

# Adaptive Measurements in the Optical Quantum Information Laboratory

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(Invited Paper)

**Abstract**—Adaptive techniques make practical many quantum measurements that would otherwise be beyond current laboratory capabilities. For example, they allow discrimination of nonorthogonal states with a probability of error equal to the Helstrom bound, measurement of the phase of a quantum oscillator with accuracy approaching (or in some cases attaining) the Heisenberg limit (HL), and estimation of phase in interferometry with a variance scaling at the HL, using only single qubit measurement and control. Each of these examples has close links with quantum information, in particular, experimental optical quantum information: the first is a basic quantum communication protocol; the second has potential application in linear optical quantum computing; the third uses an adaptive protocol inspired by the quantum phase estimation algorithm. We discuss each of these examples and their implementation in the laboratory, but concentrate upon the last, which was published most recently [Higgins *et al.*, *Nature*, vol. 450, p. 393, 2007].

**Index Terms**—Adaptive, algorithm, computing, estimation, interferometry, measurement, optical, phase, quantum.

## I. INTRODUCTION

MEASUREMENT of a quantum system has conventionally been defined in terms of an observable, which is represented by an Hermitian operator on the system's Hilbert space. However, it is now widely recognized that many realistic measurements should not be described in such terms [1]. Rather, the formalism of generalized measurements, which is described by a set of positive maps, is required. Such generalized measurements are not only necessary for describing realistic detection, but also allow for interesting protocols that con-

ventional (projective) measurements cannot achieve. A simple example is unambiguous (but probabilistic) state discrimination for nonorthogonal states [2], [3].

A powerful way to generate interesting generalized measurements from available detectors in the laboratory is by *adaptive measurement protocols*. By this we mean the following: an incomplete measurement is made on the system, and its result is used to choose the nature of the second measurement made on the system, and so on (until the measurement is complete). A complete measurement is one which leaves the system in a state independent of its initial state, and hence containing no further information of use [1]. A measurement may be incomplete by being a weak (nonprojective) measurement on the system as a whole, or by being a strong (projective) measurement but only on a subsystem, or in other ways. All types allow for adaptive protocols.

Adaptive measurements connect to quantum information, in particular experimental quantum optical information, in a number of ways. We will review three examples in Section II: distinguishing nonorthogonal states, optical phase measurement, and interferometric phase estimation. The first is a basic protocol in quantum communication, the second has potential applications in linear optics quantum computing (LOQC), and the third has been inspired by the quantum computing algorithm theory. All three have been realized in the laboratory in recent years [4]–[6]. The last, most recent, of these is analyzed in detail from a quantum information perspective in the remaining sections of the paper, which cover quantum limits to phase estimation (Section III), the quantum phase estimation algorithm (QPEA) (Section IV) and its generalization (Section V), and whether adaptive measurements are necessary in this context (Section VI).

## II. APPLICATIONS FOR ADAPTIVE MEASUREMENTS

### A. Distinguishing Nonorthogonal States

The idea of using adaptive measurements for discriminating between two nonorthogonal quantum states was introduced as early as 1973 [7]. “Dolinar’s receiver” is an optical technique, which is applicable to a traveling mode prepared in one of two possible coherent states. The object is to discriminate the preparations with minimum probability of error. The minimum possible error probability is known as the Helstrom bound [8], which in this case is  $(1/4)e^{-|\Delta\alpha|^2}$ , where  $\Delta\alpha$  is the difference between the coherent amplitudes of the two states. In this case, the Helstrom measurement could be realized simply

Manuscript received February 26, 2009; revised March 30, 2009. First published November 17, 2009; current version published December 3, 2009. This work was supported by the Australian Research Council (ARC) (CE0348250, FF0458313, DP0986503). The work of H. M. Wiseman was supported in part by the ARC Centre for Quantum Computer Technology and in part by the Centre for Quantum Dynamics, Griffith University. The work of D. W. Berry was supported in part by the ARC Centre for Quantum Computer Technology, in part by the Department of Physics, Macquarie University, and in part by the Institute for Quantum Computing. The work of S. D. Bartlett was supported by the School of Physics, University of Sydney. The works of B. L. Higgins and G. J. Pryde were supported by the Centre for Quantum Dynamics, Griffith University.

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Digital Object Identifier 10.1109/JSTQE.2009.2020810

by measuring a suitable observable on the harmonic oscillator Hilbert space. However, this observable does not correspond to any of the observables usually measured in quantum optics, such as a quadrature, or a displaced photon number operator. Indeed, the obvious scheme of measuring the quadrature  $\hat{X}_\theta$ , with  $\theta = \arg(\Delta\alpha)$ , gives a probability of error of scaling as the square root of the Helstrom bound for large  $|\Delta\alpha|$  [7].

Surprisingly, by using adaptive detection, one can precisely achieve the Helstrom bound [7]. One must measure the leading segment of the pulse, obtain a result, use that result to alter the measurement on the next segment of the pulse, and so on. The Dolinar receiver requires taking the continuous time limit for these segments of pulse, but reasonable results can be obtained as long as the pulse mode has a duration long compared to the delay in the feedback loop [9]. The Dolinar scheme requires measuring a displaced photon number operator (using a weak local oscillator and a photon counter) and altering the displacement whenever a photodetection occurs. Very precise control over the applied displacements and very fast electrooptics are required. For these reasons, the ‘‘Dolinar receiver’’ was not realized experimentally until 2007 by Geremia and coworkers [4]. This experiment clearly showed the improvement over the most obvious nonadaptive technique for mean photon numbers between 0.1 and 1.

The Dolinar receiver is most naturally and simply described as an adaptive scheme based on weak measurements, with the system being one of two coherent states (e.g.,  $|\pm\alpha\rangle$ ) of a single-mode harmonic oscillator, of spatial duration  $L$ . However, because of the unique properties of coherent states, the system can also be thought of as a series of shorter modes of length  $L/M$ , each of which is prepared in the same coherent state  $|\pm\alpha/\sqrt{M}\rangle$ . In the limit  $M \rightarrow \infty$ , each of these short modes corresponds to the system that is measured at a particular time by the detector. Thus, in this guise, the Dolinar receiver appears as an adaptive scheme based on projective measurements of subsystems from an ensemble of identically prepared systems.

One can take this analysis further. In the  $M \rightarrow \infty$  limit, the subsystems can be treated as qubits. The reason is that the mean excitation number for the adaptively displaced states scales as  $M^{-1}$ , and thus, these states have support almost entirely on the first two number states  $|0\rangle$  and  $|1\rangle$ , which are the eigenstates of the measured quantity (photon number).<sup>1</sup> If the two undisplaced global coherent states are  $|\pm\alpha\rangle$ , then to leading order the corresponding qubit states are  $|0\rangle \pm (\alpha/\sqrt{M})|1\rangle$ . The displacement in phase space (by a distance of order  $M^{-1/2}$ ) is equivalent to a qubit rotation (by an angle of order  $M^{-1/2}$ ), which makes a measurement in the  $\{|0\rangle, |1\rangle\}$  basis the desired adaptive measurement. Moreover, the Dolinar scheme can then be derived as a special case of the optimal adaptive scheme for discriminating

between two nonorthogonal qubit states, when one has multiple copies of them.

This latter problem, with arbitrary pure qubit states, was solved by Acin *et al.* [10], who gave a simple interpretation of the optimal scheme: it corresponds to making the optimal local Helstrom measurement on each copy, taking into account the probability one assigns to each of the two possible preparation procedures, updated according to all the preceding measurements. This conceptually simple procedure is not only the optimal adaptive scheme, but also reaches the Helstrom bound for the entire ensemble (as was known already for the special limit of the Dolinar protocol), i.e., contrary to what one might have thought, an entangling measurement across all subsystems is not required. Also note that the restriction to qubits in this quantum information setting is no real restriction, if (unlike in the optical case) one assumes one can make arbitrary projective measurements on a single subsystem, then one needs only two basis states to describe a system in one of two possible pure states.

