GLASNIK MATEMATIČKI
Vol. 42(62)(2007), 363 – 373

VECTORS AND TRANSFERS IN HEXAGONAL QUASIGROUP

Mea Bombardelli and Vladimir Volenec
University of Zagreb, Croatia

Abstract. Hexagonal quasigroup is idempotent, medial and semisymmetric quasigroup. In this article we define and study vectors, sum of vectors and transfers. The main result is the theorem on isomorphism between the group of vectors, group of transfers and the Abelian group from the characterization theorem of the hexagonal quasigroups.

1. Hexagonal quasigroup

Hexagonal quasigroups are defined in article [3].

Definition 1.1. *A quasigroup* $(Q, \cdot)$ *is called* hexagonal *if it is idempotent, medial and semisymmetric; i.e. if its elements a, b, c, d satisfy*

$$\begin{aligned} &a \cdot a = a \\ &(a \cdot b) \cdot (c \cdot d) = (a \cdot c) \cdot (b \cdot d) \\ &a \cdot (b \cdot a) = (a \cdot b) \cdot a = b. \end{aligned}$$

When it doesn't cause confusion, we can omit the sign "$\cdot$", e.g. instead of $(a \cdot b) \cdot (c \cdot d)$ we shall write $ab \cdot cd$.

Theorem 1.2. *In any hexagonal quasigroup* $(Q, \cdot)$ *the identities*

$$a \cdot bc = ab \cdot ac \qquad \text{and} \qquad ab \cdot c = ac \cdot bc$$

hold for all $a, b, c \in Q$. *The equalities* $ab = c$, $bc = a$ *and* $ca = b$ *are equivalent.*

The basic example of a hexagonal quasigroup studied in [3] is the following.

2000 *Mathematics Subject Classification.* 20N05.
Key words and phrases. Quasigroup, hexagonal quasigroup, vector, transfer.

EXAMPLE 1.3. The set $\mathbb{C}$ of complex numbers, with the operation $*$:

$$a * b = \frac{1 - i\sqrt{3}}{2} \cdot a + \frac{1 + i\sqrt{3}}{2} \cdot b$$

is a hexagonal quasigroup.

If we identify complex numbers with the points of the Euclidean plane, the points a, b and $a * b$ turn out to be vertices of positively oriented regular (equilateral) triangle.

Finite hexagonal quasigroups are interesting as well.

EXAMPLE 1.4. Two quasigroups defined by table 1 are hexagonal. We shall call them Q_3 and Q_4.

TABLE 1. Finite quasigroups Q_3 and Q_4

$\cdot$	A	B	C
A	A	C	B
B	C	B	A
C	B	A	C

$\cdot$	1	2	3	4
1	1	3	4	2
2	4	2	1	3
3	2	4	3	1
4	3	1	2	4

The direct product of hexagonal quasigroups is also a hexagonal quasigroup.

EXAMPLE 1.5. The product of quasigroups Q_3 and Q_4.
We shall denote the element $(B, 3) \in Q_3 \times Q_4$ by B_3, and similarly. Multiplication table in $Q_3 \times Q_4$ is then:

