

# Entanglement of identical particles and reference phase uncertainty

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We have recently introduced a measure of the bipartite entanglement of identical particles,  $E_P$ , based on the principle that entanglement should be *accessible* for use as a resource in quantum information processing. We show here that particle entanglement is limited by the lack of a reference phase shared by the two parties, and that the entanglement is constrained to reference-phase invariant subspaces. The super-additivity of  $E_P$  results from the fact that this constraint is weaker for combined systems. A shared reference phase can only be established by transferring particles between the parties, that is, with additional nonlocal resources. We show how this nonlocal operation can increase the particle entanglement.

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## I. INTRODUCTION

Entanglement is an essential resource for quantum information processing. The non separability of the wavefunction of two distinct systems is the usual hallmark of an entangled state. However, the symmetric or antisymmetric wavefunctions of collections of identical particles is *inherently* non separable. A crucial question then is how to quantify the entanglement of identical particles. The approach of Zanardi and others[1] is to calculate the entanglement of the quantum field modes,  $E_M$ , rather than the particles that occupy them. In particular,  $E_M$  can be non-zero even for the case of a single particle. An alternate approach[2] is to examine the non separability of the wavefunction beyond that required by symmetrization or antisymmetrization. The difficulty here, however, is that there is no fixed partition into distinct systems.

The approach we take[3] is to insist that the entanglement of the particles,  $E_P$ , is *accessible* in the sense that it could be transferred to regular quantum registers (e.g. qubit systems) using local operations; once transferred it can be used as a generic resource for quantum information processing. This requires strict *partite separation* and the entanglement to be accessible using *local operations* only. Transporting particles between the parties is clearly a nonlocal operation; local operations therefore preserve the local particle number at each site. Hence, these restrictions are equivalent to imposing a local particle number superselection rule.[3] Entanglement constrained by general superselection rules have been explored further by Bartlett and Wiseman.[4] A more introductory treatment can be found in Ref. [5]. Also the impact of superselection rules on nonlocality and quantum data hiding has been studied by Verstraete and Cirac.[9]

While the local transfer of the particle entanglement to regular quantum registers underpins our definition of  $E_P$  in Ref. [3], we did not explicitly show how the transfer operation might be implemented. In this paper[6] we

give an explicit demonstration of the transfer. We show how the lack of a shared reference phase reduces the entanglement of the regular quantum registers to that of  $E_P$ . Moreover, by performing a measurement of the difference between the reference phases at the two sites, it is possible to recover the entropy of entanglement of the original system. However, this requires the transport of particles from one site to the other. The essential point is that the entanglement can be recovered only by violating local particle conservation and transporting particles from one site to the other, that is, only by the use of other nonlocal resources. We establish a relation between the variance in the number of particles transported and the amount of entanglement in the quantum registers. This explicit demonstration gives further insight into the nature of particle entanglement and reference phase uncertainty.

Reference phase uncertainty has recently been discussed in relation to continuous variable teleportation[7] and communication without a shared reference frame.[8] The close connection between the application of local superselection rules and a reference system has been discussed recently by Kitaev *et al.*[10] They show how one can simulate the local violation of a superselection rule if a shared reference system is available. Their analysis is in the context of data security whereas our work here explores the implications for particle entanglement. Schuch *et al.*[11] have recently investigated the intra-conversion of sets of states under the constraint of the local superselection rule associated with particle conservation. They identify a new nonlocal resource corresponding to the variance in local particle number. The connection with our work is that we give explicit protocols for converting this resource into particle entanglement by establishing a shared reference frame.

The body of the paper is organized as follows. We begin in Section II with a brief review the definition of  $E_P$ . In Section III we describe the protocol for transferring

the entanglement of shared particles to regular quantum registers for a variety of cases. In Section IV we show how the particle entanglement can be increased by establishing a shared reference phase. We end with a discussion in Section V.

## II. ENTANGLEMENT OF IDENTICAL PARTICLES

We imagine two well-separated parties, Alice and Bob, sharing a collection of  $N$  identical particles, such as atoms or electrons etc., which are in the pure state  $|\Psi\rangle_{AB}$ . The *particle entanglement*  $E_P(|\Psi\rangle_{AB})$  of this state is the maximum entanglement that can be transferred to local quantum registers *without additional nonlocal resources*. We showed in Ref. [3] that this is given by:

$$E_P(|\Psi\rangle_{AB}) \equiv \sum_{n=0}^N P_n E(|\Psi_n\rangle_{AB}) \quad (1)$$

where

$$|\Psi_n\rangle_{AB} = \frac{\hat{Y}_n |\Psi\rangle_{AB}}{\sqrt{P_n}}, \quad (2)$$

