma_n^{(m)}x_n + \gamma_n^{(m)}\mu_n^{(m)} \\ &\quad -\alpha_n^{(m)}\gamma_n^{(m)}\mu_n^{(m)} + \alpha_n^{(m)}\gamma_n^{(m)}x_n\| \\ &= \lim_{n\to\infty} \|\alpha_n^{(m)}(T_m(PT_m)^{n-1}x_n^{(m-1)} - q + \gamma_n^{(m)}(\mu_n^{(m)} - x_n)) \\ &\quad +(1-\alpha_n^{(m)})(x_n - q + \gamma_n^{(m)}(\mu_n^{(m)} - x_n))\|. \end{aligned}$$

This together with (9), (10) and *Lemma 2.2*, gives

$$\lim_{n\to\infty} \|T_m(PT_m)^{n-1}x_n^{(m-1)} - x_n\| = 0 \tag{11}$$

In addition,

$$\begin{aligned}\|x_n - q\| &\le \|x_n - T_m(PT_m)^{n-1}x_n^{(m-1)}\| + \|T_m(PT_m)^{n-1}x_n^{(m-1)} - q\| \\ &\le \|x_n - T_m(PT_m)^{n-1}x_n^{(m-1)}\| + (1 + l_n^{(m)})\|x_n^{(m-1)} - q\|\end{aligned}$$

Taking liminf on both sides in the inequality above, by(11) we have

$$\liminf_{n\to\infty} \|x_n^{(m-1)} - q\| \geq a. \tag{12}$$

Thus, it follows from (8) and (12) that

$$\lim_{n\to\infty} \|x_n^{(m-1)} - q\| = a.$$

Since

$$\begin{aligned}
&\|x_n^{(m-1)} - q\| \\
&\quad = \|P(\alpha_n^{(m-1)} T_{m-1}(PT_{m-1})^{n-1} x_n^{(m-2)} + \beta_n^{(m-1)} x_n + \gamma_n^{(m-1)} \mu_n^{(m-1)}) - q\| \\
&\quad \leq \|\alpha_n^{(m-1)}(T_{m-1}(PT_{m-1})^{n-1} x_n^{(m-2)} - q + \gamma_n^{(m-1)}(\mu_n^{(m-1)} - x_n)) \\
&\qquad +(1-\alpha_n^{(m-1)})(x_n - q + \gamma_n^{(m-1)}(\mu_n^{(m-1)} - x_n))\| \\
&\quad \leq \|\alpha_n^{(m-1)}(1 + l_n^{(m-1)})\|x_n^{(m-2)} - q\| + (1-\alpha_n^{(m-1)})\|x_n - q\| \\
&\qquad +\gamma_n^{(m-1)}\|\mu_n^{(m-1)} - x_n\| \\
&\quad \leq \alpha_n^{(m-1)}(1 + l_n^{(m-1)})(\|x_n - q\| + d_n^{(m-2)}) + (1-\alpha_n^{(m-1)})\|x_n - q\| \\
&\qquad +\gamma_n^{(m-1)}\|\mu_n^{(m-1)} - x_n\| \\
&\quad = (1 + \alpha_n^{(m-1)} l_n^{(m-1)})\|x_n - q\| + \alpha_n^{(m-1)}(1 + l_n^{(m-1)}) d_n^{(m-2)} \\
&\qquad +\gamma_n^{(m-1)}\|\mu_n^{(m-1)} - x_n\|
\end{aligned} \tag{13}$$

where $\Sigma_{n=1}^{\infty} d_n^{(m-2)} < \infty$, taking lim on both sides in the inequality above, we have

$$\begin{aligned}
\lim_{n\to\infty} \|\alpha_n^{(m-1)}\ (T_{m-1}\ (PT_{m-1})^{n-1} x_n^{(m-2)} - q + \gamma_n^{(m-1)}(\mu_n^{(m-1)} - x_n) \\
+(1-\alpha_n^{(m-1)})(x_n - q + \gamma_n^{(m-1)}(\mu_n^{(m-1)} - x_n)\| = a.
\end{aligned}$$

Then by *Lemma 2.2*, we obtain,

$$\lim_{n\to\infty} \|T_{m-1}(PT_{m-1})^{n-1} x_n^{(m-2)} - x_n\| = 0. \tag{14}$$