$\cdot$	A_1	A_2	A_3	A_4	B_1	B_2	B_3	B_4	C_1	C_2	C_3	C_4
A_1	A_1	A_3	A_4	A_2	C_1	C_3	C_4	C_2	B_1	B_3	B_4	B_2
A_2	A_4	A_2	A_1	A_3	C_4	C_2	C_1	C_3	B_4	B_2	B_1	B_3
A_3	A_2	A_4	A_3	A_1	C_2	C_4	C_3	C_1	B_2	B_4	B_3	B_1
A_4	A_3	A_1	A_2	A_4	C_3	C_1	C_2	C_4	B_3	B_1	B_2	B_4
B_1	C_1	C_3	C_4	C_2	B_1	B_3	B_4	B_2	A_1	A_3	A_4	A_2
B_2	C_4	C_2	C_1	C_3	B_4	B_2	B_1	B_3	A_4	A_2	A_1	A_3
B_3	C_2	C_4	C_3	C_1	B_2	B_4	B_3	B_1	A_2	A_4	A_3	A_1
B_4	C_3	C_1	C_2	C_4	B_3	B_1	B_2	B_4	A_3	A_1	A_2	A_4
C_1	B_1	B_3	B_4	B_2	A_1	A_3	A_4	A_2	C_1	C_3	C_4	C_2
C_2	B_4	B_2	B_1	B_3	A_4	A_2	A_1	A_3	C_4	C_2	C_1	C_3
C_3	B_2	B_4	B_3	B_1	A_2	A_4	A_3	A_1	C_2	C_4	C_3	C_1
C_4	B_3	B_1	B_2	B_4	A_3	A_1	A_2	A_4	C_3	C_1	C_2	C_4

The characterisation theorem of hexagonal quasigroups was proven in [3].

THEOREM 1.6. *A hexagonal quasigroup* $(Q,\cdot)$ *exists if and only if an Abelian group* $(Q,+)$ *and an automorphism* φ *satisfying*

$$(\varphi\circ\varphi)(a)-\varphi(a)+a=0,\quad \forall a\in Q \tag{1.1}$$

exist. Each of the two binary operations $+$ *and* $\cdot$ *is defined by means of the other by the equalities*

$$a\cdot b=a+\varphi(b-a) \tag{1.2}$$

$$a+b=0a\cdot b0, \tag{1.3}$$

where 0 *is the neutral element of the group* $(Q,+)$.

In the rest of this article, Q will always be a hexagonal quasigroup.

2. GEOMETRY OF HEXAGONAL QUASIGROUP

In [3] and [4] some geometric terms are defined and studied in hexagonal quasigroups, motivated by the quasigroup $(\mathbb{C},*)$.

The elements of a hexagonal quasigroup are called *points*. A pair of points is called a *segment*, and a cyclic triple of points is called a *triangle*.

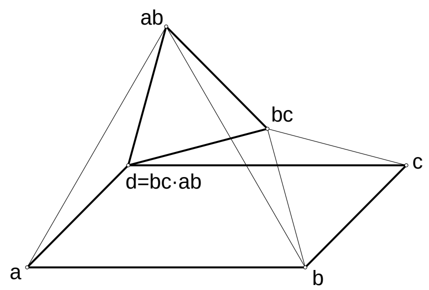

FIGURE 1. $\mathrm{Par}(a,b,c,d)$

DEFINITION 2.1. *It is said that the points* a, b, c, d *form a* parallelogram, *if* $bc\cdot ab=d$. *This is denoted by* $Par(a,b,c,d)$.

REMARK 2.2. The relation Par is defined in any medial quasigroup ([2]). According to that definition, in a hexagonal quasigroup $\mathrm{Par}(a,b,c,d)$ holds if $ax\cdot b=dx\cdot c$. It follows

$$c=(ax\cdot b)\cdot dx=(ab\cdot xb)(dx)=(ab\cdot d)(xb\cdot x)=(ab\cdot d)b$$

which is equivalent to $bc=ab\cdot d$ or $d=bc\cdot ab$, and is therefore equivalent with our definition. Hence, we may use all the results from [2].

According to [2], (Q, Par) is a parallelogram space, i.e. the following properties hold:

Par1. Any three of the four points a, b, c, d uniquely determine the fourth, such that $\mathrm{Par}(a,b,c,d)$ holds.

Par2. The statements $\mathrm{Par}(a,b,c,d)$, $\mathrm{Par}(b,c,d,a)$, $\mathrm{Par}(c,d,a,b)$, $\mathrm{Par}(d,a,b,c)$, $\mathrm{Par}(c,b,a,d)$, $\mathrm{Par}(b,a,d,c)$, $\mathrm{Par}(a,d,c,b)$ and $\mathrm{Par}(d,c,b,a)$ are equivalent.

