## Quantum error correction for continuously detected errors with any number of error channels per qubit

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It was shown by Ahn, Wiseman, and Milburn [PRA 67, 052310 (2003)] that feedback control could be used as a quantum error correction process for errors induced by weak continuous measurement, given one perfectly measured error channel per qubit. Here we point out that this method can be easily extended to an arbitrary number of error channels per qubit. We show that the feedback protocols generated by our method encode n-2 logical qubits in n physical qubits, thus requiring just one more physical qubit than in the previous case.

PACS numbers: 03.67.Pp, 42.50.Lc,03.65.Yz

Keywords: quantum error correction, quantum feedback control

Quantum error correction [1, 2, 3, 4] and quantum feedback control [5, 6] have a similar structure: a state of interest is measured, and then an operation conditioned on the measurement is performed in order to control the state. However, the end result of the control, as well as the tools used to measure and control, are different for each. Quantum feedback control has traditionally been used to control a known state using weak measurements and Hamiltonian controls, whereas quantum error correction uses projective measurements and unitary gates in order to protect an unknown quantum state.

Despite the differences between the two techniques, the similarities are sufficient to enable combining the two [7, 8, 9, 10, 11, 12]. In Ref. [12], we used feedback that was directly proportional to measured currents in order to correct for a specific error process. In particular, we assumed that the errors were *detected*: the experimenter knows precisely when and where errors have occurred because the environment that produces those errors is being continuously monitored. Given this assumption, we showed

that feedback was able to protect a stabilizer codespace perfectly for one perfectly measured channel per physical qubit, and we discussed the results when the assumption of perfect measurement is removed.

To be specific, in Ref. [12] we analyzed the situation in which given n qubits, there was a single error channel on each qubit,  $E^{(j)}$  (on qubit j). That is, the decoherence of the register is given by

$$\frac{d\rho}{dt} = \sum_{j} E^{(j)} \rho E^{(j)\dagger} - \frac{1}{2} \{ E^{(j)\dagger} E^{(j)}, \rho \} - i[H, \rho],$$
(1)

where H is an additional externally applied ("driving") Hamiltonian. Moreover, these errors could be perfectly detected in such a way that the identity of the error (when and where it occurred) was known. We found that it was always possible to find feedback Hamiltonians and "driving" Hamiltonians that together perfectly corrected both the error and the no-error evolution. Our encoding used a single stabilizer generator, i.e., encoding n-1 qubits in n.

In this paper we consider the following obvious generalization: What happens if there are multiple channels  $E^{(j),\alpha}$  on a single qubit, all of which can be detected? (Here j denotes the qubit on which the channel acts, and  $\alpha$  indexes which channel it is.) Given a certain number of

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channels per qubit, what is the smallest number of stabilizers needed to be able to use our protocol? Equivalently, given n physical qubits, how many logical qubits can be encoded? To answer that question, we will first review the main result of [12], in which there is only one channel per qubit (we will drop the  $\alpha$  index for clarity) and then present a generalization of the results to the multiple-channel case.

In the detected-channel model, evolution is given by the error Kraus operators

$$\Omega_j = (E^{(j)} + \gamma^{(j)})\sqrt{dt}$$
 (2)

and the no-error Kraus operator

$$\Omega_0 = 1 - iHdt - \frac{1}{2}E^{(j)\dagger}E^{(j)}dt$$
$$-\frac{\gamma^{(j)}}{2}(e^{-i\phi^{(j)}}E^{(j)} - e^{i\phi^{(j)}}E^{(j)\dagger})dt, (3)$$

where  $\gamma e^{-i\phi}$  is a complex parameter that describes the kind of measurement (*unraveling* of the master equation) that is being done. That is, the *average* evolution reproduces the master equation independently of  $\gamma$ :

$$\rho + d\rho = \Omega_0 \rho \Omega_0^{\dagger} + \sum_j \Omega_j \rho \Omega_j^{\dagger}. \tag{4}$$

For example,  $\gamma = 0$  for a Poisson unraveling, and  $\gamma \to \infty$  for a white-noise unraveling [13].