$\hat{Y}_n$  is the projector onto states with  $n$  particles at Alice's site and  $N-n$  at Bob's,  $P_n = {}_{AB}\langle\Psi|\hat{Y}_n|\Psi\rangle_{AB}$  is the probability of finding  $n$  particles at Alice's site,  $|\Psi_n\rangle_{AB}$  represents field modes occupied by a fixed number of particles at each site,  $E(|\Psi_n\rangle_{AB}) = S(\hat{\rho}_A^{(n)})$  is the entropy of entanglement in  $|\Psi_n\rangle_{AB}$ ,  $S(\hat{\rho})$  is the binary von Neumann entropy  $-\text{Tr}(\hat{\rho} \log_2 \hat{\rho})$ , and  $\hat{\rho}_A^{(n)}$  is the reduced density matrix  $\hat{\rho}_A^{(n)} = \text{Tr}_B[|\Psi_n\rangle_{AB}\langle\Psi_n|_{AB}]$ .<sup>1</sup> In essence, (1) results from a *local* particle number superselection rule in that the coherences between subspaces of differing local particle number are not observable by local means.

In the following section we demonstrate the transfer of the entanglement in  $|\Psi\rangle_{AB}$  to regular quantum registers. The essential features this operation are clearly revealed in the simplest system: coherently sharing a single particle between Alice and Bob in the state

$$|\Psi^{(1)}\rangle_{AB} = \frac{1}{\sqrt{2}}(|1, 0\rangle_{AB} + |0, 1\rangle_{AB}) \quad (3)$$

where  $|i, j\rangle_{AB}$  represents  $i$  particles in a field mode at Alice's site and  $j$  particles in a field mode at Bob's site. We note, in particular, that sharing a single particle and independently sharing two particles carries the following particle entanglement:[3]

$$E_P(|\Psi^{(1)}\rangle_{AB}) = 0 \quad (4)$$

$$E_P(|\Psi^{(1)}\rangle_{AB} \otimes |\Psi^{(1)}\rangle_{AB}) = \frac{1}{2}. \quad (5)$$

<sup>1</sup> To simplify the notation for projectors, we write the subscript AB outside a bracket, e.g. as  $(|\psi\rangle\langle\psi|)_{AB}$ , rather than individually on each bra and ket.

This illustrates a striking general feature of  $E_P$  in that it is super-additive. The super-additivity is a direct consequence of the inherent indistinguishability of the particles.

## III. TRANSFER PROTOCOL AND REFERENCE PHASE UNCERTAINTY

We now demonstrate the transfer protocol of the particle entanglement to regular qubit registers. We treat explicitly the case of bosons here; the modification required for the fermion case is, however, straightforward.<sup>2</sup> Let Alice have a very large number  $M \gg 1$  of identical ancillary particles in a particular field mode, i.e. the mode occupation is given by the state  $|M\rangle_A$ . An operation is then performed which shares the particles with another mode at Alice's site to produce the state

$$\sum_{n=0}^M c_n |M-n, n\rangle_A \quad (6)$$

where here  $|i, j\rangle_A$  represents  $i$  particles in one field mode and  $j$  particles in the second field mode at Alice's site, and  $c_n$  are complex amplitudes satisfying  $\sum_n |c_n|^2 = 1$ . We can rewrite this state as

$$\frac{\sqrt{M+1}}{2\pi} \int_{2\pi} |c(\theta)\rangle_A |\psi(\theta)\rangle_A d\theta \quad (7)$$

where

$$|\psi(\theta)\rangle = \sum_{n=0}^M \frac{e^{-in\theta}}{\sqrt{M+1}} |M-n\rangle = \sum_{n=0}^M \frac{e^{-i(M-n)\theta}}{\sqrt{M+1}} |n\rangle \quad (8)$$

$$|c(\theta)\rangle = \sum_{n=0}^M c_n e^{in\theta} |n\rangle. \quad (9)$$

Here  $|\psi(\theta)\rangle$  is a "truncated" phase state[12, 13] and  $|c(0)\rangle$  is a state with a large mean particle number  $\bar{N}_c = \sum_n |c_n|^2 n$  satisfying  $M \gg \bar{N}_c \gg 1$ , but otherwise arbitrary. For example,  $|c(\theta)\rangle$  could approximate a large amplitude coherent state with  $c_n \propto (\bar{N}_c^n e^{-\bar{N}_c}/n!)^{1/2}$ . A corresponding process is performed at Bob's site with his local ancillary system, resulting in the combined ancillary state

$$\frac{M+1}{(2\pi)^2} \int_{2\pi} |c(\theta)\rangle_A |\psi(\theta)\rangle_A d\theta \int_{2\pi} |c(\phi)\rangle_B |\psi(\phi)\rangle_B d\phi. \quad (10)$$

<sup>2</sup> For the fermion case we replace the state of a  $n$ -boson occupied mode  $|n\rangle$  with the state representing  $n$  fermions distributed in  $n$  different modes each of which contain a single fermion:  $|n\rangle^{(f)} \equiv |1, 1, \dots, 1, 0, \dots, 0\rangle$  where the number of modes exceeds the number of fermions. The protocol then involves operations of the same form as the boson case.