Par3. From $\mathrm{Par}(a,b,c,d)$ and $\mathrm{Par}(c,d,e,f)$ it follows $\mathrm{Par}(a,b,f,e)$.

DEFINITION 2.3. *A point m is called a* midpoint *of the segment $\{a,b\}$, if $Par(a,m,b,m)$ holds. If this is the case, we write $M(a,m,b)$.*

REMARK 2.4. A midpoint of a segment may not exist, and even when it exists, it may not be unique. E.g. in the hexagonal quasigroup $Q_3 \times Q_4$ from example 1.5 the segment $\{A_1, B_2\}$ has no midpoint, while the segment $\{A_1, B_1\}$ has four midpoints: C_1, C_2, C_3 and C_4.

However, for any two points a and m there exists unique point $b = am{\cdot}ma$ such that $\mathrm{M}(a,m,b)$.

Here we give some more results from [2] we shall need.

THEOREM 2.5. *From $Par(a_1,b_1,c_1,d_1)$ and $Par(a_2,b_2,c_2,d_2)$, it follows $Par(a_1a_2, b_1b_2, c_1c_2, d_1d_2)$.*

THEOREM 2.6. *If $M(a,m,c)$ holds, the statements $M(b,m,d)$ and $Par(a,b,c,d)$ are equivalent.*

DEFINITION 2.7. *The point m is called a* center of the parallelogram *$Par(a,b,c,d)$ if $M(a,m,c)$ and $M(b,m,d)$ hold.*

REMARK 2.8. Similarly the midpoint of a segment, a parallelogram may have no center, or have more than one center.

THEOREM 2.9. *From $Par(a,b,d,e)$ and $Par(b,c,e,f)$ it follows $Par(c,d,f,a)$.*

3. Vectors

Accordingly to [4], the relation $\sim$ defined on $Q \times Q$ by means of

$$(a,b) \sim (c,d) \Leftrightarrow \mathrm{Par}(a,b,d,c)$$

is the equivalence relation on $Q \times Q$. The equivalence class containing the pair (a,b) is denoted by $[a,b]$ and is called a *vector*. The set of all vectors is denoted by $\mathcal{V}$.

It follows immediately

COROLLARY 3.1. *The two vectors $[a,b]$ and $[c,d]$ are equal if and only if $Par(a,b,d,c)$. For any given $o \in Q$, and any vector $[a,b]$ there exists exactly one $x \in Q$ such that $[o,x] = [a,b]$.*

COROLLARY 3.2. *The statement $M(a, m, b)$ is equivalent with the equation $[a, m] = [m, b]$.*

LEMMA 3.3. *If $(x, y), (x', y') \in [a, b]$, $(y, z), (y', z') \in [c, d]$ and $(x, z) \in [u, v]$, then $(x', z') \in [u, v]$.*

PROOF. From the assumptions, it follows $\mathrm{Par}(x, y, b, a)$, $\mathrm{Par}(b, a, x', y')$, $\mathrm{Par}(y, z, d, c)$, $\mathrm{Par}(d, c, y', z')$ and $\mathrm{Par}(x, z, v, u)$. The first two statements, because of the property Par3, imply $\mathrm{Par}(x, y, y', x')$, and the other two $\mathrm{Par}(y, z, z', y')$. Therefore $\mathrm{Par}(x', x, z, z')$ also holds, and now from the last assumption it follows $\mathrm{Par}(x', z', v, u)$; i.e. $(x', z') \in [u, v]$. □

DEFINITION 3.4. *The vector $[u, v]$ is said to be the* sum of the vectors *$[a, b]$ and $[c, d]$ if $(x, y) \in [a, b]$ and $(y, z) \in [c, d]$ imply $(x, z) \in [u, v]$. If this is the case, we write $[a, b] + [c, d] = [u, v]$.*

THEOREM 3.5. *The set of all vectors $\mathcal{V}$ with the binary operation $+$ is a commutative group.*

PROOF. First note that $[x, y] + [y, z] = [x, z]$.

