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# Mann iterative scheme for nonlinear equations

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**Abstract**. *The purpose of this paper is to analyze the Mann iterative scheme with errors for solving nonlinear operator equations in real Banach spaces.*

**Key words:** *Mann iterative scheme with errors, strongly accretive mappings, strongly pseudocontractive mappings, Banach spaces*

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## 1. Introduction

Form now onward, we assume that $E$ is a real Banach space and $K$ a nonempty convex subset of $E$. Let $J$ denote the normalized duality mapping from $E$ to $2^{E^*}$ defined by

$$J(x) = \{f^* \in E^* : \langle x, f^* \rangle = ||x||^2 \text{ and } ||f^*|| = ||x||\},$$

where $E^*$ denotes the dual space of $E$ and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. We shall denote the single-valued duality map by $j$.

**Definition 1.** *A map $T : E \to E$ is called strongly accretive if there exists a constant $0 < k < 1$ such that for each $x, y \in E$ there is a $j(x-y) \in J(x-y)$ satisfying*

$$\langle Tx - Ty, j(x-y) \rangle \geq k||x-y||^2. \tag{1.1}$$

**Definition 2.** *An operator $T$ with domain $D(T)$ and range $R(T)$ in $E$ is called strongly pseudocontractive if for all $x, y \in D(T)$ there exist $j(x-y) \in J(x-y)$ and a constant $0 < k < 1$ such that*

$$\langle Tx - Ty, j(x-y) \rangle \leq (1-k)||x-y||^2. \tag{1.2}$$

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It is known that $T$ is strongly pseudocontractive if and only if $(I-T)$ is strongly accretive.

The concept of accretive mapping was first introduced independently by Browder [1] and Kato [5] in 1967. An early fundamental result in the theory of accretive mapping, due to Browder, states that the initial value problem

$$\frac{du(t)}{dt} + Tu(t) = 0, \qquad u(0) = u_0$$

is solvable if $T$ is locally Lipschitzian and accretive on $E$.

In recent years, much attention has been given to solving nonlinear operator equations in Banach spaces by using two-step and one-step iterative schemes, see [3, 6-7].

The Mann iterative scheme is defined by the sequence $\{x_n\}_{n=0}^{\infty}$ (see [6]):

$$\begin{cases} x_0 \in K, \\ x_{n+1} = (1-b_n)\,x_n + b_n T x_n, n \geq 0 \end{cases} \tag{1.3}$$

where $\{b_n\}$ is a sequence in $[0,1]$.

In 1998, Xu introduced the iterative scheme defined by (see [7]):

$$\begin{cases} x_0 \in K, \\ x_{n+1} = a_n x_n + b_n T x_n + c_n u_n,\ n \geq 0 \end{cases} \tag{1.4}$$

where $\{a_n\}, \{b_n\}, \{c_n\}$ are sequences in $[0,1]$ such that $a_n + b_n + c_n = 1$ and $\{u_n\}$ is a bounded sequence in $K$. Clearly, this iterative scheme contains (1.3) as its special case.

The purpose of this paper is to analyze the Mann iterative scheme with errors (1.4) for solving nonlinear operator equations in real Banach spaces. We also study the convergence analysis of the iterative method.

## 2. Main results

The following two Lemmas are now well known.

**Lemma 1.** *Let $J : E \to 2^E$ be the normalized duality mapping. Then for any $x, y \in E$, we have*

$$||x+y||^2 \leq ||x||^2 + 2\langle y, j(x+y)\rangle, \quad \forall j(x+y) \in J(x+y).$$

**Lemma 2.** *If there exists a positive integer $N$ such that for all $n \geq N, n \in \mathbb{N}$,*

$$\rho_{n+1} \leq (1-\theta_n)\rho_n + b_n,$$

*then*

$$\lim_{n\to\infty} \rho_n = 0,$$

*where* $\theta_n \in [0,1)$, $\sum_{n=0}^{\infty} \theta_n = \infty$, *and* $b_n = o(\theta_n)$.