The situation with mixed states is, however, much more complicated [11]. For qubits (which is now a meaningful restriction), even in the asymptotic limit, there is a clear separation in performance between

- 1) the Helstrom bound (achieved by the optimal joint measurement);
- 2) the optimal adaptive scheme (involving dynamic programming);
- 3) the locally optimal adaptive scheme (i.e., the locally optimal Helstrom measurement);
- 4) the obvious nonadaptive scheme (a majority vote from repeated unbiased measurements).

Moreover, surprisingly, the latter two actually reverse order (i.e., the locally optimal scheme performs worse than the nonadaptive scheme) for a sufficient degree of mixture.

Before leaving the topic of adaptive measurements for state discrimination, we should mention the work of Jacobs [12] on adaptive continuous measurements on a single qubit prepared in one of two nonorthogonal states  $|\phi_\pm\rangle$ . The continuous measurement limit can be thought of (as in the Dolinar case) as a sequence of weak measurements of duration  $\Delta t$ , whose disturbance also scales as  $\Delta t$ , and taking the limit  $\Delta t \rightarrow dt$ . In this case, if one measures for an infinitely long time, then one can realize the Helstrom bound for distinguishability simply by making the same weak measurements at every step of the Pauli operator proportional to  $|\phi_+\rangle\langle\phi_+| - |\phi_-\rangle\langle\phi_-|$ . However, for some finite times, more information can be obtained by using a locally optimized adaptive measurement in which the measured Pauli operator is continually rotated such that its expected value is zero at all times [12]. At sufficiently long times, this scheme always becomes worse than the simple nonadaptive scheme. It is also worth noting that if one considers the probability of error in one’s guess of which preparation was performed, rather than the mutual information between one’s record and the preparation, then the simple nonadaptive scheme always performs better [13].

<sup>1</sup>There is a technical issue in that, according to Dolinar’s protocol, the displacement diverges at the initial time if the two states are initially equally likely. However, as demonstrated in [4], a modified protocol with only a moderately large displacement obtains the great majority of the improvement offered by Dolinar’s protocol.

## B. Optical Phase Measurement

Although the Dolinar receiver was the first notable adaptive measurement scheme to be introduced theoretically, it was not the first to be realized experimentally. This honor goes to the adaptive phase measurement algorithm introduced by Wiseman and Killip [14], [15], and realized by Mabuchi and coworkers [5]. This technique is based on homodyne detection rather than photodetection, as in the Dolinar receiver. Its aim is not state discrimination (although it can be useful for that [15]), but rather measuring a physical quantity: the phase of the state. The optimal measurement to do this, i.e., a canonical phase measurement, would give a measurement result  $\phi$  with variance equal to the intrinsic phase variance of the state by definition [16]. However, this cannot be realized by standard optical measurements. For a state with initially completely unknown phase, the best standard technique is heterodyne detection [16]. This introduces an excess phase variance scaling as  $1/4\bar{n}$ , where  $\bar{n}$  is the mean number of photons in the state [14]. This scaling is known as the *standard quantum limit* (SQL).

If one knew that the phase of the state was approximately  $\varphi$ , one could make a homodyne measurement of the quadrature  $\hat{X}_\theta$  with  $\theta = \varphi + \pi/2$ , and this would be almost as good as a canonical phase measurement for many types of states.<sup>2</sup> The intuitive idea of an adaptive homodyne measurement is to begin with homodyne measurement of a random quadrature, and then to adjust the local oscillator phase  $\theta$  adaptively over time to  $\varphi(t) + \pi/2$ . Here,  $\varphi(t)$  is an estimate of the system phase based on the homodyne data so far. Interestingly, one does not want to choose  $\varphi(t)$  to be the *best* estimate of  $\phi$  at time  $t$ —that actually gives inferior performance for some states [18]. At the end of the measurement, one does want to choose the final estimate  $\phi$  to be the best estimate. For the “Mark II” scheme of Wiseman and Killip [14], the excess phase variance scales as  $1/8\bar{n}^{3/2}$ , which is far smaller than that of the best nonadaptive measurement. More complicated adaptive schemes can do even better [18], near the ultimate (Heisenberg) limit scaling of  $1/\bar{n}^2$ .

The Mark II scheme was experimentally implemented using small, coherently excited, microwave frequency sidebands of a large coherent beam [5]. In this case, the phase to be estimated was actually the phase of the microwave excitation, as carried by the optical frequency sidebands. An improvement over the best nonadaptive scheme was seen for mean photon numbers  $\bar{n}$  between about 10 and 300. Because the experiment was performed with coherent states, it is possible to think of it (like the Dolinar case) as a series of projective (quadrature) measurements performed on identically prepared weak coherent states. However, this would not be the case if the experiment was performed with a nonclassical state such as a squeezed state. In this case, it would be necessary to adopt the more natural description of a succession of weak measurements on a single mode.

It is impossible for an adaptive homodyne measurement to attain the accuracy of a canonical phase measurement in general [15]. However, there is a known exception: when the state has

support on the  $|0\rangle$  and  $|1\rangle$  photon number states. This case was actually solved in the paper that first proposed and analyzed adaptive phase measurement [17], and is the “Mark I” scheme of [14] and [15]. It has the interesting consequence that, given a single photon, it is possible to create deterministically an arbitrary superposition of the states  $|0\rangle$  and  $|1\rangle$ . This is done by first creating the “mode-entangled” [19] single-photon state  $\sqrt{\eta}|0, 1\rangle + \sqrt{1-\eta}|1, 0\rangle$ , and then, making a canonical phase measurement on one mode using the Mark I scheme. This yields a completely random result  $\phi$  (which emphasizes that in this case, there is no approximate initial phase that a nonadaptive homodyne scheme could use), and collapses the second mode into the state  $\sqrt{\eta}e^{i\phi}|0\rangle + \sqrt{1-\eta}|1\rangle$ . Since  $\phi$  is known, the phase of this superposition can then be adjusted to the desired phase.

The ability to create arbitrary superpositions of  $|0\rangle$  and  $|1\rangle$  photon states is, of course, the ability to create arbitrary photonic qubit states using “single-rail” logic [20]. It turns out that adaptive phase measurements can also be applied at other points in LOQC to make it possible to use single-rail encoding with resources that are far smaller than was previously thought possible, and not substantially larger than the resources required for conventional “dual-rail” encoding [21].

It is worth noting here that even a “conventional” linear optics quantum computation can be regarded as an enormous adaptive measurement protocol, since it involves making measurements on subsystems (single photons) of an entangled state, and manipulating the remainder of the photons in a way that depends upon the results of the prior measurements, before a final measurement that reveals the result of the computation. This is true for both the circuit architecture and the cluster-state architecture for LOQC [20]. Indeed, the 2007 four-photon cluster-state experiment from the group of Zeilinger [22] is perhaps the first adaptive measurement in which the subsystems were entangled prior to the measurement.

## C. Interferometric Phase Estimation

While adaptive optical phase measurement is potentially useful for quantum computing, the third and final applications we consider are linked to quantum computation in the opposite way, i.e., it uses an algorithm from quantum computing theory to inspire new adaptive protocols. Specifically, based on the QPEA of Cleve *et al.* [23], [24], we have devised a new family of adaptive protocols for interferometric phase estimation [6]. Moreover, we have implemented one of these algorithms, thus demonstrating Heisenberg-limited scaling for phase estimation for the first time [6].

The bulk of this paper is dedicated to explaining adaptive (and nonadaptive) measurements for interferometric phase estimation from a quantum information perspective. In this section, we will concentrate on placing it in the context of optical interferometry (although the idea could work equally well with particles—such as neutrons [25]—other than photons). A conceptual experimental diagram is shown in Fig. 1. The key differences from a standard Mach–Zehnder interferometer are:

<sup>2</sup>The  $\pi/2$  phase difference between  $\theta$  and  $\varphi$  here follows from a convention regarding the phase introduced by a beam splitter, which is generally used in papers on this topic [5], [14], [15], [17], [18].