### A. Single shared particle

Our transfer protocol is based on a method introduced by Mayers.[14] We demonstrate it first for the state of a single shared particle,  $|\Psi^{(1)}\rangle_{AB}$  in (3). Let the initial state of the two regular qubits, one at Alice's site and the other at Bob's, be  $|\underline{0}, 0\rangle_{AB}$ . We use an underline to distinguish the states of a regular qubit,  $|\underline{0}\rangle$ ,  $|\underline{1}\rangle$ , (such as two orthogonal electronic states of an atom) from the Fock states of the field modes  $|0\rangle$ ,  $|1\rangle$ ,  $\dots$ .

We will first concentrate on the integrand of the left integral in (10) for a specific value of  $\theta$ . Let this term together with the shared particle modes and a single regular qubit at Alice's site be given by

$$\begin{aligned} & \dots |\psi(\theta)\rangle_A \otimes |\Psi^{(1)}\rangle_{AB} \otimes |\underline{0}\rangle_A \\ & = \dots |\psi(\theta)\rangle_A \otimes \frac{1}{\sqrt{2}} (|1, 0\rangle_{AB} + |0, 1\rangle_{AB}) \otimes |\underline{0}\rangle_A \end{aligned} \quad (11)$$

where, for clarity, we have reordered the states and written " $\dots$ " to represent states of modes that are not of

immediate interest. Alice performs a local CNOT operation using her local shared-particle mode as the control and her local regular qubit as the target, yielding

$$\dots |\psi(\theta)\rangle_A \otimes \frac{1}{\sqrt{2}} (|1, 0\rangle_{AB} \otimes |\underline{1}\rangle_A + |0, 1\rangle_{AB} \otimes |\underline{0}\rangle_A) . \quad (12)$$

To complete her part of the protocol, Alice must disentangle her shared-particle mode from her regular qubit for this value of  $\theta$ . This entails "hiding" the shared particle in the truncated phase state  $|\psi(\theta)\rangle_A$ . Expanding the state  $|\psi(\theta)\rangle_A$  in the number basis yields

$$\dots \frac{1}{\sqrt{2(M+1)}} \sum_{n=0}^M e^{-i(M-n)\theta} \left[ |n\rangle_A \otimes |1, 0\rangle_{AB} \otimes |\underline{1}\rangle_A + |n\rangle_A \otimes |0, 1\rangle_{AB} \otimes |\underline{0}\rangle_A \right] . \quad (13)$$

Alice now applies a controlled operation with her regular qubit as the control and the mapping:

$$\dots |x\rangle_A \otimes |y, z\rangle_{AB} \otimes |\underline{0}\rangle_A \mapsto \dots |x\rangle_A \otimes |y, z\rangle_{AB} \otimes |\underline{0}\rangle_A , \quad (14)$$

$$\dots |x\rangle_A \otimes |y, z\rangle_{AB} \otimes |\underline{1}\rangle_A \mapsto \dots |x+y\rangle_A \otimes |0, z\rangle_{AB} \otimes |\underline{1}\rangle_A , \quad (15)$$

to produce the state

$$\dots \frac{1}{\sqrt{2}} \left\{ |\psi(\theta)\rangle_A \otimes (e^{-i\theta} |0, 0\rangle_{AB} \otimes |\underline{1}\rangle_A + |0, 1\rangle_{AB} \otimes |\underline{0}\rangle_A) + \frac{1}{\sqrt{M+1}} \left[ |M+1\rangle_A - e^{-i(M+1)\theta} |0\rangle_A \right] \otimes |0, 0\rangle_{AB} \otimes |\underline{1}\rangle_A \right\} . \quad (16)$$

