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THE SPANS OF FIVE STAR-LIKE SIMPLE CLOSED CURVES

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Abstract. Let X be a continuum, that is a compact, connected, nonempty metric space. The span of X is the least upper bound of the set of real numbers r which satisfy the following conditions: there exists a continuum, C, contained in $X \times X$ such that $d(x, y)$ is larger than or equal to r for all (x, y) in C and $p_1(C) = p_2(C)$, where p_1, p_2 are the usual projection maps. The following question has been asked. If X and Y are two simple closed curves in the plane and Y is contained in the bounded component of the plane minus X, then is the span of X larger than the span of Y? We define a set of simple closed curves, which we refer to as being five star-like. We answer this question in the affirmative when X is one of these simple closed curves. We calculate the spans of the simple closed curves in this collection and consider the spans of various geometric objects related to these simple closed curves.

1. Introduction

The span of a metric continuum was defined in 1964 by A. Lelek (see [3, p.209]). Variations of the span have been defined since then (cf [4, 5, 2]). In general it is difficult to evaluate the spans of a geometric object. It is of interest how these various spans, for a particular object, are related to each other. Also of interest is how the spans of related objects compare to each other. The following question by H. Cook has been particularly interesting.

If X_1 and X_2 are two simple closed curves in the plane and X_2 is contained in the bounded component of X_1, then is the span of X_1 larger than the span of X_2 ? ([1])

This question has not yet been answered in general. Given some specific conditions on either X_1 or X_2, the answer has been shown to be yes. We

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give a short summary of the results for these special cases that have been determined.

In the following results we assume that:

1. X_2 is a simple closed curve,
2. X_3 is a continuum that is contained in the bounded component of $R^2 - X_2$,
3. X_1 is a plane separating continuum that contains X_2 in one of its bounded components, and
4. the span of X is denoted by $\sigma(X)$.

If X_2 is the boundary of a convex region in the plane and X_1 is a simple closed curve, then $\sigma(X_2) < \sigma(X_1)$ ([7], see also [6]).

If X_2 is the boundary of a convex region in the plane, then
$\sigma(X_3) < \sigma(X_2) < \sigma(X_1)$ [11].

Also, $\sigma(X_3) < \sigma(X_2) < \sigma(X_1)$ when X_2 is either an "indented circle" ([8, 9, 13]), or a "concave upward symmetric" simple closed curve [10], or a simple closed curve as defined in [12]. In this paper we define a set of simple closed curves that we refer to as five star-like and show that if X_2 is a five star-like simple closed curve and X_1 and X_3 are as defined previously, then $\alpha(X_3) < \alpha(X_2) < \alpha(X_1)$, where α represents the span σ, semispan σ_0, surjective span σ^*, surjective semispan σ_0^*, symmetric span s, and surjective symmetric span s^*.

2. Preliminaries

Let X be a continuum, that is a compact, connected metric space. The span of X, $\sigma(X)$, is the least upper bound of the set of real numbers r which satisfy the following conditions: there exist a continuum, C, and continuous functions $f, g : C \to X$, such that

$$\text{dmin}(f, g) = \min\{d(f(c), g(c)) \,|\, c \in C\} \geq r$$

and

$$f(C) = g(C) \qquad \sigma$$

span

To obtain the various other spans, we replace the preceeding equation with the following:

$$f(C) \subseteq g(C) \qquad \sigma_0$$

semispan

$$f(C) = g(C) = X \qquad \sigma^*$$

surjective span

$$f(C) \subseteq g(C) = X \qquad \sigma_0^*$$

surjective semispan

$$f(C) = g(C) \qquad s$$

symmetric span

$$\text{and } \forall c \in C,\ \exists c' \quad \text{such that} \quad f(c) = g(c') \quad \text{and} \quad f(c') = g(c)$$

$$f(C) = g(C) = X \qquad\qquad s^*$$

surjective symmetric span

$$\text{and } \forall c \in C,\ \exists c' \quad \text{such that} \quad f(c) = g(c') \quad \text{and} \quad f(c') = g(c).$$

The inequalities below follow immediately from the definitions,

$$0 \le \sigma^*(X) \le \sigma(X) \le \sigma_0(X) \le \operatorname{diam} X,$$
$$0 \le \sigma^*(X) \le \sigma_0^*(X) \le \sigma_0(X) \le \operatorname{diam} X,$$
$$0 \le s(X) \le \sigma(X),$$
$$0 \le s^*(X) \le \sigma^*(X).$$

The following results are easy consequences of the various definitions.

1. If J is an arc, then $\alpha(J) = 0$, where $\alpha = \sigma, \sigma_0, \sigma^*, \sigma_0^*, s, s^*$.
2. If X is a simple closed curve, then $\sigma(X) = \sigma^*(X)$, $\sigma_0(X) = \sigma_0^*(X)$, and $s(X) = s^*(X)$.

We utilize the following theorem from [3] in the proof of Corollary 2.

Theorem L: If Y is a closed subset of the Hilbert cube I^ω and $\rho : Y \to S$ is an essential mapping of Y onto the circumference S, then

$$\inf_{s \in S}(\rho^{-1}(s), \rho^{-1}(-s)) \le \sigma(Y).$$

3. Main Results

(a) Let Q be a five-sided, convex polygon with sequentially labeled vertices Q_i' for $i = 0, 1, 2, 3, 4$, with all interior angles $\angle Q_i'$, $i = 0, 1, 2, 3, 4$, larger than $90°$. Extend each side $\overline{Q_j' Q_{j+1}'}$ of the polygon.

(b) Let Q_{j+2} be the point of intersection of $\overleftrightarrow{Q_j' Q_{j+1}'}$ and $\overleftrightarrow{Q_{j+2}' Q_{j+3}'}$ for $j = 0, 1, 2, 3, 4$, where all indices are taken modulo 5.

(c) Let X be the star shaped simple closed curve defined by $X = \cup_{j=0}^{4}(\overline{Q_j' Q_{j+1}} \cup \overline{Q_{j+1}' Q_{j+1}})$. We refer to X as a five star-like simple closed curve.

(d) Let t_j be the point on $\overline{Q_{j-1} Q_{j-3}}$ that is the closest to Q_j. Note that $t_j \in (\overline{Q_{j-2}' Q_{j-1}} - \{Q_{j-2}'\})$, since $Q_j \in \overrightarrow{Q_{j-2}' Q_{j-1}'}$ and $\angle Q_{j-2}' > 90°$.