We have

$$([x, y] + [y, z]) + [z, w] = [x, z] + [z, w] = [x, w],$$

$$[x, y] + ([y, z] + [z, w]) = [x, y] + [y, w] = [x, w],$$

hence $([x, y] + [y, z]) + [z, w] = [x, y] + ([y, z] + [z, w])$, which proves the associativity.

Since $\mathrm{Par}(x, x, y, y)$, $[x, x] = [y, y]$, $\forall x, y$. The vector $[a, a]$ will be denoted 0. Obviously, $[x, y] + 0 = [x, y] + [y, y] = [x, y]$ and $0 + [x, y] = [x, x] + [x, y] = [x, y]$, i.e. 0 is the neutral element for the operation $+$.

Since $[x, y] + [y, x] = [x, x] = 0$, the inverse of the vector $[x, y]$ is $[y, x]$.

We still need to prove the commutativity; i.e. that: $[a, b] + [c, d] = [c, d] + [a, b]$.

Let a, b, c and d be any points, and let p and q be points such that $[a, b] = [d, p]$ and $[c, d] = [b, q]$; i.e. $\mathrm{Par}(a, b, p, d)$ and $\mathrm{Par}(c, d, q, b)$. From Theorem 2.9 it follows $\mathrm{Par}(q, p, c, a)$; i.e. $[a, q] = [c, p]$.

Finally,

$$[a, b] + [c, d] = [a, b] + [b, q] = [a, q]$$

$$[c, d] + [a, b] = [c, d] + [d, p] = [c, p]$$

concludes the proof. □

DEFINITION 3.6. *We say that the* vectors $[a, b]$, $[c, d]$ and $[e, f]$ form a triangle *if there exist points p, q and r such that $[p, q] = [a, b]$, $[q, r] = [c, d]$, and $[r, p] = [e, f]$.*

LEMMA 3.7. *Vectors $[a, b]$, $[c, d]$ and $[e, f]$ form a triangle if and only if $[a, b] + [c, d] + [e, f] = 0$.*

Proof. If the vectors $[a,b]$, $[c,d]$ and $[e,f]$ form a triangle, then there exist p, q and r such that $[p,q]=[a,b]$, $[q,r]=[c,d]$ and $[r,p]=[e,f]$. Then $[a,b]+[c,d]+[e,f]=[p,q]+[q,r]+[r,q]=[p,p]=0$.

Let $[a,b]+[c,d]+[e,f]=0$. Let p be any point, and q and r such that $[p,q]=[a,b]$ and $[q,r]=[c,d]$. Then $[e,f]=-([a,b]+[c,d])=-([p,q]+[q,r])=-[p,r]=[r,p]$. ∎

Theorem 3.8. *Vectors $[a,b]$, $[c,d]$ and $[e,f]$ form a triangle if and only if $de\cdot ad=cf\cdot bc$.*

Proof. Let x be any point, and y, z, x' points such that $[a,b]=[x,y]$, $[c,d]=[y,z]$ and $[e,f]=[z,x']$. From $\mathrm{Par}(a,b,y,x)$ and $\mathrm{Par}(d,c,y,z)$ and Theorem 2.5 it follows $\mathrm{Par}(ad,bc,y,xz)$, and from $\mathrm{Par}(c,d,z,y)$ and $\mathrm{Par}(f,e,z,x')$ it follows $\mathrm{Par}(cf,de,z,yx')$. Finally, from $\mathrm{Par}(de,cf,yx',z)$ and $\mathrm{Par}(ad,bc,y,xz)$ we obtain $\mathrm{Par}(de\cdot ad,cf\cdot bc,x',x)$.