We now prove that $\lim_{n\to\infty} \|x_{n+1} - x_n^{(m-1)}\| = 0$:

$$\begin{aligned}
&\|x_{n+1} - x_n^{(m-1)}\| \\
&\quad = \|P(\alpha_n^{(m)} T_m(PT_m)^{n-1} x_n^{(m-1)} + \beta_n^{(m)} x_n + \gamma_n^{(m)} \mu_n^{(m)}) - x_n^{(m-1)}\| \\
&\quad \leq \|\alpha_n^{(m)} T_m(PT_m)^{n-1} x_n^{(m-1)} + (1 - \alpha_n^{(m)} - \gamma_n^{(m)}) x_n + \gamma_n^{(m)} \mu_n^{(m)} - x_n^{(m-1)}\| \\
&\quad \leq \alpha_n^{(m)}\|T_m(PT_m)^{n-1} x_n^{(m-1)} - x_n\| + \gamma_n^{(m)}\|\mu_n^{(m)} - x_n\| + \|x_n - x_n^{(m-1)}\| \\
&\quad \leq \alpha_n^{(m)}\|T_m(PT_m)^{n-1} x_n^{(m-1)} - x_n\| + \gamma_n^{(m)}\|\mu_n^{(m)} - x_n\| \\
&\qquad +\alpha_n^{(m-1)}\|T_{m-1}(PT_{m-1})^{n-1} x_n^{(m-2)} - x_n\| + \gamma_n^{(m-1)}\|\mu_n^{(m-1)} - x_n\|.
\end{aligned}$$

Taking lim on both sides in the inequality above, by (11) and (14) we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n^{(m-1)}\| = 0,$$

and therefore

$$\lim_{n\to\infty} \|x_n - x_{n-1}^{(m-1)}\| = 0. \tag{15}$$

Further by(11), we still have

$$\lim_{n\to\infty} \|x_{n+1} - T_m(PT_m)^{n-1}x_n^{(m-1)}\| = 0 \tag{16}$$

and

$$\begin{aligned}
&\|x_n - T_m x_n\| \\
&\quad = \|x_n - T_m(PT_m)^{n-1}x_n^{(m-1)} + T_m(PT_m)^{n-1}x_n^{(m-1)} - T_m x_n\| \\
&\quad \le \|x_n - T_m(PT_m)^{n-1}x_n^{(m-1)}\| + \|T_m(PT_m)^{n-1}x_n^{(m-1)} - T_m(PT_m)^{n-1}x_{n-1}^{(m-1)}\| \\
&\qquad\quad + \|T_m(PT_m)^{n-1}x_{n-1}^{(m-1)} - T_m x_n\| \\
&\quad \le \|x_n - T_m(PT_m)^{n-1}x_n^{(m-1)}\| + (1 + l_n^{(m)})\|x_n^{(m-1)} - x_{n-1}^{(m-1)}\| \\
&\qquad\quad + (1 + l_1^{(m)})\|T_m(PT_m)^{n-2}x_{n-1}^{(m-1)} - x_n\|.
\end{aligned} \tag{17}$$

It follows from (16) that

$$\lim_{n\to\infty} \|T_m(PT_m)^{n-2}x_{n-1}^{(m-1)} - x_n\| = 0. \tag{18}$$

In addition,

$$\begin{aligned}
&\|x_n^{(m-1)} - x_{n-1}^{(m-1)}\| \\
&\quad = \|x_n^{(m-1)} - T_{m-1}(PT_{m-1})^{n-1}x_n^{(m-2)} + T_{m-1}(PT_{m-1})^{n-1}x_n^{(m-2)} - x_{n-1}^{(m-1)}\| \\
&\quad \le \|x_n^{(m-1)} - T_{m-1}(PT_{m-1})^{n-1}x_n^{(m-2)}\| + \|T_{m-1}(PT_{m-1})^{n-1}x_n^{(m-2)} - x_n\| \\
&\qquad\quad + \|x_n - x_{n-1}^{(m-1)}\| \\
&\quad \le (\alpha_n^{(m-1)} + 2)\|T_{m-1}(PT_{m-1})^{n-1}x_n^{(m-2)} - x_n\| + \gamma_n^{(m-1)}\|\mu_n^{(m-1)} - x_n\| \\
&\qquad\quad + \|x_n - x_{n-1}^{(m-1)}\|
\end{aligned}$$

Taking lim on both sides in the inequality above, by (14) and (15) we have

$$\lim_{n\to\infty} \|x_n^{(m-1)} - x_{n-1}^{(m-1)}\| = 0. \tag{19}$$

By (11), (18) and (19), it follows from (17) that $\lim_{n\to\infty} \|x_n - T_m x_n\| = 0$. Similarly, we may show that $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0, i \in \{1, 2, \cdots, m\}$. The proof is completed. □

Theorem 3.3. *Let K be a nonempty closed convex and bounded subset of a real uniformly convex Banach space E. Let $T_i : K \to E$ be m nonself asymptotically nonexpansive mappings with sequences $k_n^{(i)} \subset [1, \infty)$ such that $\Sigma_{n=1}^{\infty}(k_n^{(i)} - 1) < \infty, k_n^{(i)} \to 1$ as $n \to \infty, i \in \{1, 2, \cdots, m\}$. Suppose that $\{\alpha_n^{(i)}\}$, $\{\beta_n^{(i)}\}$ and $\{\gamma_n^{(i)}\}$ are three real sequences in $[0, 1)$, $\{x_n\}$ is defined by (5) with the following restrictions:*

(i) $\alpha_n^{(i)} + \beta_n^{(i)} + \gamma_n^{(i)} = 1$ *for all* $i \in \{1, 2, \cdots, m\}, n \geq 1$.