(e) Let r_j be the point on $\overline{Q_{j+1} Q_{j+3}}$ that is closest to Q_j. Note that $r_j \in (\overline{Q_{j+1} Q_{j+1}'} - \{Q_{j+1}'\})$, since $Q_j \in \overrightarrow{Q_{j+1}' Q_j'}$ and $\angle Q_{j+1}' > 90°$.

(f) Note that $t_j' = d(Q_j, t_j) < d(Q_j, Q_{j+2}') = q_j'$ and $r_j' = d(Q_j, r_j) < d(Q_j, Q_{j+2}') = q_j'$.

(g) Suppose that $q_3' = \min\{q_i'\}_{i=0}^4$. We can make this assumption since we can relabel the vertices so that this is true. We refer to the number

$$\mathrm{fss}(X) = \max\{q_3', \min\{r_0', r_1', t_1', t_2'\}, \min\{r_1', t_2', q_4'\}, \min\{r_4', r_0', t_0', t_1'\}, \min\{r_4', t_0', q_2'\}\}$$

as the five star-like spread of X, where X is a five star-like simple closed curve.

THEOREM 3.1. *Let X be a five star-like simple closed curve. Then* $\sigma(X) = \sigma_0(X) = \sigma^*(X) = \sigma_0^*(X) = s(X) = s^*(X) = \mathrm{fss}(X)$.

PROOF. Let $f, g : I \to X$ be continuous functions from I, the unit interval, onto X, such that their movements are always clockwise on X and one function is constantly Q_j for some j, while the other function moves from Q_k to Q_{k+1}, passing through the point Q_k', where $k \neq j, j-1$ and $f[I] = g[I] = X$. Let $\mathcal{P}$ be the set consisting of all pairs of functions (f, g) that satisfy these conditions. We claim that $\alpha(X) = \max\{\mathrm{dmin}(f,g) \mid (f,g) \in \mathcal{P}\} = \mathrm{fss}(X)$, where $\alpha = \sigma, \sigma_0, \sigma^*, \sigma_0^*, s, s^*$. We consider two cases.

CASE A: $\max\{\mathrm{dmin}(f,g) \mid (f,g) \in \mathcal{P}\} = q_3'$

Consider the pair of functions given in Table 1.

TABLE 1. $f(t)$ and $g(t)$ versus t.

t	$f(t)$	$g(t)$
0	Q_0	Q_3
0.1	Q_1	Q_3
0.2	Q_1	Q_4
0.3	Q_2	Q_4
0.4	Q_2	Q_0
0.5	Q_3	Q_0
0.6	Q_3	Q_1
0.7	Q_4	Q_1
0.8	Q_4	Q_2
0.9	Q_0	Q_2
1.0	Q_0	Q_3

Note that

$$d(Q_j, \overline{Q_{j+2}Q_{j+2}'} \cup \overline{Q_{j+2}'Q_{j+3}}) = d(Q_j, Q_{j+2}') = q_j'.$$

Given this observation and assumption (g) in the construction of X, we see that $\mathrm{dmin}(f, g) = q_3'$.

We observe that $\alpha(X) \geq d(Q_3, Q_0') = q_3'$ for $\alpha = \sigma, \sigma_0, \sigma^*, \sigma_0^*, s, s^*$. This is true since, $\sigma(X) = \sigma^*(X)$, $\sigma_0(X) = \sigma_0^*(X)$, $f(I) = g(I) = X$, and for all $t \in [0,1]$ there is a $t' \in [0,1]$ such that $f(t) = g(t')$ and $g(t) = f(t')$.

In particular, (f, g), from Table 1, is a "better" pair than the pairs (f_1, g_1) and (f_2, g_2) given in Table 2 and Table 3.

TABLE 2. $f_1(t)$ and $g_1(t)$ versus t.

t	$f_1(t)$	$g_1(t)$
0	Q_0	Q_3
0.1	Q_0	Q_4
0.2	Q_1	Q_4
0.3	Q_2	Q_4
0.4	Q_2	Q_0
0.5	Q_3	Q_0
0.6	Q_4	Q_0
0.7	Q_4	Q_1
0.8	Q_4	Q_2
0.9	Q_0	Q_2
1.0	Q_0	Q_3

TABLE 3. $f_2(t)$ and $g_2(t)$ versus t.

t	$f_2(t)$	$g_2(t)$
0	Q_0	Q_2
0.1	Q_1	Q_2
0.2	Q_1	Q_3
0.3	Q_1	Q_4
0.4	Q_2	Q_4
0.5	Q_2	Q_0
0.6	Q_2	Q_1
0.7	Q_3	Q_1
0.8	Q_4	Q_1
0.9	Q_4	Q_2
1.0	Q_0	Q_2

Consequently, since

$$\text{dmin}(f, g) \geq \max\{\text{dmin}(f_1, g_1), \text{dmin}(f_2, g_2)\}$$

it must be the case that

$$\text{dmin}(f, g) = q_3^{'} \geq \min\{r_4^{'}, t_0^{'}\}$$

and (∗)

$$\text{dmin}(f, g) = q_3^{'} \geq \min\{t_2^{'}, r_1^{'}\}$$

This implies that $\text{fss}(X) = q_3'$.

We define a continuous function $p_j : Y \to \overrightarrow{Q_j'Q_j} \cup \overrightarrow{Q_j'Q_{j+1}}$ where $j = 0, 1, 2, 3, 4$, and Y is the closure of the bounded component of $R^2 - X$. First we define p_j on $Y \cap Y_l$ where Y_l is the closure of the bounded component of $R^2 - (\overline{Q_j'Q_{j+1}} \cup \overline{Q_{j+1}Q_{j+1}'} \cup \overline{Q_{j+1}'Q_{j+2}} \cup \overline{Q_{j+2}Q_{j+2}'} \cup \overline{Q_{j+2}'Q_{j+3}} \cup \overline{Q_{j+3}Q_j'})$. We consider two cases in defining p_j on Y_l for $j = 0, 1, 2, 3, 4$.