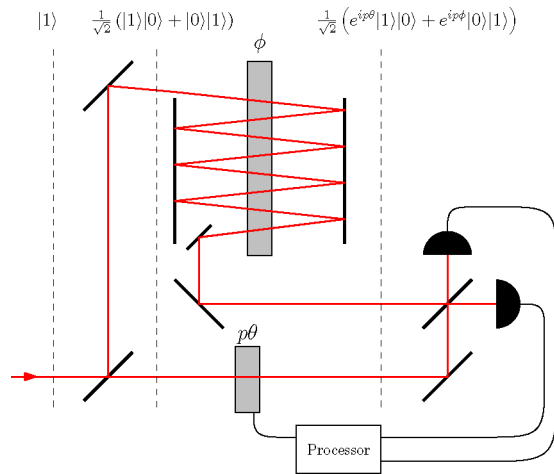


Fig. 1. Conceptual diagram of the generalized QPEA implemented using a Mach-Zehnder interferometer, shown with quantum states (expressed in the photon number basis) at key points. The large phase-shift element is configured to implement an adjustable number  $p$  of  $\phi$  phase shifts on photons passing through the upper arm (in this example,  $p = 8$ ). The small phase-shift element implements an adjustable  $p\theta$  phase shift on photons passing through the lower arm. The output of the single-photon detectors determines, via the processor, how to adjust  $\theta$  prior to the next photon input, and also the final phase estimate  $\phi_{\text{est}}$ .

- 1) the number of passes of one arm through the unknown phase shift can be greater than one, and is assumed controllable over the course of the experiment;
- 2) the phase shift in the other arm is assumed controllable over the course of the experiment.

In standard interferometry,  $N$  independent photon detections allow the unknown phase to be estimated with accuracy  $\Delta\phi = 1/\sqrt{N}$  (for large  $N$ ), which is known as the SQL. By contrast, the Heisenberg limit (HL) is quadratically better:  $\Delta\phi = \pi/N$ .

Those familiar with the QPEA may be surprised to find that applying a generalization of this algorithm to the Mach-Zehnder interferometer (as will be explored in later sections) yields only a quadratic improvement in the accuracy. The QPEA is supposed to give a binary phase readout, implying an uncertainty exponentially small in the size of the register. Also, it is at the heart of Shor's algorithm [26] that gives an exponential speedup over classical algorithms for factoring. In quantum computing theory, the QPEA is used to estimate the phase of an eigenvalue  $e^{i\phi}$  of a (typically multiqubit) unitary operator  $U$  that corresponds to some calculation. If one can do a quantum computation implementing  $U$ , then one can do (more or less) with the same resource cost a quantum computation implementing  $U^p$ , for any  $p$ . Thus, in a quantum algorithm context, the number of "passes"  $p$  is irrelevant.

If we were to follow an analogous method of resource counting in interferometry—count simply the number of *photons* irrespective of the number of times  $p$  each photon passes through the unknown phase shift—then the algorithm implemented in [6] would have yielded a  $\Delta\phi$  exponentially small in  $N$ . This violates the long-established HL scaling [8], [27] with  $\Delta\phi$  of order  $1/N$ . The (correct) Heisenberg scaling is obtained by not counting the photons, but rather the total number of *photon passes* through

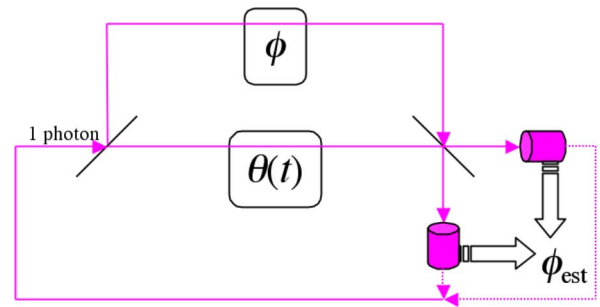


Fig. 2. Arbitrarily accurate phase estimation may be done with a single photon if quantum nondemolition measurements are allowed. After detection, the photon (wherever it is found) is redirected back into the interferometer.

the unknown phase shift [28], i.e., one should count each photon involving a  $p$ -pass interferometer (as shown in Fig. 1) as using  $p$  resources, and not 1.

This method of counting is not justified merely by giving the expected HL. Rather, it has a number of other justifications. First, photon number is not a sensible resource. For example, if one allows for nondemolition photon number measurements, one could repeat a standard interferometry experiment arbitrarily many times, thus obtaining an arbitrarily small uncertainty, using just a single photon, as shown in Fig. 2. Other nonlinear optical processes allow for scenarios where it is not even clear how to define the number of photons [29]. Second, one practical reason for caring about photon number is if one has a sample that is extremely sensitive to light. In this situation, what is relevant is clearly the number of photon passes through the sample, not anything else. Third, if one counted photons rather than photon passes, this would ignore the extra time it takes for a photon to make  $p$  passes, rather than 1 pass, through the sample. In the asymptotic regime of arbitrarily large  $N$  (and hence large  $p$ ), this is necessarily the time that will determine the duration of the experiment.

It might be thought that the technique of using  $p$  passes of a single photon is not really a measurement of  $\phi$  at all, but rather a measurement of  $p\phi$ . From this, one would, of course, expect a sensitivity in  $\phi$  that scales as  $p$  from a single photon measurement, and so, an overall uncertainty in  $\phi$  scaling inversely with the total number of photon passes is just as expected. This works only if one already knows the phase approximately, and the deviations one is trying to detect are much smaller than  $2\pi/p$ . This is quite different from the fundamental problem of estimating a completely unknown phase  $\phi$ , which is the task to which the HL pertains [8], [27]. In the latter case, one cannot simply set  $p$  to be as large as it can be, and then measure  $p\phi$ , because this would yield information only about  $\phi$  modulo  $2\pi/p$ . Rather, even if one sets  $p$  to high values at some stage of the experiment, at some other stage(s), it must be set to one in order to pin down a single value for  $\phi$  within the range  $[0, 2\pi)$ . That is to say, for the task we are interested in, the measured quantity is the variable  $\phi$  modulo  $2\pi$ , not the variable  $p\phi$  modulo  $2\pi$ . The optimal way to vary  $p$  over the course of the experiment is then the crucial issue and the QPEA suggests an answer (or at least a starting point [6]).

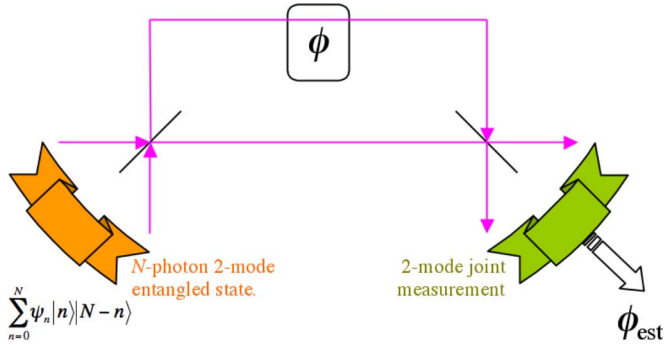


Fig. 3. Single-pass Mach-Zehnder interferometer with arbitrary inputs and arbitrary measurements. It is known that this can achieve the HL  $\Delta\phi \sim \pi/N$  for an  $N$ -photon input state.

In many practical examples, the strategy of fixing  $p$  at a value as large as possible works because one is trying to detect small changes to an already known phase. This is the case in gravitational wave interferometry and atomic clocks. In the former case, the maximum number of passes is set by the geometry of the experiment and the quality of the mirrors, and this determines the accuracy [30]. In the latter case, the decoherence time of the atoms determines the maximum number of “passes” (Rabi cycles) the atoms can undergo, and the SQL is determined by this and the number of atoms in the sample [31]. In both of these cases, one is interested in minimizing the error, and hence, one would ideally measure  $p\phi$  with a large ensemble  $M$  of identical systems (photons or atoms). Thus, the overall scaling for the uncertainty would be like  $M^{-1/2}p^{-1}$ , i.e., one does not get scaling at the HL if one fixes  $p$  and sets  $N = Mp$ . In practice, one cannot even increase  $M$  without limit in these experiments. For gravitational wave interferometry, there is an optimal number of photons for which the noises from photon counting and from radiation pressure balance [30]. For atomic clocks, increasing the cloud density similarly leads to collisional energy shifts (i.e., phase noise). In summary, in both of these cases, the limits are set by practical, rather than fundamental, considerations.