(ii) $\Sigma_{n=1}^{\infty} \gamma_n^{(i)} < \infty$ *for all* $i \in \{1, 2, \cdots, m\}, n \geq 1$.

If one of T_i *is completely continuous, and* $\bigcap_{i=1}^m F(T_i) \neq \emptyset$*, then* $\{x_n\}$ *converges strongly to a common fixed point of* $\{T_i\}_{i=1}^m$.

Proof. By *Lemma 3.1*, the sequence $\{x_n\}$ is bounded. In addition, by *Lemma 3.2*, $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$, then $\{T_i x_n\}_{i=1}^m$ are also bounded. If T_1 is completely continuous, there exists subsequence $\{T_1 x_{n_j}\}$ of $\{T_1 x_n\}$ such that $T_1 x_{n_j} \to q$ as $j \to \infty$. It follows from *Lemma 3.2* that $\lim_{j\to\infty} \|x_{n_j} - T_i x_{n_j}\| = 0$. Then $\lim_{j\to\infty} \|x_{n_j} - q\| = 0$. So by *Lemma 2.3* we have $q \in \bigcap_{i=1}^m F(T_i)$. Furthermore, by *Lemma 3.1* we get that $\lim_{j\to\infty} \|x_n - q\|$ exists. Thus $\lim_{j\to\infty} \|x_n - q\| = 0$. The proof is completed. □

Theorem 3.4. *Let* K *be a nonempty closed convex and bounded subset of a real uniformly convex Banach space* E *satisfying Opial's condition. Let* $T_i : K \to E$ *be* m *nonself asymptotically nonexpansive mappings with sequences* $k_n^{(i)} \subset [1, \infty)$ *such that* $\Sigma_{n=1}^{\infty}(k_n^{(i)} - 1) < \infty, k_n^{(i)} \to 1$ *as* $n \to \infty, i \in \{1, 2, \cdots, m\}$. *Suppose that* $\{\alpha_n^{(i)}\}$, $\{\beta_n^{(i)}\}$ *and* $\{\gamma_n^{(i)}\}$ *are three real sequences in* $[0, 1)$, $\{x_n\}$ *is defined by (5) with the following restrictions:*

(i) $\alpha_n^{(i)} + \beta_n^{(i)} + \gamma_n^{(i)} = 1$ *for all* $i \in \{1, 2, \cdots, m\}, n \geq 1$.

(ii) $\Sigma_{n=1}^{\infty} \gamma_n^{(i)} < \infty$ *for all* $i \in \{1, 2, \cdots, m\}, n \geq 1$.

If $\bigcap_{i=1}^m F(T_i) \neq \emptyset$*, then* $\{x_n\}$ *converges weakly to a common fixed point of* $\{T_i\}_{i=1}^m$.

Proof. For any $q \in \bigcap_{i=1}^m F(T_i)$, it follows from *Lemma 3.1* that $\lim_{j\to\infty} \|x_n - q\|$ exists. We now prove that $\{x_n\}$ has a unique weak subsequential limit in $\bigcap_{i=1}^m F(T_i)$.

Firstly, let q_1 and q_2 be weak limits of subsequence $\{x_{n_j}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$, repectively. By *Lemma 3.2* and *2.3* we know that $q_1, q_2 \in \bigcap_{i=1}^m F(T_i)$. Secondly, assume $q_1 \neq q_2$, then by Opial's condition we obtain

$$\begin{aligned} \lim_{n\to\infty} \|x_n - q_1\| &= \lim_{k\to\infty} \|x_{n_k} - q_1\| < \lim_{k\to\infty} \|x_{n_k} - q_2\| = \lim_{j\to\infty} \|x_{n_j} - q_2\| \\ &< \lim_{k\to\infty} \|x_{n_k} - q_1\| = \lim_{n\to\infty} \|x_n - q_1\|, \end{aligned} \tag{20}$$

which is a contradiction, hence $q_1 = q_2$. Then $\{x_n\}$ converges weakly to a common fixed point of $\{T_i\}_{i=1}^m$. The proof is completed. □

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