Next Bob repeats these operations using his truncated phase state and another regular qubit in the state  $|\underline{0}\rangle_B$  at his site as follows. We first consider the integrand of the right integral in (10) for a specific value of  $\phi$ . We also reorder the states and include only states of modes that are of immediate interest:

$$\begin{aligned} & \dots \frac{1}{\sqrt{2}} \left\{ |\psi(\theta)\rangle_A \otimes (e^{-i\theta} |0, 0\rangle_{AB} \otimes |\underline{1}\rangle_A + |0, 1\rangle_{AB} \otimes |\underline{0}\rangle_A) \right. \\ & \left. + \frac{1}{\sqrt{M+1}} \left[ |M+1\rangle_A - e^{-i(M+1)\theta} |0\rangle_A \right] \otimes |0, 0\rangle_{AB} \otimes |\underline{1}\rangle_A \right\} \otimes |\underline{0}\rangle_B \otimes |\psi(\phi)\rangle_B . \end{aligned} \quad (17)$$

Bob performs a local CNOT operation using his local shared-particle mode as the control and his regular qubit as the target. He then performs a controlled operation analogous to (14) and (15) using his regular qubit as the control and his truncated phase state  $|\psi(\phi)\rangle_B$  as the target. This gives the state

$$\begin{aligned} & \dots \frac{1}{\sqrt{2}} \left\{ |\psi(\theta)\rangle_A \otimes |0, 0\rangle_{AB} \otimes (e^{-i\theta} |\underline{1}, 0\rangle_{AB} + e^{-i\phi} |\underline{0}, 1\rangle_{AB}) \otimes |\psi(\phi)\rangle_B \right. \\ & + \frac{1}{\sqrt{M+1}} |\psi(\theta)\rangle_A \otimes |0, 0\rangle_{AB} \otimes |\underline{0}, 1\rangle_{AB} \otimes \left[ |M+1\rangle_B - e^{-i(M+1)\phi} |0\rangle_B \right] \\ & \left. + \frac{1}{\sqrt{M+1}} \left[ |M+1\rangle_A - e^{-i(M+1)\theta} |0\rangle_A \right] \otimes |0, 0\rangle_{AB} \otimes |\underline{1}, 0\rangle_{AB} \otimes |\psi(\phi)\rangle_B \right\} . \end{aligned} \quad (18)$$

Here, and in the following, we write the state  $|\underline{n}\rangle_A \otimes |\underline{m}\rangle_B$  of the regular qubits as  $|n, m\rangle_{AB}$ , for convenience. For the limiting case of large  $M$  the statistical weighting of the last two terms, being of order  $2/(M+1)$ , becomes vanishingly small. We ignore these terms for the remainder of this paper. We now trace over all particle modes as our interest

lies only in the regular qubits. Recalling that the state being considered is part of the integrands in (10), we find that we need to evaluate integrals of the following form

$$I = \int_{2\pi} \sum_{n=0}^M \sum_{m=0}^M \langle n|\psi(\theta)\rangle \langle \psi(\theta')|n\rangle \langle m|c(\theta)\rangle \langle c(\theta')|m\rangle e^{ik\theta'} \frac{d\theta'}{2\pi} \quad (19)$$

where  $k$  is a non-negative integer. Using the expansions of  $|\psi(\theta)\rangle$  and  $|c(\theta)\rangle$  in terms of the Fock states in (8) and (9) shows that this expression is simply

$$I = \sum_{m=k}^M \frac{|c_m|^2}{(M+1)} e^{ik\theta} \approx \frac{1}{M+1} e^{ik\theta}. \quad (20)$$

We have assumed here that the state  $|c(\theta)\rangle$  has negligible overlap with  $|n\rangle$  for  $n \leq k$ ; this is the case, for example, if  $|c(\theta)\rangle$  approximates a large amplitude coherent state with  $|c_n|^2 \propto \bar{N}_c^n e^{-\bar{N}_c}/n!$ . Armed with this result, we find that the qubit registers on their own are left in the mixed state:

$$\begin{aligned} & \frac{1}{2} \int_{2\pi} \int_{2\pi} (e^{-i\theta} |\underline{1}, 0\rangle_{AB} + e^{-i\phi} |\underline{0}, 1\rangle_{AB}) \\ & \quad \times ({}_{AB}\langle \underline{1}, 0| e^{i\theta} + {}_{AB}\langle \underline{0}, 1| e^{i\phi}) \frac{d\theta}{2\pi} \frac{d\phi}{2\pi} \\ & = \frac{1}{2} (|\underline{1}, 0\rangle \langle \underline{1}, 0| + |\underline{0}, 1\rangle \langle \underline{0}, 1|)_{AB}. \end{aligned} \quad (21)$$

As predicted in Ref. [3] and shown in (4), there is no entanglement here. *The origin of the loss of entanglement can therefore be attributed to the unknown phase difference  $\theta - \phi$  that emerges in the transfer protocol, i.e. to the lack of a shared reference phase.*