Accordingly to Lemma 3.7 the vectors $[a,b]$, $[c,d]$ and $[e,f]$ form a triangle if and only if $[x,y]+[y,z]+[z,x']=0$, i.e. if and only if $x=x'$, which is equivalent to $de\cdot ad=cf\cdot bc$. ∎

From this proof (from $\mathrm{Par}(de\cdot ad,cf\cdot bc,x',x)$) we obtain:

Corollary 3.9. *The sum of the vectors $[a,b]$, $[c,d]$ and $[e,f]$ is a vector $[x,x']$, where*

$$x'=(de\cdot ad)(cf\cdot bc)\cdot x(de\cdot ad).$$

Theorem 3.10. *Let a, b, c be any points, a_1 and c_1 points such that $M(b,a_1,c)$ and $M(a,c_1,b)$ hold, and b_1 the point for which $Par(a_1,b,c_1,b_1)$ hold. Then $M(a,b_1,c)$ and the vectors $[a,a_1]$, $[b,b_1]$ and $[c,c_1]$ form a triangle.*

Proof. From $\mathrm{Par}(a,c_1,b,c_1)$ and $\mathrm{Par}(b,c_1,b_1,a_1)$ it follows that $\mathrm{Par}(a,c_1,a_1,b_1)$, i.e. one has $[a,b_1]=[c_1,a_1]$. From $\mathrm{Par}(c,a_1,b,a_1)$ and $\mathrm{Par}(b,a_1,b_1,c_1)$ it follows $\mathrm{Par}(c,a_1,c_1,b_1)$, i.e. $[c_1,a_1]=[b_1,c]$. Hence $[a,b_1]=[b_1,c]$, i.e. $\mathrm{M}(a,b_1,c)$.

To prove the other part of the statement we need to check that $b_1c\cdot ab_1=bc_1\cdot a_1b$.

From $\mathrm{Par}(a_1,b,c_1,b_1)$ we have $bc_1\cdot a_1b=b_1$, so the righthand side of the upper equation equals b_1.

From $\mathrm{M}(a,b_1,c)$ it follows

$$c=ab_1\cdot b_1a,\qquad ab_1=b_1a\cdot c$$

$$a=cb_1\cdot b_1c,\qquad b_1c=a\cdot cb_1$$

$$b_1c\cdot ab_1=(a\cdot cb_1)(b_1a\cdot c)=(a\cdot b_1a)(cb_1\cdot c)=b_1b_1=b_1,$$