**Theorem 1.** *Let* $T : E \to E$ *be a uniformly continuous and strongly pseudocontractive mapping with a bounded range. Let* $p$ *be a fixed point of* $T$ *and let the Mann iterative scheme* $\{x_n\}_{n=0}^{\infty}$ *be defined by*

$$x_{n+1} = a_n x_n + b_n T x_n + c_n u_n, \ n \geq 0, \tag{2.1}$$

*where* $\{a_n\}, \{b_n\}, \{c_n\}$ *are sequences in* $[0,1]$ *such that* $a_n + b_n + c_n = 1$ *satisfying the conditions:*

*a)* $\sum_{n=0}^{\infty} b_n = \infty$,

*b)* $c_n = 0(b_n)$,

*c)* $\lim\limits_{n\to\infty} b_n = 0$,

*and* $\{u_n\}$ *is a bounded sequence in* $E$. *Then the sequence* $\{x_n\}_{n=0}^{\infty}$ *converges strongly to the unique fixed point* $p$ *of* $T$.

**Proof.** Since $p$ is a fixed point of $T$, then the set of fixed points $F(T)$ of $T$ is nonempty. We will show that $p$ is the unique fixed point of $T$. Suppose there exists $q \in F(T)$. Then, from the definition of strongly pseudocontractive mapping,

$$||p-q||^2 = \langle p-q, j(p-q)\rangle = \langle Tp - Tq, j(p-q)\rangle \leq (1-k)||p-q||^2.$$

Since $k \in (0,1)$, it follows that $||p-q||^2 \leq 0$, which implies the uniqueness.

Since $T$ has a bounded range, we set

$$M_1 = \|x_0 - p\| + \sup_{n\geq 0} \|Tx_n - p\| + \sup_{n\geq 0} \|u_n - p\|.$$

Obviously $M_1 < \infty$.

It is clear that $||x_0 - p|| \leq M_1$. Let $||x_n - p|| \leq M_1$. Next we will prove that $||x_{n+1} - p|| \leq M_1$.

Consider

$$\begin{aligned}
\|x_{n+1} - p\| &= \|a_n x_n + b_n T x_n + c_n u_n - p\| \\
&= \|a_n(x_n - p) + b_n(Tx_n - p) + c_n(u_n - p)\| \\
&\leq (1-b_n)\|x_n - p\| + b_n \|Tx_n - p\| + c_n \|u_n - p\| \\
&\leq (1-b_n)M_1 + b_n \|Tx_n - p\| + c_n \|u_n - p\| \\
&\leq \|x_0 - p\| + (1-b_n)[\sup_{n\geq 0} \|Tx_n - p\| + \sup_{n\geq 0} \|u_n - p\|] \\
&\quad + b_n \|Tx_n - p\| + b_n \|u_n - p\| \\
&\leq M_1.
\end{aligned}$$

So, from the above discussion, we can conclude that the sequence $\{x_n\}_{n=0}^{\infty}$ is bounded. Let $M_2 = \sup\limits_{n\geq 0}||x_n - p||$.

Denote $M = M_1 + M_2$. Obviously $M < \infty$.

A real valued function $f$ defined on an interval (or on any convex subset $C$ of some vector space) is called **generalized convex** if for any three points $x$, $y$ and $z$ in its domain $C$ and any $a$, $b$, $c$ in $[0,1]$; $a+b+c=1$, we have

$$f(a+b+c) \leq af(x)+bf(y)+cf(z). \tag{2.2}$$

The real function $f:[0,\infty)\to[0,\infty)$, $f(t)=t^2$ is increasing and generalized convex. For all $a$, $b$, $c \in [0,1]$ with $a+b+c=1$ and $t_1$, $t_2$, $t_3>0$, we have

$$(at_1+bt_2+ct_3)^2 \leq at_1^2+bt_2^2+ct_3^2. \tag{2.3}$$

Consider

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|a_nx_n+b_nTx_n+c_nu_n-p\|^2\\ &= \|a_n(x_n-p)+b_n(Tx_n-p)+c_n(u_n-p)\|^2\\ &\leq [(1-b_n)\|x_n-p\|+b_n\|Tx_n-p\|+c_n\|u_n-p\|]^2\\ &\leq (1-b_n)\|x_n-p\|^2+M^2b_n+M^2c_n. \end{aligned}\tag{2.4}$$