Case p_j, Y_l, 1: $t_{j+2}' \leq r_{j+1}'$

Let $t \in \overline{t_{j+2}Q_{j+1}}$ and let L_t be the line that is parallel to $\overline{Q_{j+2}t_{j+2}}$ and passes through the point t. For each $y \in Y \cap L_t$, let $p_j(y) = t$. It may be the case $Q_{j+1} = t_{j+2}$. If so, then $L_t \cap Y = \{Q_{j+1}, Q_{j+2}\}$ and $\text{diam}\,(p_j^{-1}\{t\}) = d(Q_{j+2}', t_{j+2}) = t_{j+2}'$. Otherwise, $\overline{Q_{j+1}t_{j+2}} \cup \overline{t_{j+2}Q_{j+2}} \cup \overline{Q_{j+2}Q_{j+1}}$ forms a right triangle and for each t, $\text{diam}\,(p_j^{-1}\{t\}) \leq t_{j+2}'$.

Let $h_1 : [0,1] \to \overline{Q_j't_{j+2}}$ be the homeomorphism such that $h_1(t) = (1-t)t_{j+2} + tQ_j'$.

Let $h_2 : [0,1] \to \overline{Q_{j+2}Q_{j+3}}$ be the homeomorphism such that $h_2(t) = (1-t)Q_{j+2} + tQ_{j+3}$.

Let L_t be the line connecting $h_1(t)$ and $h_2(t)$ for each $t \in [0,1]$. For each $y \in Y \cap L_t$ let $p_j(y) = h_1(t)$. Note that $\text{diam}\,(p_j^{-1}(\{h_1(t)\})) \leq \max\{t_{j+2}', q_{j+3}'\}$.

Case p_j, Y_l, 2: $r_{j+1}' < t_{j+2}'$

Let $t \in \overline{r_{j+1}Q_{j+2}}$ and let L_t be the line that is parallel to $\overline{r_{j+1}Q_{j+1}}$ and passes through the point t. Let t' be the point such that $\{t'\} = L_t \cap ((\overrightarrow{Q_j'Q_{j+1}} - \overline{Q_j'Q_{j+1}}) \cup Q_{j+1})$. For each $y \in Y \cap L_t$ let $p_j(y) = t'$. Note that for each t', $\text{diam}\,(p_j^{-1}\{t'\}) \leq r_{j+1}'$.

Let $h_1 : [0,1] \to \overline{Q_j'Q_{j+1}}$ be the homeomorphism such that $h_1(t) = (1-t)Q_{j+1} + tQ_j'$.

Let $h_2 : [0,1] \to \overline{r_{j+1}Q_{j+3}}$ be the homeomorphism such that $h_2(t) = (1-t)r_j + tQ_{j+3}$.

Let L_t be the line connecting $h_1(t)$ and $h_2(t)$. For each $y \in L_t \cap Y$ let $p_j(y) = h_1(t)$. Note that for each t, $\text{diam}\,(p_j^{-1}(\{h_1(t)\})) \leq \max\{r_{j+1}', q_{j+3}'\}$.

The definition of p_j on $Y \cap Y_r$, where Y_r is the closure of the bounded component of $R^2 - (\overline{Q_j'Q_j} \cup \overline{Q_jQ_{j+4}'} \cup \overline{Q_{j+4}'Q_{j+4}} \cup \overline{Q_{j+4}Q_{j+3}'} \cup \overline{Q_{j+3}'Q_{j+3}} \cup \overline{Q_{j+3}Q_j'})$ is defined in a similar manner in the two corresponding cases.

Case p_j, Y_r, 1: $r_{j+4}' \leq t_j'$.

In this case for $y \in Y \cap Y_r$, $\text{diam}\,(p_j^{-1}(\{p_j(y)\})) \leq \max\{r_{j+4}', q_{j+3}'\}$

Case p_j, Y_r, 2: $t_j' < r_{j+4}'$

In this case for $y \in Y \cap Y_r$, $\text{diam}\,(p_j^{-1}(\{p_j(y)\})) \leq \max\{t_j', q_{j+3}'\}$.

We see that for $y \in Y_l$, $\operatorname{diam}(p_j^{-1}(\{p_j(y)\})) \leq \max\{\min\{r'_{j+1}, t'_{j+2}\}, q'_{j+3}\}$. For $y \in Y_r$, $\operatorname{diam}(p_j^{-1}(\{p_j(y)\})) \leq \max\{\min\{r'_{j+4}, t'_j\}, q'_{j+3}\}$. Consequently, for $y \in Y$,

$$\operatorname{diam}(p_j^{-1}(\{p_j(y)\})) \leq \max\{\min\{r'_{j+1}, t'_{j+2}\}, \min\{r'_{j+4}, t'_j\}, q'_{j+3}\}.$$

Let $f^*, g^* : C \to Z$ be any two continuous functions from a continuum C into a continuum $Z \subseteq Y$, such that $f^*(C) \subseteq g^*(C) \subseteq Z$. Consider $p_j \circ f^*$, $p_j \circ g^* : C \to (\overrightarrow{Q'_j Q_j} \cup \overrightarrow{Q'_j Q_{j+1}})$. The image of $p_j \circ g^*(C)$ is an arc and $p_j \circ f^*(C) \subseteq p_j \circ g^*(C)$. Since all the spans of an arc are zero, there is a $c \in C$ such that $p_j \circ f^*(c) = p_j \circ g^*(c)$. Consequently, $d(f^*(c), g^*(c)) \leq \operatorname{diam}(p_j^{-1}\{p_j(g^*(c))\}) \leq \max\{\min\{r'_{j+1}, t'_{j+2}\}, \min\{r'_{j+4}, t'_j\}, q'_{j+3}\}$ and

$$\sigma_0(Z) \leq \max\{\min\{r'_{j+1}, t'_{j+2}\}, \min\{r'_{j+4}, t'_j\}, q'_{j+3}\}.$$

In this case (i.e. case A) when $\operatorname{fss}(X) = \operatorname{dmin}(f, g)$ where f and g are defined in Table 1, we conclude that $\sigma_0(X) \leq q'_{j+3}$ by taking $Z = X \subseteq Y$, $j = 0$, and using $(*)$. Given the inequalities relating the various spans and the fact that for each $t \in [0, 1]$ there is a $t' \in [0, 1]$ such that $g(t) = f(t')$ and $f(t) = g(t')$, we conclude that $\sigma(X) = \sigma_0(X) = \sigma^*(X) = \sigma_0^*(X) = s(X) = s^*(X) = \operatorname{fss}(X) = q'_3$. This completes case A.