## B. Independently sharing two particles

This situation can be contrasted with the result of independently sharing two particles, that is when Alice and Bob share the state

$$\begin{aligned} & |\Psi^{(1)}\rangle_{AB} \otimes |\Psi^{(1)}\rangle_{AB} \\ & = \frac{1}{\sqrt{2}} (|1, 0\rangle_{AB} + |0, 1\rangle_{AB}) \otimes \frac{1}{\sqrt{2}} (|1, 0\rangle_{AB} + |0, 1\rangle_{AB}) \end{aligned} \quad (22)$$

Carrying out the above transfer operations on the first shared particle results in the state represented by the first line of (18) with probability  $P = 1 - 2/(M+1)$ :

$$\begin{aligned} & \dots \frac{1}{\sqrt{2}} |\psi(\theta)\rangle_A \otimes |0, 0\rangle_{AB} \\ & \quad \otimes (e^{-i\theta} |\underline{1}, 0\rangle_{AB} + e^{-i\phi} |\underline{0}, 1\rangle_{AB}) \otimes |\psi(\phi)\rangle_B. \end{aligned} \quad (23)$$

Repeating the operations on the second shared particle using the truncated phase states  $|\psi(\theta)\rangle_A$  and  $|\psi(\phi)\rangle_B$  and two additional regular qubits (one at Alice's site and the other at Bob's) results in the reduced density operator for the four regular qubits as

$$\begin{aligned} & \int_{2\pi} \int_{2\pi} [ |R(\theta, \phi)\rangle \otimes |R(\theta, \phi)\rangle ] [ \langle R(\theta, \phi)| \otimes \langle R(\theta, \phi)| ] \frac{d\theta}{2\pi} \frac{d\phi}{2\pi} \\ & = \frac{1}{4} [ |\underline{00}, 11\rangle \langle \underline{00}, 11| + |\underline{11}, 00\rangle \langle \underline{11}, 00| + (|\underline{10}, 01\rangle + |\underline{01}, 10\rangle) ( \langle \underline{10}, 01| + \langle \underline{01}, 10| ) ]_{AB} \end{aligned} \quad (24)$$

where

$$|R(\theta, \phi)\rangle = \frac{1}{\sqrt{2}} (e^{-i\theta} |\underline{1}, 0\rangle_{AB} + e^{-i\phi} |\underline{0}, 1\rangle_{AB}) \quad (25)$$

and we have written the joint state  $|i, j\rangle_{AB} \otimes |n, m\rangle_{AB}$  as  $|\underline{in}, \underline{jm}\rangle_{AB}$ . Each of the parties, Alice and Bob, can perform a local measurement to determine if the states of their two regular qubits at their site are equal or different; the result of the measurement is equally likely to be

$$\frac{1}{2} (|\underline{00}, 11\rangle \langle \underline{00}, 11| + |\underline{11}, 00\rangle \langle \underline{11}, 00|)_{AB} \quad (26)$$

or

$$\frac{1}{2} [ (|\underline{10}, 01\rangle + |\underline{01}, 10\rangle) ( \langle \underline{10}, 01| + \langle \underline{01}, 10| ) ]_{AB}, \quad (27)$$

respectively. The entanglement in the first result is zero whereas it is 1 ebit in the second, and so the average entanglement is 1/2 ebit. This agrees exactly with (5) and Ref. [3].

We note that the subspace spanned by the states  $\{|\underline{10}, 01\rangle_{AB}, |\underline{01}, 10\rangle_{AB}\}$  is invariant to arbitrary shifts of the local reference phases. For example, the reference phase shifts given by  $|\underline{0}\rangle_A \mapsto |\underline{0}\rangle_A$ ,  $|\underline{1}\rangle_A \mapsto e^{i\theta} |\underline{1}\rangle_A$ ,  $|\underline{0}\rangle_B \mapsto |\underline{0}\rangle_B$ ,  $|\underline{1}\rangle_B \mapsto e^{i\phi} |\underline{1}\rangle_B$  map an arbitrary state

of this subspace,  $\alpha|10,01\rangle_{AB} + \beta|01,10\rangle_{AB}$ , to the state  $e^{i(\theta+\phi)}(\alpha|10,01\rangle_{AB} + \beta|01,10\rangle_{AB})$ , which differs only by an overall phase factor from the original state. Clearly, in the absence of a shared reference phase, the transferred entanglement is constrained to such *reference-phase invariant subspaces*. Comparing (21) and (27) we conclude that *the super-additivity of  $E_F$  is due to this constraint being weaker for the combined system*.