Now from *Lemma 1* for all $n\geq 0$, we obtain

$$\begin{aligned}\|x_{n+1}-p\|^2 &= ||a_nx_n+b_nTx_n+c_nu_n-p||^2\\ &= ||a_n(x_n-p)+b_n(Tx_n-p)+c_n(u_n-p)||^2\\ &\leq (1-b_n)^2||x_n-p||^2+2b_n\langle Tx_n-p, j(x_{n+1}-p)\rangle\\ &\quad +2c_n\langle u_n-p, j(x_{n+1}-p)\rangle\\ &\leq (1-b_n)^2||x_n-p||^2+2b_n\langle Tx_{n+1}-p,\ j(x_{n+1}-p)\rangle\\ &\quad +2b_n\langle Tx_n-Tx_{n+1}, j(x_{n+1}-p)\rangle+2M^2c_n\\ &\leq (1-b_n)^2||x_n-p||^2+2b_n(1-k)||x_{n+1}-p||^2\\ &\quad +2b_n\|Tx_n-Tx_{n+1}\|\,\|x_{n+1}-p\|+2M^2c_n\\ &\leq (1-b_n)^2||x_n-p||^2+2b_n(1-k)||x_{n+1}-p||^2+2b_nd_n\\ &\quad +2M^2c_n, \end{aligned}\tag{2.5}$$

where

$$d_n = M\|Tx_n-Tx_{n+1}\|. \tag{2.6}$$

From (2.1) we have

$$\begin{aligned}\|x_n-x_{n+1}\| &= \|b_n(x_n-Tx_n)-c_n(u_n-x_n)\|\\ &\leq b_n\|x_n-Tx_n\|+c_n\|u_n-x_n\|\\ &\leq 2M(b_n+c_n). \end{aligned}\tag{2.7}$$

From the conditions (b-c) and (2.7), we obtain

$$\lim_{n\to\infty}\|x_n-x_{n+1}\|=0,$$

and the uniform continuity of $T$ leads to

$$\lim_{n\to\infty} \|Tx_n - Tx_{n+1}\| = 0,$$

implying

$$\lim_{n\to\infty} d_n = 0. \tag{2.8}$$

Substituting (2.4) in (2.5), and with the help of condition (b) we get

$$\begin{aligned}\|x_{n+1} - p\|^2 \leq & [(1-b_n)^2 + 2(1-k)b_n(1-b_n)]||x_n - p||^2 \\ & + 2b_n[M^2(1-k)(b_n + c_n) + d_n + M^2 t_n]. \end{aligned} \tag{2.9}$$

Now by $\lim\limits_{n\to\infty} b_n = 0$, there exists a natural number $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have $b_n \leq \frac{1}{2}$. From (2.9), we get

$$\begin{aligned}\|x_{n+1} - p\|^2 \leq & (1-k\alpha_n)||x_n - p||^2 \\ & + 2b_n[M^2(1-k)(b_n + c_n) + d_n + M^2 t_n]. \end{aligned} \tag{2.10}$$

Now with the help of (a-c), (2.8) and *Lemma 2*, from (2.10) we obtain that

$$\lim_{n\to\infty} ||x_n - p|| = 0,$$

completing the proof. □

**Theorem 2.** *Let $T : E \to E$ be a Lipschitzian and strongly pseudocontractive mapping with a bounded range. Let $p$ be a fixed point of $T$ and let the Mann iterative scheme $\{x_n\}_{n=0}^{\infty}$ be defined by*

$$x_{n+1} = a_n x_n + b_n T x_n + c_n u_n, \ n \geq 0$$

*where $\{a_n\}, \{b_n\}, \{c_n\}$ are sequences in $[0,1]$ such that $a_n + b_n + c_n = 1$ satisfying the conditions:*

*a)* $\sum_{n=0}^{\infty} b_n = \infty$,

*b)* $c_n = 0(b_n)$,

*c)* $\lim\limits_{n\to\infty} b_n = 0$,

*and $\{u_n\}$ is a bounded sequence in $E$. Then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique fixed point $p$ of $T$.*