Case B: $\max\{\operatorname{dmin}(f, g) \,|\, (f, g) \in \mathcal{P}\} > q'_3$

Let $(f, g) \in \mathcal{P}$ such that $\max\{\operatorname{dmin}(f, g) \,|\, (f, g) \in \mathcal{P}\} = \operatorname{dmin}(f, g) > q'_3$. Since $r'_3 < q'_3$ and $t'_3 < q'_3$, the pair (f, g) can not include the following (forbidden) steps as given in Table 4 and Table 5.

Table 4. Forbidden steps

t	$f(t)$	$g(t)$	t	$f(t)$	$g(t)$	t	$f(t)$	$g(t)$
t_1	Q_4	Q_3	t_1	Q_0	Q_3	t_1	Q_1	Q_3
t_2	Q_0	Q_3	t_2	Q_1	Q_3	t_2	Q_2	Q_3

Table 5. Forbidden steps

t	$f(t)$	$g(t)$	t	$f(t)$	$g(t)$	t	$f(t)$	$g(t)$
t_1	Q_3	Q_4	t_1	Q_3	Q_0	t_1	Q_3	Q_1
t_2	Q_3	Q_0	t_2	Q_3	Q_1	t_2	Q_3	Q_2

For any pair of functions, f and g, we consider all the possible starting values of f and g, that is $f(0)$ and $g(0)$, and determine what steps are possible for $t > 0$ given the restriction $\operatorname{dmin}(f, g) > d(Q_3, Q'_1) = q'_3$. Because of the symmetry of this process, we consider either $f(0) = Q_j$ and $g(0) = Q_k$ where $k \neq j$, or $f(0) = Q_k$ and $g(0) = Q_j$ for $k \neq j$, but not both. All these possibilities are summarized in the tables below. For $n \in N$, we let $0 = t_0 < t_1 < t_2 < \cdots < t_n < 1$.

TABLE 6. Left to right: Patterns I, II, III, IV .

t	$f(t)$	$g(t)$
0	Q_0	Q_2
t_1	Q_1	Q_2
t_2	Q_1	Q_3
t_3	Q_1	Q_4
t_4	Q_1	Q_0
t_5	Q_2	Q_0

t	$f(t)$	$g(t)$
0	Q_0	Q_2
t_1	Q_1	Q_2
t_2	Q_1	Q_3
t_3	Q_1	Q_4
t_4	Q_2	Q_4
t_5	Q_2	Q_0

t	$f(t)$	$g(t)$
0	Q_0	Q_2
t_1	Q_0	Q_3
t_2	Q_0	Q_4
t_3	Q_1	Q_4
t_4	Q_1	Q_0
t_5	Q_2	Q_0

t	$f(t)$	$g(t)$
0	Q_0	Q_2
t_1	Q_0	Q_3
t_2	Q_0	Q_4
t_3	Q_1	Q_4
t_4	Q_2	Q_4
t_5	Q_2	Q_0

TABLE 7.

t	$f(t)$	$g(t)$
0	Q_0	Q_1
t_1	Q_0	Q_2

TABLE 8. Left to right: Patterns V and VI.

t	$f(t)$	$g(t)$
0	Q_4	Q_0
t_1	Q_4	Q_1
t_2	Q_4	Q_2
t_3	Q_0	Q_2

t	$f(t)$	$g(t)$
0	Q_4	Q_0
t_1	Q_4	Q_1
t_2	Q_0	Q_1
t_3	Q_0	Q_2

TABLE 9.

t	$f(t)$	$g(t)$
0	Q_3	Q_0
t_1	Q_4	Q_0

We start with $f(0) = Q_0$ and $g(0) = Q_2$ in Table 6. We see that there are four possibilities: pattern I, II, III, or IV. In Table 7, where $f(0) = Q_0$ and $g(0) = Q_1$, there is only one possible second step. After this step, the functions would follow one of the patterns I, II, III, or IV in Table 6. In Table 8, we start with $f(0) = Q_4$ and $g(0) = Q_0$. We see that there are two possibilities, patterns V and VI. Each of these ends with $f(t_3) = Q_0$ and $g(t_3) = Q_2$. These functions would then follow either pattern I, II, III, or IV

Table 10. Left to right: Patterns VII and VIII.

t	$f(t)$	$g(t)$
0	Q_4	Q_1
t_1	Q_4	Q_2
t_2	Q_0	Q_2

t	$f(t)$	$g(t)$
0	Q_4	Q_1
t_1	Q_0	Q_1
t_2	Q_0	Q_2

Table 11.

t	$f(t)$	$g(t)$
0	Q_2	Q_1
t_1	Q_3	Q_1
t_2	Q_4	Q_1

Table 12.

t	$f(t)$	$g(t)$
0	Q_3	Q_1
t_1	Q_4	Q_1

Table 13.

t	$f(t)$	$g(t)$
0	Q_3	Q_2
t_1	Q_4	Q_2
t_2	Q_0	Q_2

in Table 6. In Table 9 we have $f(0) = Q_3$ and $g(0) = Q_0$. There is only one possible second step, that is $f(t_1) = Q_4$ and $g(t_1) = Q_0$. These functions then would follow either pattern V or VI in Table 8. After this, the functions would follow one of the patterns, I, II, III, or IV in Table 6. In Table 10, we have $f(0) = Q_4$ and $g(0) = Q_1$. There are two possibilities, patterns VII or VIII. Both of these patterns end with $f(t_2) = Q_0$ and $g(t_2) = Q_2$. These functions would then follow one of the patterns I, II, III, or IV in Table 6. Similarly, the functions in Table 11 would subsequently follow one of the patterns VII or VIII in Table 10, then follow one of the patterns I, II, III, or IV in Table 6. The functions in Table 12 would follow either pattern VII or VIII in Table 10, then either pattern I, II, III, or IV in Table 6. The pairs of functions in tables 13 and 14 would each follow one of the patterns I, II, III, or IV in Table 6.