It should be noted that the earlier remarks relating to measuring  $\phi$  modulo  $2\pi/p$  apply to NOON states with  $N = p$  in *exactly* the same way as they do to a single photon with  $p$  passes. A NOON state is a state of the form  $|N, 0\rangle + |0, N\rangle$  (expressed using the number basis for the two arms of the interferometer shown in Fig. 3 [32]). For large  $p$ , an  $N = p$  NOON state has an advantage over a single photon with  $p$  passes in that the time it takes to pass through the interferometer does not scale with  $p$ . It has the obvious disadvantage of being far more difficult to produce—so thus,  $N = 4$  is the largest NOON state demonstrated (and that only in a postselected sense) [33]. It is also far more difficult to detect—it requires photon-number-resolving detectors with loss smaller than  $O(1/N)$ . If it were possible to generate and detect “high NOON” states, then the algorithms here could be applied directly to that case as well. The equivalence comes from the fact that all of the phase information in a NOON state after it has passed through the final beam splitter is contained in the parity of the photon number at one detector (which is why absurdly high efficiencies are required). Note that

although the multipass technique can tolerate detector losses of order unity, it is sensitive to absorption within the sample in *precisely* the same way as is the NOON state technique.

From the aforementioned comparison between NOON states and multiple passes of a single photon, one might argue that there are really two resources:  $N$  (the number of photon passes in both cases) and  $T$  (the time taken for the experiment). In the limit  $N \rightarrow \infty$ , the duration  $T$  for the NOON-state protocol increases only logarithmically with  $N$  (as explained in the following sections), and for an optimal multiphoton entangled state (see Section III-C),  $T$  need not scale with  $N$  at all. In the multipass case, by contrast,  $T$  necessarily scales as  $N$ , as discussed before. In saying that our technique attains the HL, we mean only that it reaches the minimum possible phase variance for a given  $N$ , i.e., to obtain a phase estimate with a binary expansion of  $\log_2(N)$  bits, with an error in only the least significant bit, we are prepared to accept an experiment of exponentially (in  $\log_2 N$ ) long duration. This can be avoided only by moving the exponential cost associated with  $N$  from the time to the state itself—the NOON state (and its relatives) contain *exponentially many photons*, and hence contain an exponentially large amount of energy. In practice, photons are fast and small, so  $N$  would have to become very large for either time or energy to become important.<sup>3</sup> On the other hand, the NOON-state protocol also requires an *exponentially efficient detector*. As noted before, one requires  $1 - \eta$  to be exponentially (in  $\log_2 N$ ) small, in contrast to the best current detectors where  $1 - \eta$  barely qualifies as “small” at all. Thus, it is hard to see why the resource  $T$  should be regarded as more fundamental than these other resources.

Finally, it might be thought that a multipass interferometer is somehow changing the rules of the game, while using a NOON state does not reflect such changes, i.e., one might argue that the rules allow arbitrary preparation as inputs to the interferometer, and arbitrary processing of the output, as shown in Fig. 3, but not changing of the beam paths through the sample. However, it is simple to bypass this objection. If arbitrary preparations and measurements are allowed, then one allowed scheme is not to measure the output modes, but rather simply to redirect them back as the input modes, as shown in Fig. 4. The input beam splitter simply cancels the effect of the output beam splitter, and the second passage of the photon through the unknown phase shift is exactly as in a two-pass interferometer. This can be repeated as many times as desired to give a multipass interferometer.

We implemented a generalized QPEA in [6] using multipassed single photons with a common-spatial-mode polarization interferometer, which is more stable than a Mach-Zehnder interferometer. The two arms of the interferometer were the right-circular and left-circular polarization modes, and the unknown phase  $\phi$  was implemented as the orientation angle of a birefringent half-wave plate. The number of passes was varied from 32 to 1, in powers of 2, using electromechanical devices. We implemented standard interferometry (verifying the SQL), the QPEA (surprisingly also scaling as the SQL, as will be discussed later), and a generalized QPEA in which  $M = 6$  photons

<sup>3</sup>This, of course, ignores the *difficulty* of converting  $N\hbar\omega$  of energy into a NOON state, which is clearly enormous for large  $N$ , but hard to quantify.

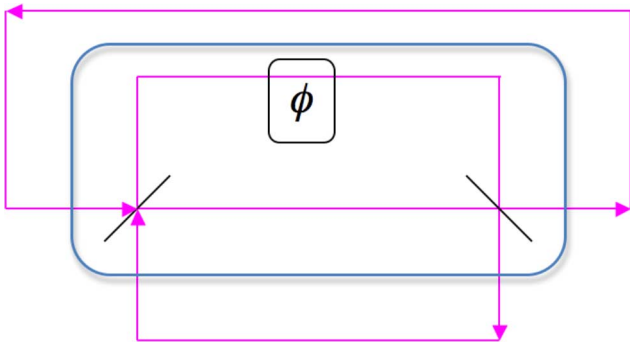


Fig. 4. Realizing a multipass Mach-Zehnder interferometer from a single-pass one (inside the round-cornered box) by recycling the outputs into the inputs. Note that this is done coherently unlike in Fig. 2.

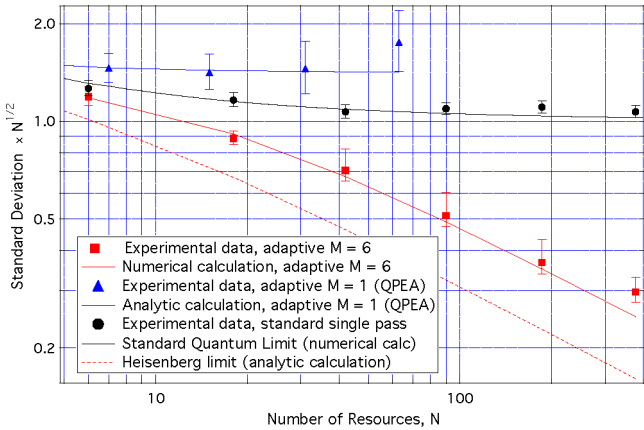


Fig. 5. Experimental results from [6] of standard deviations of phase estimates for varying numbers of resources  $N$ . We compare theoretical predictions (lines) and measured values (points, each representing 1000 estimates) for standard phase estimation, the QPEA ( $M = 1$ ), and our generalized QPEA ( $M = 6$ ) algorithms. Error bars denote 95% confidence intervals. Our algorithm clearly gives better phase estimates than both the SQL and the QPEA limit.

(rather than  $M = 1$  as in the QPEA) were sent down consecutively for each pass configuration. This last technique gave a  $\Delta\phi$  scaling the same as the HL, with a multiplicative overhead of only 1.56, as predicted by theory (see Fig. 5).

### III. QUANTUM LIMITS TO PHASE ESTIMATION

#### A. Rules and Representations

In this and following sections, we will analyze phase estimation algorithms (adaptive and otherwise) from a purely quantum information perspective, i.e., we consider general qubits rather than photons. In this context, the rules of the game are as follows.

- 1) We have a gate that performs the unitary operation  $\exp(i\phi |1\rangle\langle 1|)$  on a specific sort of qubit.
- 2) We have an indefinite supply of these qubits.
- 3) The parameter  $\phi$  is initially *completely unknown*.
- 4) We are allowed only  $N$  applications of the gate.
- 5) We aim to minimize the variance in our best estimate  $\phi_{\text{est}}$  of  $\phi$ .

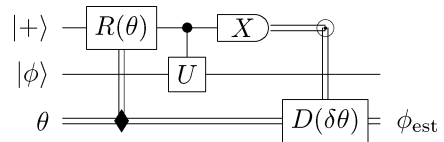
Technically, we use the Holevo variance measure  $V_H = \langle \exp[i(\phi - \phi_{\text{est}})] \rangle^{-2} - 1$  [14], [34], as this respects the cyclic

nature of phase. The variance is the most robust figure of merit, in that if it scales well, then all other measures will also scale well, but not *vice versa* [35].