so, the left hand side also equals b_1. ∎

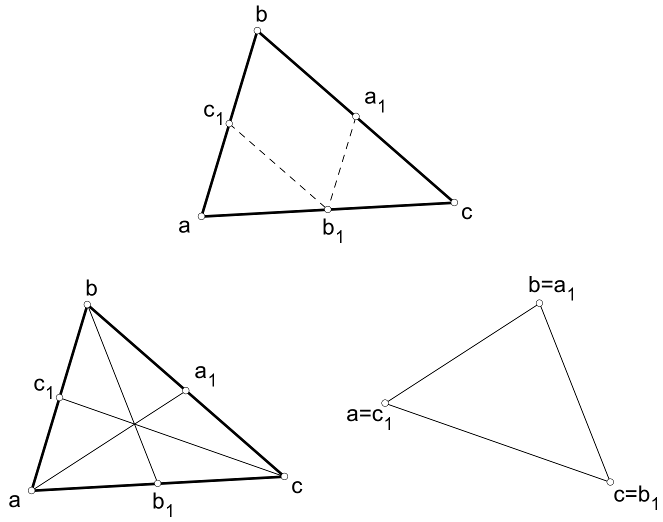

FIGURE 2. Theorem 3.10

REMARK 3.11. In the quasigroup $(\mathbb{C}, *)$ this theorem proves the known fact that medians of any triangle (as vectors) form a triangle. However the statement: "If $\mathrm{M}(b, a_1, c)$, $\mathrm{M}(c, b_1, a)$ and $\mathrm{M}(a, c_1, b)$, then the vectors $[a, a_1]$, $[b, b_1]$ and $[c, c_1]$ form a triangle" is not valid in every hexagonal quasigroup. E.g., in the quasigroup $Q_3 \times Q_4$, for $a = A_1$, $b = B_1$, $c = C_1$, $a_1 = A_2$, $b_1 = B_2$ and $c_1 = C_2$ the assumptions are satisfied, but the sum of the vectors $[A_1, A_2]$, $[B_1, B_2]$ and $[C_1, C_2]$ equals $[A_1, A_2] \neq 0$.

Let us prove one more theorem about parallelograms:

THEOREM 3.12. *From* $Par(a_1, b_1, a_2, c_1)$*,* $Par(a_2, b_2, a_3, c_2)$*,* $Par(a_3, b_3, a_4, c_3)$*,* $Par(a_4, b_4, a_1, c_4)$ *and* $Par(b_1, b_2, b_3, b_4)$ *it follows* $Par(c_1, c_2, c_3, c_4)$*.*

PROOF. From the definition of the vectors, $\mathrm{Par}(a, b, c, d)$ is equivalent to $[a, b] = [d, c]$. Using the properties of vectors from the assumptions we obtain:

$$\begin{aligned} [c_1, c_2] &= [c_1, a_2] + [a_2, c_2] = [a_1, b_1] + [b_2, a_3] \\ &= [a_1, b_4] + [b_4, b_1] + [b_2, b_3] + [b_3, a_3] = [a_1, b_4] + [b_3, a_3] \\ &= [c_4, a_4] + [a_4, c_3] = [c_4, c_3], \end{aligned}$$

and finally $\mathrm{Par}(c_1, c_2, c_3, c_4)$. ∎

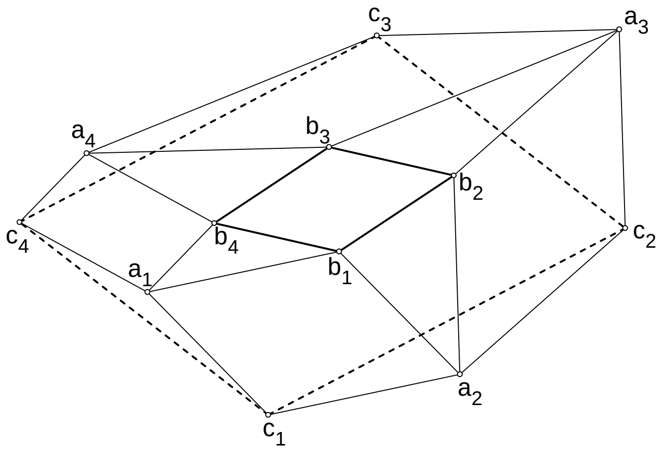

FIGURE 3. Theorem 3.12

4. TRANSFERS

In the article [1] Lettrich and Perenčaj studied functions $L_a(x) = ax$, $R_a(x) = xa$ and $T_{a,b}(x) = L_{ab} \circ R_a(x) = ab \cdot xa$ in a structure called R–structure. In our terminology, the R–structure is a hexagonal quasigroup in which no two different elements commute. The geometric meaning of $T_{a,b}$ in the quasigroup $(\mathbb{C}, *)$ is the transfer by vector $[a, b]$. We shall repeat some results from [1], and prove some new for any hexagonal quasigroup.

DEFINITION 4.1. *The function* $T_{a,b} : Q \to Q$*,*

$$T_{a,b}(x) = ab \cdot xa$$

is called transfer *by the vector* $[a, b]$.

We have immediately

LEMMA 4.2. *For any* $a, b, x \in Q$ *the statement* $Par(x, a, b, T_{a,b}(x))$*, and the equation* $[x, T_{a,b}(x)] = [a, b]$ *are valid.*

THEOREM 4.3. *The following statements are equivalent*

1° $T_{a,b}(x) = T_{c,d}(x)$*, for some* $x \in Q$
2° $T_{a,b} = T_{c,d}$
3° $Par(a, b, d, c)$
4° $[a, b] = [c, d]$.