We have considered all possible combinations for $f(0)$ and $g(0)$. Note that $f(0) = Q_3$ and $g(0) = Q_4$ is not possible, since for t_1, $f(t_1) = Q_3$, $g(t_1) = Q_0$, and $\text{dmin}(f, g) \leq r_3 < d(Q_3, Q_0^{'})$. We see that all pairs of functions f and g, satisfying our conditions, contain one of the patterns I, II, III, or IV. So, if

Table 14.

t	$f(t)$	$g(t)$
0	Q_4	Q_2
t_1	Q_0	Q_2

there is a pair of functions $(f, g) \in \mathcal{P}$ such that $\text{dmin}(f, g) > d(Q_3, Q_0^{'}) = q_3^{'}$, then f and g contain one of the patterns I, II, III, or IV. So, for any pair, $(f, g) \in \mathcal{P}$, we are now considering, we see that:

$$\text{dmin}(f, g) \leq \max\{\min \text{I}, \min \text{II}, \min \text{III}, \min \text{IV}\}$$

where

$$\begin{aligned} d(Q_i, t_i) &= t_i^{'}, \quad d(Q_i, r_i) = r_i^{'}, \quad d(Q_i, Q_{i+2}^{'}) = q_i^{'}, \quad \text{and} \\ \min \text{I} &= \min\{r_0^{'}, r_1^{'}, t_1^{'}, t_2^{'}, q_1^{'}\} = \min\{r_0^{'}, r_1^{'}, t_1^{'}, t_2^{'}\} \\ \min \text{II} &= \min\{r_1^{'}, t_2^{'}, q_1^{'}, q_2^{'}, q_4^{'}\} = \min\{r_1^{'}, t_2^{'}, q_4^{'}\} \\ \min \text{III} &= \min\{r_0^{'}, r_4^{'}, t_0^{'}, t_1^{'}, q_0^{'}\} = \min\{r_0^{'}, r_4^{'}, t_0^{'}, t_1^{'}\} \\ \min \text{IV} &= \min\{r_4^{'}, t_0^{'}, q_0^{'}, q_2^{'}, q_4^{'}\} = \min\{r_4^{'}, t_0^{'}, q_2^{'}\}. \end{aligned}$$

Note that patterns I and III are mirror images of each other, as are II and IV.

If the "best" f and g is not as given in Table 1, then since all other pairs, f and g, contain one of the patterns I, II, III, or IV, then the best f and g must be one of the pairs of functions given in Table 15. Hence, the "best" pair f and g will be from either Table I, II, III, or IV in Table 15, and

$$\text{dmin}(f, g) = \max\{\min \text{I}, \min \text{II}, \min \text{III}, \min \text{IV}\},$$

where $\text{dmin}(f, g) = \min$ I for Table I, $\text{dmin}(f, g) = \min$ II for Table II, $\text{dmin}(f, g) = \min$ III for Table III, and $\text{dmin}(f, g) = \min$ IV for Table IV. Consequently, $\alpha(X) \geq \max\{\min$ I, min II, min III, min IV$\}$, where $\alpha = \sigma, \sigma_0, \sigma^* \sigma_0^*, s, s^*$ since the functions f and g satisfy the requirements for all of the spans. We will show that $\sigma_0(X) \leq \max\{\min$ I, min II, min III, min IV$\}$ and consequently that $\alpha(X) = \max\{\min$ I, min II, min III, min IV$\}$ for $\alpha = \sigma, \sigma_0, \sigma^*, \sigma_0^*, s, s^*$.

Since I and III are mirror images of each other as are II and IV, we need to examine only two cases,

$$\min \text{I} \quad = \quad \max\{\min \text{I}, \min \text{II}, \min \text{III}, \min \text{IV}\}$$

and

$$\min \text{II} \quad = \quad \max\{\min \text{I}, \min \text{II}, \min \text{III}, \min \text{IV}\}$$

Case 1: min II $= \max\{\min$ I, min II, min III, min IV$\} = \min\{r_1^{'}, t_2^{'}, q_4^{'}\}$.

Suppose that $\max\{\min$ I, min II, min III, min IV$\} = \min$ II, that is, the best pair of functions f and g is from Table II. So, $\text{dmin}(f, g) = \min$

TABLE 15. Left to right Table I, Table II, Table III, Table IV.

t	$f(t)$	$g(t)$
0	Q_0	Q_2
0.1	Q_1	Q_2
0.2	Q_1	Q_3
0.3	Q_1	Q_4
0.4	Q_1	Q_0
0.5	Q_2	Q_0
0.6	Q_2	Q_1
0.7	Q_3	Q_1
0.8	Q_4	Q_1
0.9	Q_0	Q_1
1.0	Q_0	Q_2

t	$f(t)$	$g(t)$
0	Q_0	Q_2
0.1	Q_1	Q_2
0.2	Q_1	Q_3
0.3	Q_1	Q_4
0.4	Q_2	Q_4
0.5	Q_2	Q_0
0.6	Q_2	Q_1
0.7	Q_3	Q_1
0.8	Q_4	Q_1
0.9	Q_4	Q_2
1.0	Q_0	Q_2

t	$f(t)$	$g(t)$
0	Q_0	Q_2
0.1	Q_0	Q_3
0.2	Q_0	Q_4
0.3	Q_1	Q_4
0.4	Q_1	Q_0
0.5	Q_2	Q_0
0.6	Q_3	Q_0
0.7	Q_4	Q_0
0.8	Q_4	Q_1
0.9	Q_0	Q_1
1.0	Q_0	Q_2

t	$f(t)$	$g(t)$
0	Q_0	Q_2
0.1	Q_0	Q_3
0.2	Q_0	Q_4
0.3	Q_1	Q_4
0.4	Q_2	Q_4
0.5	Q_2	Q_0
0.6	Q_3	Q_0
0.7	Q_4	Q_0
0.8	Q_4	Q_1
0.9	Q_4	Q_2
1.0	Q_0	Q_2

II $= \min\{r_1', t_2', q_4'\}$ and $\alpha(X) \geq \min\{r_1', t_2', q_4'\}$, where $\alpha = \sigma, \sigma_0, \sigma^*, \sigma_0^*, s$, and s^*. We will now show that $\alpha(X) = \min\{r_1', t_2', q_4'\} = \mathrm{dmin}(f, g)$, where $\alpha = \sigma, \sigma_0, \sigma^*, \sigma_0^*, s$, and s^*.