In quantum information language, a photon entering one port of the interferometer is represented by preparation of the qubit in (say) the logical  $|0\rangle$  state. The initial beam splitter acts as a Hadamard transform  $H$ , yielding  $H|0\rangle = (|0\rangle + |1\rangle)/\sqrt{2} = |+\rangle$ . The unknown phase shift in the upper arm is represented by the unitary operator  $\exp(i\phi |1\rangle\langle 1|)$ , while the known phase shift in the lower arm is represented by  $\exp(i\theta |0\rangle\langle 0|)$ . The final beam splitter again transforms from the logical ( $Z$ ) basis to the  $|\pm\rangle$  basis (the  $X$  basis). Thus, if the photonic qubit is measured, then this amounts to a measurement of  $X$  prior to the final beam splitter. Estimating the phase on the basis of passing a single photon once through the interferometer, and measuring it, is therefore described by the circuit

$$|0\rangle \rightarrow H \rightarrow e^{i\theta|0\rangle\langle 0|} \rightarrow e^{i\phi|1\rangle\langle 1|} \rightarrow X$$

For compact notation in later circuit diagrams, and also to connect more closely to the QPEA of Shor’s algorithm, we change to a representation where  $\exp(i\phi |1\rangle\langle 1|)$  is represented by the controlled unitary gate  $|1\rangle\langle 1| \otimes U + |0\rangle\langle 0| \otimes I$ , where  $U$  acts on a state  $|\phi\rangle$  (which could have any Hilbert space dimension) with  $U|\phi\rangle = e^{i\phi}|\phi\rangle$ , and  $I$  is the identity operator on this space. We also treat the auxiliary phase  $\theta$  as a real-number-valued classical register, which controls (indicated by a  $\blacklozenge$  symbol) the gate  $R(\theta) \equiv \exp(i\theta |0\rangle\langle 0|)$ . Thus, we rewrite the aforementioned circuit as



Here, if we take  $\theta$  to be random (but known), this ensures an estimate  $\phi_{\text{est}}$  with an accuracy which is independent of the true value  $\phi$ . Here,  $\phi_{\text{est}}$  is shifted (indicated by the gate  $D$ ) from  $\theta$  by an amount  $\delta\theta$  depending in some way (indicated by  $\odot$ ) on the classical result. The advantage of treating  $\theta$  in this way will become apparent.

#### B. Standard Quantum Limit

The SQL pertains when we simply repeat the aforementioned single-qubit circuit  $N$  times, i.e., we have  $N$  qubits, independently prepared, independently measured, and with  $\exp(i\phi |1\rangle\langle 1|)$  applied once on each ( $p = 1$ ). To ensure uniform sampling,  $\theta_{\text{init}}$  is random, and  $\theta$  is incremented by  $\pi/N$  between one qubit and the next. The case  $N = 4$  is shown in Fig. 6. This yields the SQL for accuracy, which is given by

$$\text{SQL} = V[\phi_{\text{est}}] \sim \frac{1}{N}, \quad \text{for } N \gg 1. \quad (1)$$

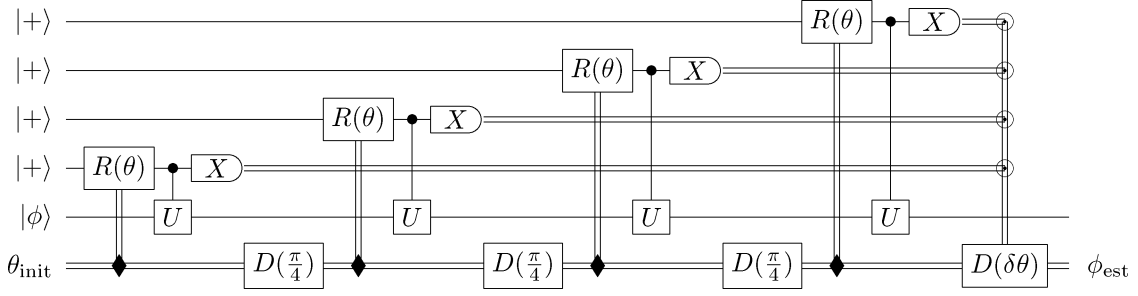


Fig. 6. Circuit representation of “standard” interferometry with a controllable auxiliary phase. This defines the SQL. Here,  $N = 4$ .

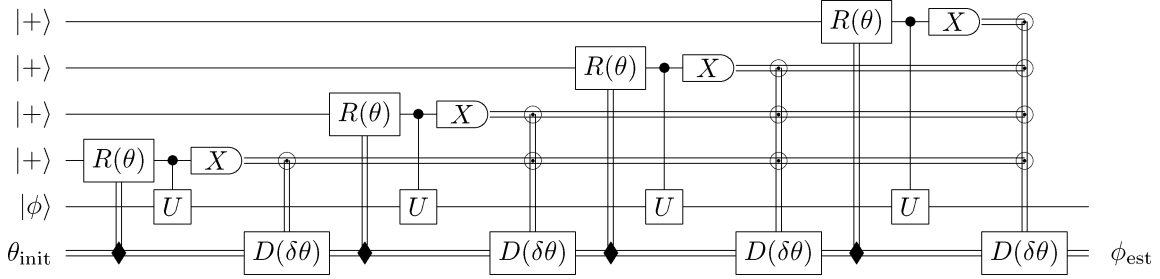
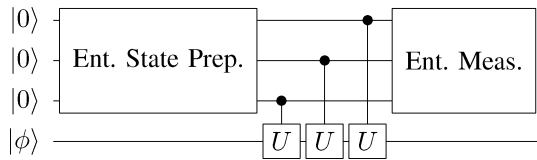


Fig. 7. Circuit representation of “standard” interferometry with an adaptively controlled auxiliary phase. The adaptation does not improve the asymptotic accuracy beyond the SQL.

We can also allow for  $\theta$  to be controlled adaptively between one qubit and the next, as shown in Fig. 7. The obvious procedure, suggested in [36], is to choose  $\theta$  so as to minimize the *expected* variance after the measurement whose result is influenced by this new  $\theta$ , which entails averaging over the two possible results of the measurement. This local optimization gives a slightly more accurate measurement for small  $N$ , but makes no difference asymptotically [35]. A global optimization does even better for small  $N$ , but again essentially no difference for large  $N$  [35].

### C. Heisenberg Limit

With no restrictions on how the qubits are prepared or measured, an obvious approach is to use  $N$  qubits, prepared in a suitable entangled state, and measured by a suitable entangling measurement. In this case, the initial and final beam splitters (Hadamards) can be absorbed into the state preparation and measurement, and so are irrelevant to the problem. So too is the auxiliary phase. Each qubit controls the phase gate once. For example, with  $N = 3$ , we have



Although the phase gates are shown as acting sequentially, they can be imagined to act simultaneously. This circuit is equivalent to Fig. 3, where arbitrary preparation and measurement allow the interferometer to achieve its ultimate performance,

the HL, which is given by [36]

$$\text{HL} = V[\phi_{\text{est}}] = \tan^2 \left( \frac{\pi}{N+2} \right). \quad (2)$$

This is attained by using the canonical phase measurement, described by projection onto phase states

$$|\Phi\rangle \propto \sum_{n=0}^N e^{i\Phi n} |n, N-n\rangle_S \quad (3)$$

and the optimal input state [36], [37]

$$|\psi_{\text{opt}}\rangle \propto \sum_{n=0}^N \sin \left[ \frac{(n+1)\pi}{N+2} \right] |n, N-n\rangle_S. \quad (4)$$

Here  $|n, N-n\rangle_S$  is a symmetrized state in which  $n$  qubits are in state  $|1\rangle$  and  $N-n$  in state  $|0\rangle$ . (For identical and indistinguishable bosons such as photons, in two modes, this symmetry is enforced by the quantum statistics.) Note that this entangled state differs from the NOON state, which has the form  $|N, 0\rangle_S + |0, N\rangle_S$  in this notation. The NOON state gives the maximum Fisher information (equal to  $N$ ) from a single measurement, but has an appalling variance (the Holevo variance is, in fact, infinite) because it only detects changes in  $\phi$  modulo  $2\pi/N$ . The optimal state  $|\psi_{\text{opt}}\rangle$  has only a moderately smaller Fisher information—about  $0.36 N$  [35]—and the minimum variance.