**Theorem 3.** *Let $T : E \to E$ be a uniformly continuous and strongly accretive operator. For a given $f \in E$, let $x^*$ denote the unique solution of the equation $Tx = f$. Define the operator $H : E \to E$ by $Hx = f + x - Tx$, and suppose that the range of $H$ is bounded. For any $x_0 \in E$ let $\{x_n\}_{n=0}^{\infty}$ in $E$ be the* Mann *iterative scheme defined by*

$$x_{n+1} = a_n x_n + b_n H x_n + c_n u_n, \ n \geq 0$$

*where $\{a_n\}, \{b_n\}, \{c_n\}$ are sequences in $[0,1]$ such that $a_n + b_n + c_n = 1$ satisfying the conditions:*

*a)* $\sum_{n=0}^{\infty} b_n = \infty$,

*b)* $c_n = 0(b_n)$,

*c)* $\lim\limits_{n\to\infty} b_n = 0$,

*and* $\{u_n\}$ *is a bounded sequence in* $E$. *Then the sequence* $\{x_n\}_{n=0}^{\infty}$ *converges strongly to the unique solution of* $Tx = f$.

**Proof.** Observe that the operator $H$ has bounded range and $x^*$ is a fixed point of $H$. Furthermore, for any $x, y \in E$,

$$\begin{aligned}\langle Hx - Hy, j(x-y)\rangle &= \langle (f + x - Tx) - (f + y - Ty), j(x-y)\rangle \\ &= ||x-y||^2 - \langle Tx - Ty, j(x-y)\rangle \\ &\le (1-k)||x-y||^2,\end{aligned}$$

so that the rest of the argument now follows as in the proof of *Theorem 1.* □

**Theorem 4.** *Let* $T : E \to E$ *be a Lipschitzian and strongly accretive operator. For a given* $f \in E$, *let* $x^*$ *denote the unique solution of the equation* $Tx = f$. *Define the operator* $H : E \to E$ *by* $Hx = f + x - Tx$, *and suppose that the range of* $H$ *is bounded. For any* $x_0 \in E$ *let* $\{x_n\}_{n=0}^{\infty}$ *in* $E$ *be the* Mann *iterative* scheme *defined by*

$$x_{n+1} = a_n x_n + b_n H x_n + c_n u_n,\ n \ge 0$$

*where* $\{a_n\}, \{b_n\}, \{c_n\}$ *are sequences in* $[0,1]$ *such that* $a_n + b_n + c_n = 1$ *satisfying the conditions:*

*a)* $\sum_{n=0}^{\infty} b_n = \infty$,

*b)* $c_n = 0(b_n)$,

*c)* $\lim\limits_{n\to\infty} b_n = 0$,

*and* $\{u_n\}$ *is a bounded sequence in* $E$. *Then the sequence* $\{x_n\}_{n=0}^{\infty}$ *converges strongly to the unique solution of* $Tx = f$.

**Remark 1.** *The results for the Mann iterative scheme (1.3) are now straight forward.*

## References

[1] F. E. Browder, *Nonlinear mappings of nonexpansive and accretive type in Banach spaces*, Bull. Amer. Math. Soc. **73**(1967), 875-882.

[2] F. E. Browder, *Nonlinear operators and nonlinear equations of evolutions in Banach spaces*, Proc. Sympos. Pure Math. **18(2)**, Amer. Math. Soc., Providence, R. I., 1976.

[3] S. Ishikawa, *Fixed points by a new iteration method*, Proc. Amer. Math. Soc. **44**(1974), 147-150.

[4] G. G. Johnson, *Fixed points by mean value iteration*, Proc. Amer. Math. Soc. **34**(1972), 193-194.

[5] T. Kato, *Nonlinear semigroup and evolution equations*, J. Math. Soc. Japan **19**(1967), 508-520.

[6] W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4**(1953), 506-510.

[7] Y. Xu, *Ishikawa and Mann iteration process with errors for nonlinear strongly accretive operator equations*, J. Math. Anal. Appl. **224**(1998), 91-101.