Let $f^*, g^* : C \to Z$ be continuous functions from a continuum C into a continuum $Z \subseteq Y$ such that $f^*(C) \subseteq g^*(C)$. We consider three subcases.

CASE 1A: min II $= r_1'$, $r_1' \leq t_2'$, and $r_1' \leq q_4'$.

Observe that min IV $= \min\{r_4', t_0', q_2'\} \leq$ min II $= r_1' \leq t_2' < q_2'$, so min IV $\neq q_2'$ and min IV $= \min\{r_4', t_0'\}$. So, either $r_4' \leq r_1' = \mathrm{dmin}(f, g)$ or $t_0' \leq r_1' = \mathrm{dmin}(f, g)$. We define $p_0 : Y \to \overrightarrow{Q_0', Q_0} \cup \overrightarrow{Q_0'Q_1}$ based on $r_1' \leq t_2'$ and either $r_4' \leq r_1'$ or $t_0' \leq r_1'$. Since $r_1' \leq t_2'$, we define p_0 on Y_l by using case p_0, Y_l, 1 when $r_1' = t_2'$ and case p_0, Y_l, 2 when $r_1' < t_2'$. Also, either $r_4' \leq t_0'$ or $t_0' < r_4'$. We define p_0 on Y_r by case p_0, Y_r, 1 when $r_4' \leq t_0'$ and by case p_0, Y_r, 2 when $t_0' < r_4'$. Consider $p_0 \circ f^*$, $p_0 \circ g^* : C \to \overrightarrow{Q_0'Q_0} \cup \overrightarrow{Q_0'Q_1}$. Since the range of $p \circ g^*$ is an interval and $p_0 \circ f^*(C) \subset p_0 \circ g^*(C)$, we see that there is a $c \in C$ such that $p_0 \circ f^*(c) = p_0 \circ g^*(c)$ and $d(f^*(c), g^*(c)) \leq \max\{\min\{t_0', r_4'\}, q_3', r_1'\} = r_1'$.

CASE 1B: min II $= t_2'$, $t_2' < r_1'$, and $t_2' \leq q_4'$.

Observe that min IV $= \min\{r_4', t_0', q_2'\} \leq$ min II $= t_2' < q_2'$, so min IV $\neq q_2'$, and min IV $= \min\{r_4', t_0'\}$. So, either $r_4' \leq t_2' = \mathrm{dmin}(f, g)$ or $t_0' \leq t_2' = \mathrm{dmin}(f, g)$. We define $p_0 : Y \to \overrightarrow{Q_0'Q_0} \cup \overrightarrow{Q_0'Q_1}$ based on $t_2' < r_1'$ and either $r_4' \leq t_2'$ or $t_0' \leq t_2'$. Since $t_2' < r_1'$, we define p_0 on Y_l by using case p_0, Y_l, 1. Also, either $r_4' \leq t_0'$ or $t_0' < r_4'$. We define p_0 on Y_r by case p_0, Y_r, 1 when $r_4' \leq t_0'$ and by case p_0, Y_r, 2 when $t_0' < r_4'$. Again we conclude that there is a $c \in C$ such that $d(f^*(c), g^*(c)) \leq \max\{\min\{t_0', r_4'\}, q_3', t_2'\} = t_2'$.

CASE 1C: min II $= q_4'$, $q_4' < r_1'$, and $q_4' < t_2'$.

Observe that min I $=\min\{r_0^{'}, r_1^{'}, t_1^{'}, t_2^{'}\} \leq$ min II $= q_4^{'} < r_1^{'}$ and $q_4^{'} < t_2^{'}$, min I $= \min\{r_0^{'}, t_1^{'}\}$, so either $r_0^{'} \leq q_4^{'} = \text{dmin}(f,g)$ or $t_1^{'} \leq q_4^{'} = \text{dmin}(f,g)$. We define $p_1 : Y \to \overrightarrow{Q_1^{'}Q_1} \cup \overrightarrow{Q_1^{'}Q_2}$ based on either $r_0^{'} \leq t_1^{'}$ or $t_1^{'} < r_0^{'}$ for $y \in Y_r$ and either $t_3^{'} \leq r_2^{'}$ or $r_2^{'} < t_3^{'}$ for $y \in Y_l$. Note that $t_3^{'} < q_3^{'} < q_4^{'}$. So we see that there is a $c \in C$ such that $d(f^*(c), g^*(c)) \leq \max\{\min\{r_0^{'}, t_1^{'}\}, \{\min\{r_2^{'}, t_3^{'}\}, q_4^{'}\} = q_4^{'}$.

Therefore, in case 1 we conclude that $\sigma_0(X) \leq \text{dmin}(f,g) =$ min II. Since X is a simple closed curve we know that $\sigma(X) = \sigma^*(X)$ and $\sigma_0(X) = \sigma_0^*(X)$. Also, we see that for all $t \in [0,1]$ there is a $t^{'} \in [0,1]$ such that $f(t) = g(t^{'})$ and $f(t^{'}) = g(t)$. Consequently, we see that $\sigma(X) = \sigma_0(X) = \sigma^*(X) = \sigma_0^*(X) = s(X) = s^*(X) =$ min II.

Case 2: min I $=\max\{$min I, min II, min III, min IV$\} = \min\{r_0^{'}, r_1^{'}, t_1^{'}, t_2^{'}\}$.

Suppose that min I $=\max\{$min I, min II. min III, min IV$\}$, that the best pair of functions f and g is from Table I. So, $\text{dmin}(f,g) =$ min I $= \min\{r_0^{'}, r_1^{'}, t_1^{'}, t_2^{'}\}$ and $\alpha(X) \geq \min\{r_0^{'}, r_1^{'}, t_1^{'}, t_2^{'}\}$ where $\alpha = \sigma, \sigma_0, \sigma^*, \sigma_0^*, s$ and s^*. We will now show that $\alpha(X) = \min\{r_0^{'}, r_1^{'}, t_1^{'}, t_2^{'}\} = \text{dmin}(f,g)$ where $\alpha = \sigma, \sigma_0, \sigma^*, \sigma_0^*, s$ and s^*.