In order to find feedback Hamiltonians and driving Hamiltonians that together perfectly correct both the error and no-error evolution, a sufficient condition that needs to be met is

$$\langle \psi_i | D^{(j)} | \psi_k \rangle = 0, \tag{5}$$

where  $|\psi_i\rangle, |\psi_k\rangle$  are orthogonal states in the codespace, and  $D^{(j)}$  is the traceless part of  $(E^{(j)} + \gamma^{(j)})^{\dagger}(E^{(j)} + \gamma^{(j)})$ . Equation (5) is just a variant of the Knill-Laflamme condition for correcting errors [3], applied to the case in which the time and position of the error are known. It is satisfied when the codespace is generated by a stabilizer S satisfying

$$0 = \{S, D^{(j)}\}. \tag{6}$$

Since it is always possible to find another Hermitian traceless one-qubit operator  $s^{(j)}$  such

that  $\{s^{(j)}, D^{(j)}\} = 0$ , it then follows that we may pick the single stabilizer generator

$$S = s^{(1)} \otimes \dots \otimes s^{(n)} \tag{7}$$

so that the stabilizer group is  $\{1, S\}$ .

The identification of Eqn. (5) with the Knill-Laflamme condition, combined with feedback results from [5], show that it is possible to correct the error using feedback. Furthermore, the no-error evolution given in (3) can be corrected by applying a driving Hamiltonian as follows:

$$H = \sum_{j} \frac{i}{2} D^{(j)} S + \frac{i \gamma^{(j)}}{2} (e^{-i\phi^{(j)}} E^{(j)} - e^{i\phi^{(j)}} E^{(j)^{\dagger}}).$$
(8)

Putting this Hamiltonian in the total no-error Kraus operator in (3) gives, with a = 1 + O(dt),

$$\Omega_0 = a1 - \frac{1}{2} \sum_j D^{(j)} (1 - S) dt. \tag{9}$$

The second term here is zero on the codespace, so the no-error evolution does not disturb the codespace.

For multiple channels (denoted by  $\alpha$ ) on a given qubit, the expressions in this previous work can easily be generalized. We are assuming that the time scale of correction is fast compared to the time scale of decoherence; therefore, different errors do not interfere with one another, and all the expressions in our paper behave well (i.e., linearly). We should also note here that implicit in the idea that all errors are detected is the assumption that, therefore, given such a detection we know not only when and where (j) the error has occurred, but also what the error is  $(\alpha)$ . In other words, given a detection we can determine the error Kraus operator  $\Omega_j^{\alpha}$  that has been applied.

Given the above assumptions, to generalize to multiple-channel protocols we must merely check whether for all  $\alpha$  and j it is true that

$$\langle \psi_i | D^{(j),\alpha} | \psi_k \rangle = 0.$$
 (10)

If (10) holds, the corresponding errors  $E^{(j),\alpha}$  will be correctable, and we will see that this condition also makes it possible to find a driving Hamiltonian such that the no-jump errors are also corrected.

Let us first consider the case when there are two channels on a single qubit:  $\alpha = 1, 2$ . When there are two channels, a Bloch-sphere analysis shows that it is possible to find a single S such that  $\{S, D^{(j),\alpha}\} = 0$ . Let us consider qubit 1: since  $D^{(1),1}$  and  $D^{(1),2}$  are traceless, they can be represented by two vectors on the Bloch sphere. In fact,  $D^{(1),1}$  and  $D^{(1),2}$  define a plane intersecting the Bloch sphere; now we pick  $s^{(1)}$  to be the operator that corresponds to the vector on the Bloch sphere that is orthogonal to that plane. Since it is possible to find a unitary rotation that takes  $s^{(1)}$  to  $\sigma_Z$  as well as  $D^{(1),1}$  and  $D^{(1),2}$  to linear combinations of  $\sigma_X$  and  $\sigma_Y$ , this operator must anticommute with  $D^{(1),1}$  and  $D^{(1),2}$ . Doing the same for the other physical qubits, we pick the single stabilizer generator

$$S = s^{(1)} \otimes \dots \otimes s^{(n)} \tag{11}$$

so that the stabilizer group is  $\{1, S\}$  as before. Again, this procedure encodes n-1 qubits in n.