In the asymptotic limit

$$\text{HL} = V[\phi_{\text{est}}] \sim \left( \frac{\pi}{N} \right)^2, \quad \text{for } N \gg 1 \quad (5)$$

which is quadratically better than the SQL. The HL scaling implies that to obtain  $K+1$  bits of precision for  $\phi_{\text{est}}$ , we require of order  $N = 2^{K+1}$  qubits, i.e., it is exponentially costly in

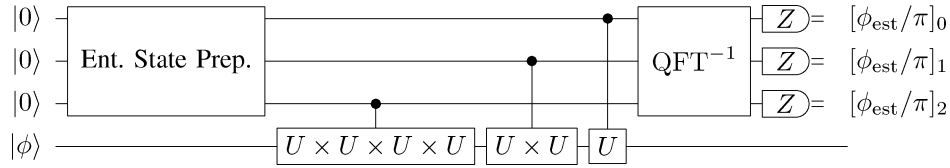


Fig. 8. Circuit representation of Heisenberg-limited interferometry using a binary encoding. Here, the resource count is  $N = 4 + 2 + 1 = 7$ .

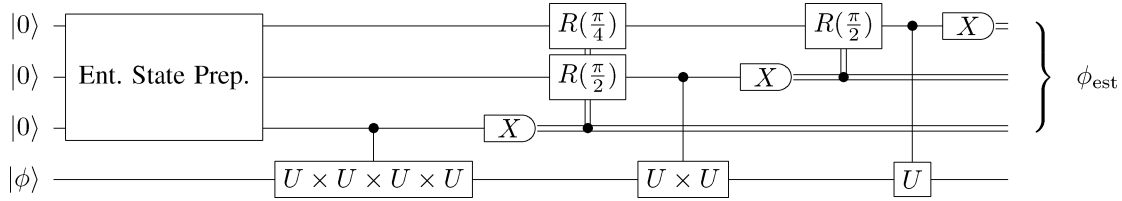


Fig. 9. Circuit representation of Heisenberg-limited interferometry, as in Fig. 8, but using the adaptive measurement scheme of Griffiths and Niu.

“spatial” resources.<sup>4</sup> However, this is not necessary. The fact that the optimal state and measurement can be written using a symmetrized basis with only  $N + 1$  basis states (out of a total Hilbert space dimension of  $2^N$  for  $N$  qubits) allows an alternate representation. Assuming  $N = 2^{K+1} - 1$ , we can use just  $K + 1$  qubits, and define a new  $|n\rangle$  as the logical state of a register of qubits with  $n$  being a binary string of length  $K + 1$ . The price to be paid is that the evolution generated by the phase gates

$$\sum_{n=0}^N \sin \left[ \frac{(n+1)\pi}{N+2} \right] |n\rangle \rightarrow \sum_{n=0}^N e^{in\phi} \sin \left[ \frac{(n+1)\pi}{N+2} \right] |n\rangle \quad (6)$$

now requires an exponential number of phase shifts on the “most significant qubits” in the binary representation of  $n$ , i.e., we have swapped an exponential cost in spatial resources for an exponential cost in time resources, if each phase gate is assumed to take a fixed time.

The canonical phase measurement, described by the phase states (3), is a measurement in a basis conjugate to the logical basis. The transformation from one basis to the other is exactly what is achieved by the quantum Fourier transform [26]. Thus, using the binary representation rather than the symmetric representation, the HL can be achieved by the circuit shown in Fig. 8. In this instance, there are  $K + 1 = 3$  qubits, but  $N = 2^{K+1} - 1 = 7$ . The estimate  $\phi_{\text{est}}$  is read out from the results of  $Z$  measurements as shown, and we are using the notation  $r = [r]_0[r]_1[r]_2 \dots$ . The  $k$ th qubit ( $k = 0, 1, \dots, K$ ) “passes” the phase gate  $2^k$  times. As stressed before, even though we represent (for instance) four applications of the phase gate by a single controlled  $U^4$  gate, this must be regarded as using four resources.

It is a remarkable fact, first pointed out by Griffiths and Niu [38], that the  $\text{QFT}^{-1}$  can be achieved by local (i.e., single-qubit) measurement and control. This can be seen by moving the measurements back through the  $\text{QFT}^{-1}$  and the controlled phase gates, using the gate commutation properties. The control is often called feedforward, but since each controlled qubit

and measured qubit are entangled prior to the measurement, the control is arguably feedback based on a partial measurement of a multiqubit system. Indeed, we will use this terminology even when the qubits are independently prepared, because they are still *correlated* (from the point of view of the experimenter) due to the action of the phase gate with unknown phase  $\phi$ . In any case, this adaptive scheme makes the measurement component of this Heisenberg-limited protocol far easier to implement experimentally (see Fig. 9).

#### IV. QUANTUM PHASE ESTIMATION ALGORITHM

Although the Griffiths–Niu technique makes the measurement easy to implement, attaining the exact HL still requires creating a multiqubit entangled state, which is hard. This suggests exploring what happens if we replace the entangled state by independent qubits as in the standard scheme. This yields the QPEA [23], [24]. If we mediate the control steps via the auxiliary phase  $\theta$  and introduce a random  $\theta_{\text{init}}$  to ensure equal accuracy for all  $\phi$ , then the QPEA is represented by the circuit in Fig. 10. This looks almost identical to the adaptive version of the standard protocol, as shown in Fig. 7. The difference is in the multiple gate applications on a single qubit; the QPEA with  $K + 1$  register qubits again uses  $N = 2^K + 2^{K-1} + \dots + 1 = 2^{K+1} - 1$  resources, whereas the standard scheme uses  $K + 1$  resources.

Since the QPEA gives  $K + 1$  bits of  $\phi_{\text{est}}/\pi$  and  $N \sim 2^{K+1}$ , would we not expect

$$\text{QPEA} : V[\phi_{\text{est}}] \propto \left( \frac{\pi}{2^{K+1}} \right)^2 \sim \left( \frac{\pi}{N} \right)^2 \quad (\text{the HL})? \quad (7)$$

Contrary to this expectation, an exact calculation [6], [39] gives

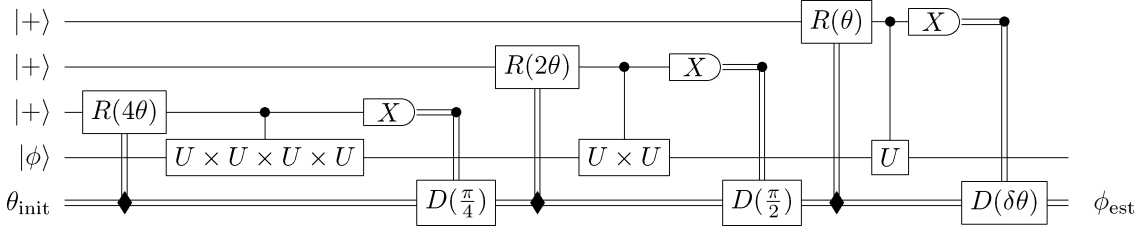
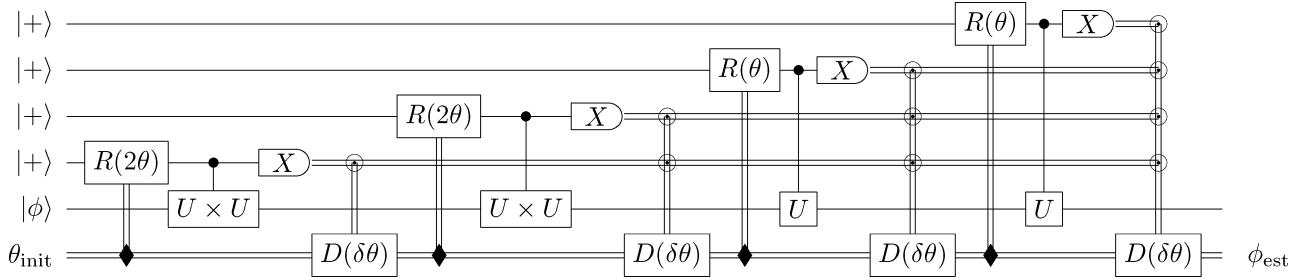
$$\text{QPEA} : V[\phi_{\text{est}}] \sim \frac{2}{N} \propto \frac{1}{N} \quad (\text{the SQL}). \quad (8)$$

So what went wrong? Why does the algorithm not only fail to attain the HL, but actually do worse than the SQL? The short answer is outliers. The distribution  $P(\phi_{\text{est}})$  is sharply peaked around  $\phi$ . The half-width at half-maximum height is given by

$$\text{QPEA} : (\text{HWHM})^2 \simeq \left( \frac{2.81}{N} \right)^2. \quad (9)$$

<sup>4</sup>In the context of optical interferometry, this corresponds to the exponential energy cost of NOON-like states discussed in Section II-C.