Let $f^*, g^* : C \to Z$ be continuous functions from a continuum C into a continuum $Z \subseteq Y$ such that $f^*(C) \subseteq g^*(C)$. We consider four subcases.

Case 2A: min I $= r_0^{'}$, $r_0^{'} \leq t_1^{'}$, $r_0^{'} \leq r_1^{'}$, and $r_0^{'} \leq t_2^{'}$.

Observe that min II $= \min\{r_1^{'}, t_2^{'}, q_4^{'}\} \leq$ min I $= r_0^{'}$. We can assume that $r_0^{'} \neq r_1^{'}$, since this case has been covered in case 1A, so $r_0^{'} < r_1^{'}$. We can assume that $r_0^{'} \neq t_2^{'}$, since this case has been covered in case 1B, so $r_0^{'} < t_2^{'}$. So, we assume that min II $= q_4^{'} \leq$ min I $= r_0^{'}$. So, we have that $q_4^{'} \leq r_0^{'} = \text{dmin}(f,g)$. We can assume that $q_4^{'} \neq r_0^{'}$, since this is covered in case 1C. So, $q_4^{'} < r_0^{'} = \text{dmin}(f,g)$. We define $p_1 : Y \to \overrightarrow{Q_1^{'}Q_1} \cup \overrightarrow{Q_1^{'}Q_2}$ based on case p_1, Y_r, 1 for $y \in Y_r$ since $r_0^{'} \leq t_1^{'}$ and based on either case p_1, Y_l, 1 if $t_3^{'} \leq r_2^{'}$ or case p_1, Y_l, 2 if $r_2^{'} < t_3^{'}$ for $y \in Y_l$. Consider $p_1 \circ f^*$, $p_1 \circ g^* : C \to \overrightarrow{Q_1^{'}Q_1} \cup \overrightarrow{Q_1^{'}Q_2}$. Since the range is an interval there is a $c \in C$ such that $p_1 \circ f^*(c) = p_1 \circ g^*(c)$ and $d(f^*(c), g^*(c)) \leq \max\{\min\{t_3^{'}, r_2^{'}\}, q_4^{'}, r_0^{'}\} = r_0^{'}$.

Case 2B: min I $= t_1^{'}$, $t_1^{'} < r_0^{'}$, $t_1^{'} \leq r_1^{'}$, and $t_1^{'} \leq t_2^{'}$.

Observe that min II $= \min\{r_1^{'}, t_2^{'}, q_4^{'}\} \leq$ min I $= t_1^{'}$. We can assume that $t_1^{'} \neq r_1^{'}$, since this is already covered in case 1A, so $t_1^{'} < r_1^{'}$. We can assume that $t_1^{'} \neq t_2^{'}$, since this is already covered in case 1B, so $t_1^{'} < t_2^{'}$. So we assume that min II $= q_4^{'} \leq$ min I $= t_1^{'}$. So, we have that $q_4^{'} \leq t_1^{'} = \text{dmin}(f,g)$. We can assume that $q_4^{'} \neq t_1^{'}$, since this is covered in case 1C. So, $q_4^{'} < t_1^{'} = \text{dmin}(f,g)$. We define $p_1 : Y \to \overrightarrow{Q_1^{'}Q_1} \cup \overrightarrow{Q_1^{'}Q_2}$ based on case p_1, Y_r, 2 for $y \in Y_r$ since $t_1^{'} < r_0^{'}$ and based on either case p_1, Y_l, 1 if $t_3^{'} \leq r_2^{'}$ or case p_1, Y_l, 2 if $r_2^{'} < t_3^{'}$ for $y \in Y_l$. Consider $p_1 \circ f^*$, $p_1 \circ g^* : C \to \overrightarrow{Q_1^{'}Q_1} \cup \overrightarrow{Q_1^{'}Q_2}$. Since the range is an interval there is a $c \in C$ such that $p_1 \circ f^*(c) = p_1 \circ g^*(c)$ and $d(f^*(c), g^*(c)) \leq \max\{\min\{t_3', r_2'\}, q_4', t_1'\} = t_1'$.

CASE 2C: $\min \mathrm{I} = r_1'$, $r_1' < r_0'$, $r_1' < t_1'$, and $r_1' \leq t_2'$.

This case has been covered in case 1A.

CASE 2D: $\min \mathrm{I} = t_2'$, $t_2' < r_0', t_2' < r_1'$, and $t_2' < t_1'$.

This case has already been covered in case 1B.

Therefore in case 2 we conclude that $\sigma_0(X) \leq \mathrm{dmin}(f,g) = \min \mathrm{I}$. Also, $\sigma(X) = \sigma^*(X)$ and $\sigma_0(X) = \sigma_0^*(X)$ since X is a simple closed curve and for all $t \in [0,1]$ there is a $t' \in [0,1]$ such that $f(t) = g(t')$ and $f(t') = g(t)$. Consequently, $\sigma(X) = \sigma_0(X) = \sigma^*(X) = \sigma_0^*(X) = s(X) = s^*(X) = \min \mathrm{I}$.

The other three cases can be proved in a similar manner.

Hence, $\sigma(X) = \sigma_0(X) = \sigma^*(X) = \sigma_0^*(X) = s(X) = s^*(X) = \max\{q_{j+3}', \min \mathrm{I}, \min \mathrm{II}, \min \mathrm{III}, \min \mathrm{IV}\} = \mathrm{fss}(X)$. □

THEOREM 3.2. *Let Z be a continuum such that $Z \subseteq Y$, where Y is the closure of the bounded component of $R^2 - X$ and X is a five star-like simple closed curve, then $\alpha(Z) \leq \mathrm{fss}(X)$, where $\alpha = \sigma, \sigma_0, \sigma^*, \sigma_0^*, s, s^*$.*

PROOF. Since each of the functions p_i for $i = 0, 1, 2, 3, 4$ were defined for any continuum $Z \subseteq Y$ where Y is the closure of the bounded component of $R^2 - X$, we see that for any such Z, $\alpha(Z) \leq \mathrm{fss}(X)$, where $\alpha = \sigma, \sigma_0, \sigma^*, \sigma_0^*, s$, and s^*. □

COROLLARY 3.3. *Let Z be a simple closed curve such that $Z \subseteq Y$ where Y is the closure of the bounded component of $R^2 - X$ and X is a five star-like simple closed curve, then*

$$\sigma(Z) \leq \sigma(X) \ .$$

So, in this situation, the question by Howard Cook is answered in the affirmative.