### C. The general case

The transfer protocol can easily be generalized to multi-occupied field modes where the  $n$ -particle state  $|n\rangle$  is mapped to the state  $|\underline{n}\rangle$  of a regular quantum register. Here  $\{|\underline{n}\rangle : n = 0, 1, \dots\}$  is an orthogonal basis set. We write a general pure state representing  $N$  particles shared between Alice and Bob as

$$|\Psi\rangle_{AB} = \sum_{n=0}^N g_n |\psi_n\rangle_{AB} \quad (28)$$

where  $g_n$  are complex amplitudes and  $|\psi_n\rangle_{AB}$  represents a state comprising  $n$  particles at Alice's site and the remainder at Bob's site. In A we show that the final state of the regular quantum registers after the transfer protocol is

$$\sum_{n=0}^N |g_n|^2 (|\underline{\psi}_n\rangle\langle\underline{\psi}_n|)_{AB} \quad (29)$$

where  $|\underline{\psi}_n\rangle_{AB}$  is the regular quantum register version of the shared particles state  $|\psi_n\rangle_{AB}$ . Each of the terms in the sum of (29) belongs to a different reference-phase invariant subspace. It is possible to make a local measurement which projects onto these subspaces. Thus the entanglement of (29) is given by (1) with  $|\psi_n\rangle$  replaced by  $|\underline{\psi}_n\rangle$ . In other words, the entanglement transferred to the quantum registers is in exact agreement with our definition of  $E_F$ . Moreover, this shows that the transferred entanglement is constrained to reference-phase invariant subspaces, in general.

## IV. ENTANGLEMENT AND REDUCED REFERENCE PHASE UNCERTAINTY

The foregoing suggests that the particle entanglement can be increased by fixing the phase difference between the two sites. Indeed, consider the case of sharing a single particle which results in the mixed state in (21) in the absence of a known phase difference  $\theta - \phi$ . The entanglement in this state can be increased if we reduce the uncertainty in the phase difference so that the implicit phase distributions in the integral in (21) are no longer flat. One way of doing this is to perform a phase-difference measurement between the two sites. In B we show that the state of the regular qubits following an

ideal phase-difference measurement of the ancillary states  $|c(\theta)\rangle_A \otimes |c(\phi)\rangle_B$  in (10) is given by

$$\frac{1}{2} \left( |\underline{1,0}\rangle\langle\underline{1,0}| + C|\underline{1,0}\rangle\langle\underline{0,1}| + C^*|\underline{0,1}\rangle\langle\underline{1,0}| + |\underline{0,1}\rangle\langle\underline{0,1}| \right)_{AB} \quad (30)$$

where

$$C = \int_{2\pi} \int_{2\pi} P_\varphi(\theta, \phi) e^{i(\phi-\theta)} d\theta d\phi \quad (31)$$

and  $P_\varphi(\theta, \phi)$ , which is defined in (B9), represents the resolution of the phase-difference measurement for a measured difference of  $\varphi$ . The entanglement of formation[15] of the mixed state (30) is given by

$$E_F = -p \log_2(p) - (1-p) \log_2(1-p) \quad (32)$$

where  $p = \frac{1}{2}(1 + \sqrt{1 - |C|^2})$ . For  $|C| \approx 1$  we find

$$E_F \approx 1 - \frac{1 - |C|^2}{\ln 2} \quad (33)$$

Any resolution of the phase difference requires a minimum variance  $\langle \Delta \hat{N}_{Tr}^2 \rangle$  in the number of particles transported from one site to the other. We can relate  $E_F$  to the variance in particle number using the Heisenberg-Robertson uncertainty relation for phase and number operators. In C we find that the optimum strategy gives

$$|C|^2 \leq \frac{4\langle \Delta \hat{N}_{Tr}^2 \rangle}{1 + 4\langle \Delta \hat{N}_{Tr}^2 \rangle} \quad (34)$$

Thus, from (33), an upper bound for the entanglement of formation is given approximately by

$$E_F \leq 1 - \frac{1}{4\langle \Delta \hat{N}_{Tr}^2 \rangle \ln 2} \quad (35)$$

in the limit that  $\langle \Delta \hat{N}_{Tr}^2 \rangle \gg 1$ .