 Fig. 10. QPEA for  $K = 3$ , so that  $N = 7$ .

 Fig. 11. Our generalized QPEA for the case  $M = 2$  and  $K = 1$  (so that  $N = 6$ ).

But the distribution has high wings, giving SQL scaling for the variance. Specifically [39],

$$P_{\text{QPEA}}(\delta\phi) = \frac{\sin^2[(N+1)(\delta\phi/2)]}{2\pi(N+1)\sin^2[\delta\phi/2]} \quad (10)$$

where  $\delta\phi = \phi_{\text{est}} - \phi$ . In the wings,  $P_{\text{QPEA}}(\delta\phi)$  has an envelope that falls only like  $(\delta\phi)^{-2}$ . This is a consequence of the fact that we are using not the optimal state (4) with a wide but smooth number distribution, but rather a state with a “flat” number distribution

$$|\psi_{\text{flat}}\rangle = (N+1)^{-1/2} \sum_{n=0}^N |n\rangle. \quad (11)$$

It is the sharp cutoff of the number coefficients that leads to the poor localization of phase, the conjugate variable to number.

Although outliers in  $\phi_{\text{est}}$  are important for phase estimation (where our figure of merit is the variance), they are not important for quantum computing applications. There, all one cares about is getting the right answer (to the number of bits of precision one has) from the algorithm with some reasonably high probability, and the QPEA works fine for this purpose [23], [24]. Indeed, if  $\phi/\pi$  had an exact binary expansion in  $K+1$  bits, then we could remove the random  $\theta_{\text{init}}$ , and the QPEA with  $K+1$  qubits would be guaranteed to find  $\phi$  exactly; the variance would be zero. While this assumption may be relevant in some quantum computing applications, it is contrary to the rules of the game (Section III-A) for phase estimation.

## V. GENERALIZED QPEA

One way to understand the high wings of the QPEA distribution is that if an error in an insignificant bit occurs, it propagates upwards into the more significant bits through the feedback protocol. This suggests a way to remove such errors: by repeat-

ing each measurement some number  $M$  times, as previously suggested in various settings [28], [40]–[42]. Recall that in the QPEA, the  $k$ th qubit ( $k = 0, 1, \dots, K$ ) controls the phase gate  $2^k$  times. We generalize this by having  $M$  qubits for each  $2^k$ -fold application, so that the total number of passes through the phase gate is  $N = M \times (2^{K+1} - 1)$ .

With this generalization, it is no longer clear how to change the auxiliary phase  $\theta$  between measurements. Giovannetti *et al.* [28], considering the same problem as here, imply (when discussing a NOON-state realization rather than the equivalent binary-encoding implementation) that the adaptation of  $\theta$  is unnecessary; we will return to this point in Section VI. We prefer to keep the adaptation of  $\theta$  because it is an integral part of the QPEA. We use the adaptive algorithm of [36] to make the *locally optimal* measurement, as explained in Section III-B. For  $M = 1$ , this *exactly* reproduces the QPEA, which is why we regard this family of algorithms (parametrized by  $M$ ) as the natural generalization of the QPEA.

For  $M > 1$ , our generalized algorithm no longer realizes an optimal phase measurement; the Griffiths and Niu trick of realizing an optimal phase measurement by local measurement and control only works for a single copy of the quantum register with binary-encoded phase. In fact, for  $M = 2$ , as illustrated in Fig. 11, the measurement is far worse than optimal. As we have shown analytically [6], [39], the measurement in this case introduces so much noise that the estimate has a variance scaling at the SQL. (In this, it is similar to the heterodyne measurement of Section II-B.) This is so despite the fact that the relevant state in this case  $|\psi_{\text{flat}}\rangle^{\otimes 2}$  would give a nearly Heisenberg-limited phase estimate if one could implement an optimal measurement. For  $M = 3$ , numerical simulations show that the nonoptimal measurement introduces an excess noise variance with scaling consistent with  $N^{-3/2}$ , intermediate between the SQL and the HL. (In this, it is similar to the Mark II measurement of Section II-B.) For  $M \geq 4$ , numerical simulations show that the measurement

allows, as for the  $M = 1$  case,<sup>5</sup> estimation with accuracy scaling at the HL. All of these numerical simulations were performed up to  $N > 10^6$ , far into the asymptotic regime.

If one increased  $M$  indefinitely, with  $K$  being fixed, one could not hope to achieve Heisenberg-limited scaling. Intuitively, this is because the Heisenberg-limited sensitivity comes from having a maximum number of passes  $p = 2^K$ , which scales linearly with  $N$ . Alternatively, it can be understood from the fact that the state  $|\psi_{\text{flat}}\rangle^{\otimes M}$  ceases to have a broad number distribution (and hence a narrow phase distribution) as  $M$  increases. In fact, it can be shown analytically that for large  $M$ , the multiplicative overhead above the HL increases like  $M$ . Thus, there must be an optimal value of  $M$  equal to 4 or higher. Numerically, we found the best results to be for  $M = 5$  [39]

$$M = 5 \text{ GQPEA} : V[\phi_{\text{est}}] \simeq \left(\frac{4.8}{N}\right)^2 \quad (12)$$

compared to  $(\pi/N)^2$  for the HL, i.e., even though the state  $|\psi_{\text{flat}}\rangle^{\otimes 5}$  is not the optimal state, and the locally optimal adaptive measurement is not a canonical phase measurement, the multiplicative overhead on  $\Delta\phi$  is less than 1.53. Although theoretically,  $M = 5$  is the optimal choice for minimizing the variance, it still yields more outliers than in the case for larger  $M$ . Experimentally, this means not that it is hard to measure with a precision scaling at the HL, but rather it is hard to prove this precision because of the difficulty of obtaining reliable statistics. For this reason, in [6], we experimentally implemented the generalized QPEA for  $M = 6$ , for which the multiplicative overhead on  $\Delta\phi$  is about 1.56.

## VI. ARE ADAPTIVE ALGORITHMS NECESSARY?

### A. Less Adaptive “Hybrid” Algorithm

Our generalized QPEA (see Fig. 11) involves *Bayesian feedback*, in which every past result contributes to determining the auxiliary phase increment  $\delta\theta$ , according to the Berry–Wiseman protocol [36]. This is to be contrasted with the original QPEA (see Fig. 10) in which only the *immediately preceding result* is required. As noted before, the QPEA would achieve the HL if it were not for outliers. This suggests that we could get to the HL more simply by augmenting the (simple adaptive) QPEA with the (nonadaptive) SQL algorithm to remove outliers. The reasoning is that the nonadaptive SQL algorithm, consisting basically of a large number of independent trials, would be expected to have a probability distribution for the estimate that is roughly Gaussian, and so would have exponentially suppressed wings, and as long as a reasonable fraction of the resources is still devoted to the QPEA, its narrow peak would hopefully remain intact.