COROLLARY 3.4. *Let Y be a plane separating continuum and X be a five star-like simple closed curve contained in the closure of a bounded component of $R^2 - Y$, then*

$$\sigma(X) \leq \sigma(Y).$$

PROOF. Let Y be a plane separating continuum such that X is contained in one of the bounded components of $R^2 - Y$. We can assume that the origin, $\mathcal{O}$, is in the bounded component of $R^2 - Q$. There is an $r > 0$ such that the circle with center the origin and radius r, $C(\mathcal{O}, r)$, is such that Y is contained in the bounded component of $R^2 - C(\mathcal{O}, r)$. We will show that $\sigma(Y) \geq \sigma(X)$.

We will consider the case where $\sigma(X) = \min \mathrm{II}$. The lines $\overleftrightarrow{Q_2Q_1}$ and $\overleftrightarrow{Q_2Q_4}$ separate the bounded component of $R^2 - C(\mathcal{O}, r)$ into four wedges. We let V_0 be the closure of the wedge that contains Q_4' and Q_0' and W_0 be the closure of the wedge that is opposite V_0. The set $V_0 - (\overline{Q_4Q_4'} \cup \overline{Q_4'Q_0} \cup \overline{Q_0Q_0'} \cup \overline{Q_0'Q_1})$ has two components. Let W_0' be the closure of the component that intersects $C(\mathcal{O}, r)$. For each $z \in W_0 \cap Y$, let $r(z) \in \overleftrightarrow{Q_2z} \cap W_0 \cap C(\mathcal{O}, r)$ and for each $z \in W_0' \cap Y$, let $r(z) \in \overleftrightarrow{Q_2z} \cap W_0' \cap C(\mathcal{O}, r)$. The lines $\overleftrightarrow{Q_4Q_2}$ and $\overleftrightarrow{Q_4Q_1}$ separate the bounded component of $R^2 - C(\mathcal{O}, r)$ into four wedges. Let V_1 be the closure of the wedge that contains Q_1'. Let W_1 be the closure of the wedge that is opposite V_1. The set $\overline{Q_1Q_1'} \cup \overline{Q_1'Q_2}$ separates V_1 into two components. Let W_1' be the closure of the component that intersects $C(\mathcal{O}, r)$. For $z \in W_1 \cap Y$, let $r(z) \in \overleftrightarrow{Q_4z} \cap W_1 \cap C(\mathcal{O}, r)$, and for $z \in W_1' \cap Y$, let $r(z) \in \overleftrightarrow{Q_4z} \cap W_1' \cap C(\mathcal{O}, r)$. Similarly, using the lines $\overleftrightarrow{Q_1Q_4}$ and $\overleftrightarrow{Q_1Q_2}$, we define the sets W_2 and W_2'. Note that $Q_1 \in W_2$ and $Q_2', Q_3' \in W_2'$. We define r for $z \in W_2 \cap Y$ by $r(z) \in \overleftrightarrow{Q_1z} \cap W_2 \cap C(\mathcal{O}, r)$ and for $z \in W_2' \cap Y$ by $r(z) \in \overleftrightarrow{Q_1z} \cap W_2' \cap C(\mathcal{O}, r)$. Observe that for either $x \in W_0$ and $y \in W_0' \cap \overleftrightarrow{Q_2x}$, or $x \in W_1$ and $y \in W_1' \cap \overleftrightarrow{Q_4x}$, or $x \in W_2$ and $y \in W_2' \cap \overleftrightarrow{Q_1x}$, $d(x, y) \geq \min$ II.

We can rotate $X \cup Y$ in the plane about the origin so that the ray $\overrightarrow{\mathcal{O}x}$ corresponds to the positive x-axis, where $\{x\} = W_0 \cap W_2' \cap \overleftrightarrow{Q_1Q_2} \cap C(\mathcal{O}, r)$. Let θ_j for $j = 0, 1, 2, 3, 4, 5, 6$, be angles in the rotated position such that $0 = \theta_0 < \theta_1 < \theta_2 < \theta_3 < \theta_4 < \theta_5 < \theta_6 = 2\pi$, where $re^{i\theta_0}, re^{i\theta_3} \in \overleftrightarrow{Q_1Q_2} \cap C(\mathcal{O}, r)$, $re^{i\theta_1}, re^{i\theta_4} \in \overleftrightarrow{Q_2Q_4} \cap C(\mathcal{O}, r)$, and $re^{i\theta_2}, re^{i\theta_5} \in \overleftrightarrow{Q_4Q_1} \cap C(\mathcal{O}, r)$. Let $f : [0, 2\pi] \to [0, 2\pi]$ be a piecewise linear, surjective function, such that $f(\theta_j) = j(\frac{\pi}{3})$, and for $\theta \in [\theta_j, \theta_{j+1}], f(\theta) = j(\frac{\pi}{3}) + (\frac{\theta - \theta_j}{\theta_{j+1} - \theta_j})(\frac{\pi}{3})$ for each $j = 0, 1, 2, 3, 4, 5$.

Let $h : C(\mathcal{O}, r) \to C(\mathcal{O}, r)$ be the homeomorphism given by $h(re^{i\theta}) = h(re^{if(\theta)})$.

We define $p : Y \to C(\mathcal{O}, r)$ by $p(y) = h \circ r(y) = h(re^{i\theta y}) = re^{if(\theta y)}$ where $r(y) = re^{i\theta y}$. We see that p is an essential map from Y onto $C(\mathcal{O}, r)$. Also, for each $y, z \in Y$ such that $p(y)$ and $p(z)$ are diametrically opposed to each other on $C(\mathcal{O}, r)$, $d(p^{-1}\{p(y)\}, p^{-1}\{p(z)\}) \geq \min$ II.

So in this case, by Theorem L, $\sigma(Y) \geq \sigma(X) = \min$ II. The cases where $\sigma(X) = \min\{q_i\}_{i=0}^4, \min$ I, min III, or min IV can be shown in a similar manner. We conclude that in all cases $\sigma(Y) \geq \sigma(X)$. □

When Y is a simple closed curve, the question by Howard Cook is answered in the affirmative in this situation.

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