PROOF. From the definition of vector, 3° $\Leftrightarrow$ 4°. Obviously, from 2° follows 1°. Let us prove 1° $\Rightarrow$ 3° and 3° $\Rightarrow$ 2°.

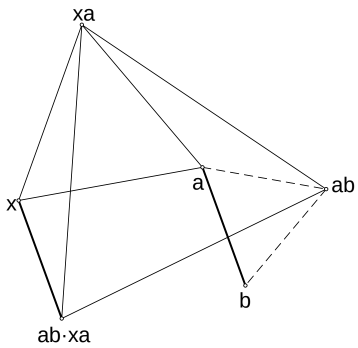

FIGURE 4. Transfer by the vector $[a, b]$

Let 1° hold. Let $y = T_{a,b}(x) = T_{c,d}(x)$. Then $\mathrm{Par}(a, b, y, x)$ and $\mathrm{Par}(c, d, y, x)$ (i.e. $\mathrm{Par}(y, x, c, d)$) and because of the property Par3 it follows $\mathrm{Par}(a, b, c, d)$; i.e. 3°.

Let 3° hold, i.e. let $\mathrm{Par}(a, b, d, c)$ and let $T_{a,b}(x) = y$. Then $\mathrm{Par}(x, a, b, y)$, i.e. $\mathrm{Par}(y, x, a, b)$. It follows $\mathrm{Par}(y, x, c, d)$ and $T_{c,d}(x) = y$. Hence, 2° holds. $\square$

COROLLARY 4.4. *Let T be a transfer such that $T(a) = b$. Then $T = T_{a,b}$.*

PROOF. $T_{a,b}(a) = ab \cdot aa = ab \cdot a = b = T(a)$. Because of the implication 1° $\Rightarrow$ 2° from Theorem 4.3 $T_{a,b} = T$. $\square$

COROLLARY 4.5. *Transfer with a fixed point is the identity.*

PROOF. Let T be a transfer, and x point such that $T(x) = x$. From the corollary 4.4 it follows $T = T_{x,x} =$ identity. $\square$

THEOREM 4.6. *For any points a, b, c, the equation $T_{b,c} \circ T_{a,b} = T_{a,c}$ holds.*

PROOF. Let $x \in Q$, $y = T_{a,b}(x)$, $z = T_{b,c}(y)$. We need to prove $T_{a,c}(x) = z$. Since $\mathrm{Par}(a, b, y, x)$ and $\mathrm{Par}(b, c, z, y)$, it follows $\mathrm{Par}(a, c, z, x)$, i.e. $\mathrm{Par}(x, a, c, z)$, which is, because of Lemma 4.2, equivalent with $z = T_{a,c}(x)$. $\square$

COROLLARY 4.7. *$(T_{a,b})^{-1}$ is $T_{b,a}$.*

PROOF. From Theorem 4.6 we have $T_{a,b} \circ T_{b,a} = T_{a,a}$, which proves the statement because $T_{a,a}$ is the identity. $\square$

THEOREM 4.8. *The set of all transfers $\mathcal{T}$ with composition $\circ$ as binary operation is a commutative group which acts strictly transitively on the set Q.*

Proof. Strict transitivity follows from Corollary 4.4.

Accordingly to the above results, $(\mathcal{T}, \circ)$ is a group, so we need only to prove the commutativity.