Investigating this hybrid scheme analytically and numerically reveals that the optimal division of resources is  $(2/3)N$  for the QPEA and  $(1/3)N$  for the standard scheme [29]. However,

<sup>5</sup>Heisenberg-limited accuracy is not attained in the  $M = 1$  case because the prepared *state* does not have a Heisenberg-limited phase variance.

contrary to expectation, we find that

$$\text{Hybrid} : V[\phi_{\text{est}}] = \frac{w(N)}{N^{3/2}} \quad (13)$$

where  $w(N)$  is a function that increases very slowly with  $N$  (from 4.83 at  $N = 5$  to 6.17 at  $N = 767$ ) [29], i.e., the hybrid scheme delivers a scaling *intermediate* between the SQL and the HL. Nevertheless, it is interesting that by combining two measurement schemes, both of which give a variance scaling at the SQL, one obtains a much better scaling.

### B. Nonadaptive Local Algorithm

The aforementioned hybrid algorithm is mostly nonadaptive—it has only  $K = O(\log N)$  adaptive measurements out of  $O(N)$  measurements in total. As noted, it surpasses the SQL in accuracy. In it, the number of qubits with  $2^k$  passes through the phase gate is  $M(K, k) = 1 + \delta_{k,0}2^K$ . This raises the question: can one get to the HL with *no feedback* by choosing a *smoother* function  $M(K, k)$  that still (like the hybrid scheme) assigns more qubits to smaller  $k$  values? Attacking this question analytically suggests considering functions of the form [29]

$$M(K, k) = M_K + \lfloor \mu(K - k) \rfloor \quad (14)$$

where  $\mu$  is a positive constant. For each value of  $k$ , the  $M(K, k)$  qubits are measured independently using the auxiliary phase  $\theta$  only to ensure an unbiased measurement. For example, it can be incremented by  $\pi/M(K, k)$ . This is illustrated in Fig. 12 with  $M_K = 1$  and  $\mu = 2$ .

Numerically, we find the best results to be for  $M_K = 2$  and  $\mu = 3$  [29]

$$\text{NALA} : V[\phi_{\text{est}}] \simeq \left(\frac{6.4}{N}\right)^2 \quad (15)$$

i.e., the overhead on  $\Delta\phi$  is less than 2.03. This is not greatly bigger than the lowest known overhead for an adaptive scheme of 1.53 for the  $M = 5$  GQPEA. It is important to note that numerical simulations for a nonadaptive scheme with  $M$  being fixed do *not* show scaling at the HL. This is contrary to the claim (rather casually made) in [28] that the HL for the variance can be attained from measuring each bit in the binary expansion of  $\phi$  by making  $\nu$  nonadaptive measurements (their  $\nu$  is our  $M$ ) for each bit, and then taking “the limit of large  $\nu$ .” It is not clear what this limit is meant to be;  $\nu$  certainly cannot be arbitrarily large as this would lead back to the SQL, as discussed in Section V.<sup>6</sup> Our numerics show that nonadaptive measurements with a large but fixed  $M$  seem to give a scaling close to the HL up to a point, but as  $N$  is increased (by increasing  $K$ ) beyond that point, the variance ceases to scale as  $N^{-2}$ .

<sup>6</sup>Giovannetti *et al.* [28] perhaps miss this point because they appear to ignore the  $\nu$  repetitions in their count of the resources, in deriving “the Heisenberg limit  $1/N$  [as] the ultimate bound to precision in phase measurements.” Indeed, they call  $N$  the “number of probes” (i.e., register qubits), rather than the number of applications of the phase gate. This is despite the fact that they also say that “Instead of a parallel strategy on  $N$  probes, one can employ a sequential strategy on a single probe,” while at the same time saying that “one finds the same  $1/N$  precision scaling . . . for sequential strategies.”

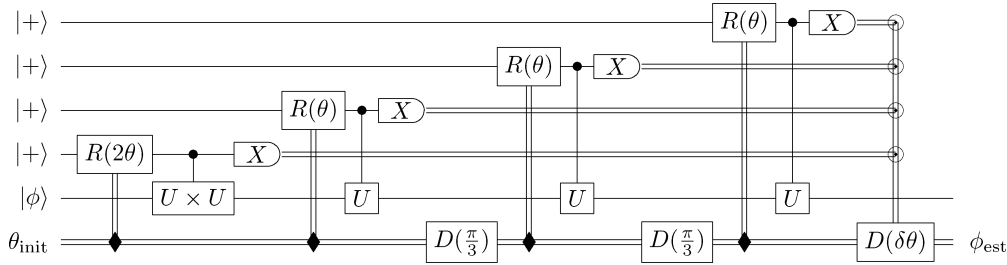


Fig. 12. Nonadaptive algorithm with  $M(K, k) = M_K + \mu(K - k)$ . Here,  $K = 1$ ,  $M_K = 1$ , and  $\mu = 2$ , so that  $N = 5$ .

In addition to this numerical evidence, the aforementioned algorithm using  $M(K, k)$  of the form (14) is the only known nonadaptive algorithm on single qubits that has been rigorously proven to attain the HL. The proof [29], [39] involves values of  $M_K$  and  $\mu$  that are known (from numerics) not to be optimal, but which allow one to rigorously bound all the contributions to the variance, using Chernoff's theorem. To apply this theorem, one needs repeated *identical* measurements, so the proof assumes using just two values of  $\theta$ : 0 and  $\pi/2$ . Specifically, for  $M(K, k) = 18 + \lfloor 16 \ln(2) \times (K - k) \rfloor$ , we prove that

$$\text{NALA} : V[\phi_{\text{est}}] \lesssim \left(\frac{150}{N}\right)^2. \quad (16)$$

The large overhead in this case shows that one would not want to use these parameters in practice, but it does prove rigorously that adaptive measurements are not necessary to attain the HL in interferometry using only single-qubit preparation and measurement.

## VII. CONCLUSION

Recent years have seen the addition of adaptive measurements to the arsenal of techniques used in quantum optics information laboratories to probe the quantum world. The notable protocols that we have discussed here are the Dolinar receiver [7], which is realized in [4], the Mark II phase measurement of [14] and [15], which is realized in [5], and the generalized QPEA proposed and realized in [6]. These protocols are all intimately related to quantum information through quantum communication, quantum computation, and quantum algorithm theory, respectively. In this context, the recent quantum computing cluster-state experiment of [22] should also be mentioned, as perhaps the first adaptive measurement in which the adapted measurement was performed on a subsystem that was initially entangled with the first measured subsystem.

The bulk of this paper has concentrated on the work of [6], which is unique in that the adaptive measurement protocol was inspired by quantum algorithm theory, but serves a purpose quite different from quantum computing, namely estimating a completely unknown phase shift  $\phi$  in one arm of an interferometer with a fixed number of photon passes through the interferometer. We analyzed this from a quantum information perspective (e.g., replacing photons by qubits and phase shifts by controlled unitaries) to make the connection to quantum algorithm theory as explicit as possible.

To attain exactly the HL for the variance of the estimate  $V[\phi_{\text{est}}] \sim (\pi/N)^2$ , the most efficient (in terms of minimizing

the number of qubits in the register, and the number of entangling gates performed on the register) protocol requires all of the following:

- 1) preparing an entangled state of  $O(\log N)$  qubits;
- 2) multiple applications of the controlled unitary gate by any given qubit;
- 3) adaptive measurements (control of individual qubits based on prior results).

We have shown numerically, and experimentally in [6], that using a generalized QPEA, one can dispense with the entangled state preparation, and still achieve Heisenberg-limited scaling, and indeed come very close to the HL

$$\text{GQPEA} : V[\phi_{\text{est}}] \sim \left(\frac{1.53\pi}{N}\right)^2. \quad (17)$$

Given that it is impossible, using only single-qubit measurement and controls, to produce simultaneously the optimal state and the optimal measurement, our phase estimation algorithm must be close to the best achievable with this restriction.

Finally, we briefly discussed some very recent results [29], [39] showing analytically and numerically that one can achieve Heisenberg-limited scaling, although with an increased overhead, even without the adaptive measurements. However, this requires a sophisticated partitioning of resources contrary to some claims in the literature (e.g., [28]).

## ACKNOWLEDGMENT

The authors acknowledge enlightening discussions with A. Doherty, who is a coauthor with some of us on the to-be-published paper mentioned in Section II-A, and with M. Mitchell, who is a coauthor with us on the two very recent papers discussed in Section VI.

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