As an example, let the ancillary states  $|c(\theta)\rangle_A \otimes |c(\phi)\rangle_B$  in (10) approximate two coherent states of not necessarily the same amplitude, and imagine transporting one of these ancillary modes from one site to the other to allow an ideal phase-difference measurement between them. In this case, the variance is given by  $\langle \Delta \hat{N}_{Tr}^2 \rangle = \langle \hat{N}_{Tr} \rangle$  where  $\langle \hat{N}_{Tr} \rangle$  is the mean particle number transported between the sites and so for the optimum strategy  $E_F \leq 1 - 1/(4\langle \hat{N}_{Tr} \rangle \ln 2)$ . In fact, a direct calculation of (31), using periodic Gaussian distributions to approximate the phase distributions of the coherent states[12] and assuming that the local coherent state has a much larger amplitude than the one which is transported, gives  $|C|^2 \approx e^{-1/4\langle \hat{N}_{Tr} \rangle} \approx 1 - 1/4\langle \hat{N}_{Tr} \rangle$  for  $\langle \hat{N}_{Tr} \rangle \gg 1$ . Thus, from (33), using coherent states to establish a shared reference phase gives the entanglement of formation as

$$E_F \approx 1 - \frac{1}{4\langle \hat{N}_{Tr} \rangle \ln 2} \quad (36)$$

for  $\langle \hat{N}_{Tr} \rangle \gg 1$ . This value represents the upper bound in (35). Clearly  $E_F$  approaches 1 ebit as  $\langle \hat{N}_{Tr} \rangle$ , the mean number transported, increases.

## V. DISCUSSION

Only manipulations by local operations and classical communication are permissible when quantifying the accessible entanglement in a system. Operations that change local particle number are therefore forbidden and this gives rise to a local particle-number superselection rule. This concept underlies the definition of  $E_P$ , the entanglement of identical particles.[3]  $E_P$  quantifies the amount of accessible entanglement in a system of identical particles, where the accessibility implies that the entanglement is able to be transferred to regular quantum registers such as qubits, and be used as a generic resource in quantum information processing.

In this paper we have shown that the process of transferring the entanglement of shared particles to quantum register *in the absence of any shared nonlocal resources* necessarily involves *random phase differences between the two sites*. The unknown nature of these phase differences leads to a reduction in the transferred entanglement. Any non zero entanglement remaining after the transfer is constrained to *reference-phase invariant subspaces*. Moreover, the super-additivity of  $E_P$  can be attributed to this *constraint being weaker for combined systems* compared to the individual systems. We also showed that the entanglement can be recovered by establishing a shared reference phase for the two sites. This operation, however, requires the transport of particles between the sites, that is, it is a non local operation. In other words, establishing a shared reference phase violates the restriction to local operations and the local superselection rule, and in doing so increases the accessible entanglement. We gave a general expression that relates the transferred entanglement to the variance in the number of particles transported for the case of a single shared particle.

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### APPENDIX A

We describe here the details of the protocol that transfers the particle entanglement of the arbitrary  $N$ -particle state given by (28) into the state (29) of regular quantum registers. The ket  $|\psi_n\rangle_{AB}$  in (28) represents the state of  $n$  particles at Alice's site and  $N - n$  at Bob's site, which we write here as

$$|\psi_n\rangle_{AB} = \sum d_{u_1^{(n)}, u_2^{(n)}, \dots, v_1^{(n)}, v_2^{(n)}, \dots} \times |u_1^{(n)}, u_2^{(n)}, \dots\rangle_A \otimes |v_1^{(n)}, v_2^{(n)}, \dots\rangle_B . \quad (\text{A1})$$

Here  $d_{\dots}$  are complex amplitudes,  $|k_1, k_2, \dots\rangle_Z$  represents a set of field modes at site  $Z \in \{A, B\}$  with corresponding

occupations  $k_1, k_2, \dots$ , and the sets of non-negative integers  $u_1^{(n)}, u_2^{(n)}, \dots$  and  $v_1^{(n)}, v_2^{(n)}, \dots$  have the property that

$$\sum_{m=0}^N u_m^{(n)} = n, \quad \sum_{m=0}^N v_m^{(n)} = N - n . \quad (\text{A2})$$

We imagine a corresponding set of regular quantum registers located at each site and initially in the state  $|0, 0, \dots\rangle_A \otimes |0, 0, \dots\rangle_B$ . The system at Alice's site can be written in part as

$$\dots |\psi(\theta)\rangle_A \otimes |u_1^{(n)}, u_2^{(n)}, \dots\rangle_A \otimes |0, 0, \dots\rangle_A . \quad (\text{A3})$$

Alice performs a unitary operation which transforms her quantum registers to

$$\dots |\psi(\theta)\rangle_A \otimes |u_1^{(n)}, u_2^{(n)}, \dots\rangle_A \otimes |\underline{u_1^{(n)}, u_2^{(n)}, \dots}\rangle_A . \quad (\text{A4})$$

She then "hides" the  $n$  shared particles in her truncated phase state as before; this leaves her system in a state closely approximated by

$$\dots e^{-in\theta} |\psi(\theta)\rangle_A \otimes |0, 0, \dots\rangle_A \otimes |\underline{u_1^{(n)}, u_2^{(n)}, \dots}\rangle_A . \quad (\text{A5})$$