The next step is to consider three channels. Unfortunately, for three channels on a single qubit, it is not in general possible to find a single s which anticommutes with all the D operators of the channels; this is reflected by the fact that the Bloch sphere is three-dimensional, and so given three arbitrary vectors, it is not possible in general to find a fourth vector perpendicular to all three.

However, we can do almost as well. Let us return to (5) again. In fact for (5) to be true, it suffices to decompose any given error operator, D, as  $D = \vec{d} \cdot \vec{\sigma}$  and to require

$$\langle \psi_i | d_l \sigma_l | \psi_k \rangle = 0 \,\forall \, l. \tag{12}$$

If our stabilizers are the two stabilizers of the familiar four-qubit code for the erasure channel [14],

$$S_1 = XXXX,$$

$$S_2 = ZZZZ,$$
(13)

we can see that for any l one of these two, call it  $S_{i(l)}$ , will satisfy

$$\{S_{j(l)}, \sigma_l\} = 0 \tag{14}$$

no matter what D is, and thus (12) holds.

In this case, with a = 1 + O(dt) as before, we have

$$\Omega_0 = a1 - \frac{D}{2}dt - \frac{\gamma}{2}(e^{-i\phi}E - e^{i\phi}E^{\dagger})dt - iHdt.$$
(15)

Let

$$H = \sum_{l} \frac{i}{2} (d_{l}\sigma_{l}) S_{j(l)} + \frac{i\gamma}{2} (e^{-i\phi} E - e^{i\phi} E^{\dagger}),$$
 (16)

where  $S_{j(l)}$  is defined as in (14). Then

$$\Omega_0 = a1 - \frac{1}{2} \sum_{l} d_l \sigma_l (1 - S_{j(l)}), \qquad (17)$$

which leaves the codespace invariant. This analysis is true for each additional error channel we introduce. Thus no matter how many error channels there are, as long as we can detect all of them and know which error has happened and where the error has happened, we can correct for the error and the no-error evolution. This code encodes two logical qubits in four physical ones.

In fact, this reasoning applies for n qubits, where n is even, given the two stabilizers

$$S_1 = X^{\otimes n}$$

$$S_2 = Z^{\otimes n}.$$
(18)

Using these stabilizers with the constant Hamiltonian found above, it is possible to encode n-2 qubits in n.

This protocol, of course, borrows heavily from the stabilizer formalism of the quantum erasure code. Indeed, the quantum erasure code can be generalized using the stabilizers in (18) in the same way, with the same scaling of n-2 logical qubits in n physical ones; as far as we know this scaling has not been explicitly noted in the literature. On the other hand, our protocol differs from the erasure code in that we have made a different and more restrictive assumption about the error model; as a result, we only need to perform local measurements instead of a highly nonlocal stabilizer measurement. To elaborate, the quantum erasure code makes the same assumption that the position and time of the error are both known. In the protocol given here, we make the further assumption that we know what error has occurred in that measuring the error tells us what error has occurred. This information about the error comes precisely from the detection of the local measurements performed by the environment. As in [12], these results indicate that if dominant error processes can be monitored, using that information can be the key to correcting them, and that the overhead in encoding is minimal (just two physical qubits).

## Acknowledgments

C. A. would like to thank the Centre for Quantum Computation for its hospitality and acknowledge the support of an Institute for Quantum Information fellowship.

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