Let T_1 and T_2 be transfers. Let o be any point, and let $a = T_1(o)$, $b = T_2(a)$ and $c = T_2(o)$. From the corollary 4.4 we obtain $T_1 = T_{o,a}$ and $T_2 = T_{o,c}$, and also $T_2 = T_{a,b}$. Now from $T_{a,b} = T_{o,c}$ it follows $\mathrm{Par}(a, b, c, o)$ and $T_{o,a} = T_{c,b}$. Finally,

$$\begin{aligned} T_2 \circ T_1 &= T_{a,b} \circ T_{o,a} = T_{o,b} \\ T_1 \circ T_2 &= T_{o,a} \circ T_{o,c} = T_{c,b} \circ T_{o,c} = T_{o,b}. \end{aligned}$$

□

Theorem 4.9. *The groups $(\mathcal{T}, \circ)$ and $(\mathcal{V}, +)$ are isomorphic.*

Proof. Let $\mathcal{F} : \mathcal{V} \to \mathcal{T}$ be a function defined by $\mathcal{F}([a, b]) = T_{a,b}$. Since

$$\mathcal{F}([a, b]) = \mathcal{F}([c, d]) \Leftrightarrow T_{a,b} = T_{c,d} \Leftrightarrow \mathrm{Par}(a, b, d, c) \Leftrightarrow [a, b] = [c, d],$$

we conclude that the function $\mathcal{F}$ is well-defined and injective. It is obviously surjective.

Let o, a, b and c be points such that $[o, a] + [o, b] = [o, c]$. Then $[o, a] = [b, c]$, and therefore:

$$\begin{aligned} \mathcal{F}([o, a] + [o, b]) &= \mathcal{F}([o, c]) = T_{o,c} = T_{b,c} \circ T_{o,b} \\ &= \mathcal{F}([b, c]) \circ \mathcal{F}([o, b]) = \mathcal{F}([o, a]) \circ \mathcal{F}([o, b]). \end{aligned}$$

Hence, $\mathcal{F}$ is an isomorphism. □

Accordingly to Theorem 1.6, for any hexagonal quasigroup $(Q, \cdot)$, and any point $o \in Q$, with $a + b = oa \cdot bo$ the structure $(Q, +)$ is a Abelian group, and its automorphism $\varphi(a) = oa$ satisfies (1.1).

Note that $f(a) = [o, a]$ is an isomorphism between groups $(Q, +)$ and $(\mathcal{V}, +)$. Indeed, f is bijection because of the property Par1, and

$$f(a) + f(b) = [o, a] + [o, b] = [o, a + b] = f(a + b),$$

since the addition in Q is defined so that $\mathrm{Par}(o, a, a + b, b)$.

We have proved:

Theorem 4.10. *Let $(Q, \cdot)$ be a hexagonal quasigroup, and $(Q, +)$ the Abelian group defined as in theorem 1.6, let $(\mathcal{V}, +)$ be the group of vectors, and $(\mathcal{T}, \circ)$ the group of transfers in the quasigroup $(Q, \cdot)$. Then the groups $(\mathcal{V}, +)$, $(\mathcal{T}, \circ)$ and $(Q, +)$ are isomorphic.*

References

[1] J. Lettrich and J. Perenčaj, *Algebraicke studium struktury rovnostrannych trojuholnikov*, Prace a studie VD, Seria matematicko-fyzikalna, čislo **1**, Žilina (1974), 113–120 (in Czech).

[2] V. Volenec, *Geometry of medial quasigroups*, Rad Jugoslav. Akad. Znan. Umjet. **421** (1986), 79-91.

[3] V. Volenec, *Hexagonal quasigroups*, Arch. Math. (Brno) **27a** (1991), 113-122.

[4] V. Volenec, *Regular triangles in hexagonal quasigroups*, Rad Hrvatske Akad. Znan. Umjet. **467** (1994), 85-93.

M. Bombardelli
Department of Mathematics
University of Zagreb
10000 Zagreb
Croatia
E-mail: Mea.Bombardelli@math.hr

V. Volenec
Department of Mathematics
University of Zagreb
10000 Zagreb
Croatia
E-mail: volenec@math.hr

Received: 20.3.2006.
Revised: 4.7.2006.