Bob repeats these operations at his site. The end result of Alice's and Bob's actions is a state of the form

$$\dots e^{-in\theta} |\psi(\theta)\rangle_A \otimes |0, 0, \dots\rangle_A \otimes |\underline{u_1^{(n)}, u_2^{(n)}, \dots}\rangle_A \\ \otimes e^{-i(N-n)\phi} |\psi(\theta)\rangle_B \otimes |0, 0, \dots\rangle_B \otimes |\underline{v_1^{(n)}, v_2^{(n)}, \dots}\rangle_B . \quad (\text{A6})$$

Including the integrals over the phase angles  $\theta$  and  $\phi$  and the remaining particle modes, and then tracing over the particle modes yields the final state of the regular quantum registers; this is given by (29) with

$$|\underline{\psi}_n\rangle_{AB} = \sum d_{u_1^{(n)}, u_2^{(n)}, \dots, v_1^{(n)}, v_2^{(n)}, \dots} \\ \times |\underline{u_1^{(n)}, u_2^{(n)}, \dots}\rangle_A \otimes |\underline{v_1^{(n)}, v_2^{(n)}, \dots}\rangle_B . \quad (\text{A7})$$

### APPENDIX B

In this appendix we derive the state of the regular qubits in (18) following an ideal phase-difference measurement of the ancillary states  $|c(\theta)\rangle_A \otimes |c(\phi)\rangle_B$  in (10). An ideal phase-difference measurement is described by the POVM[12, 16]

$$\hat{\Pi}^{(-)}(\varphi) = \int_{2\pi} \hat{\Pi}_A(\theta') \otimes \hat{\Pi}_B(\theta' + \varphi) d\theta' , \quad (\text{B1})$$

where  $\varphi$  represents the measured value of the phase difference and  $\hat{\Pi}_Z(\theta)$  is the POVM representing an ideal measurement of the phase of a field mode at site  $Z \in \{A, B\}$ :

$$\hat{\Pi}_Z(\theta) = \frac{1}{2\pi} \sum_{n,m=0}^{\infty} e^{i(n-m)\theta} (|n\rangle\langle m|)_Z . \quad (\text{B2})$$

The completeness of these POVMs is given by

$$\int_{2\pi} \hat{\Pi}^{(-)}(\varphi) d\varphi = \hat{\mathbf{1}}_A \otimes \hat{\mathbf{1}}_B, \quad (\text{B3})$$

$$\int_{2\pi} \hat{\Pi}_Z(\varphi) d\varphi = \hat{\mathbf{1}}_Z \quad (\text{B4})$$

where  $\hat{\mathbf{1}}_Z$  is the identity operator for the mode at site  $Z \in \{A, B\}$ . While it is impossible to realize these measurements exactly,<sup>3</sup> nevertheless they can be implemented, in principle, with arbitrary precision.[12, 17] Consider the full state represented by the first line of (18):

$$\frac{M+1}{(2\pi)^2} \int_{2\pi} \int_{2\pi} |c(\theta)\rangle_A \otimes |c(\phi)\rangle_B \otimes |\psi(\theta)\rangle_A \otimes |0, 0\rangle_{AB}$$

<sup>3</sup> The same can be said, for example, for position measurements: measurements of position can be made with arbitrary precision, however, the position POVM  $|x\rangle\langle x|$ , where  $|x\rangle$  is an eigenstate

$$\otimes \frac{1}{\sqrt{2}} (e^{-i\theta} |1, 0\rangle_{AB} + e^{-i\phi} |0, 1\rangle_{AB}) \otimes |\psi(\phi)\rangle_B d\theta d\phi. \quad (\text{B5})$$

Tracing over the shared particle modes and the modes in the truncated phase states, and using an argument similar to that used to derive results (19) and (20) shows that the state of the remaining parts of the system is given by

of the position operator, can never be implemented exactly.

$$\int_{2\pi} \int_{2\pi} \left[ |c(\theta)\rangle_A \otimes |c(\phi)\rangle_B \otimes \frac{(e^{-i\theta} |1, 0\rangle_{AB} + e^{-i\phi} |0, 1\rangle_{AB})}{\sqrt{2}} \right] \left[ {}_A\langle c(\theta)| \otimes {}_B\langle c(\phi)| \otimes \frac{({}_A\langle 1, 0| e^{i\theta} + {}_A\langle 0, 1| e^{i\phi})}{\sqrt{2}} \right] \frac{d\theta d\phi}{2\pi 2\pi}. \quad (\text{B6})$$

The state of the regular qubits after an ideal phase-difference measurement has given the result  $\varphi$  is found by forming the product of  $\hat{\Pi}^{(-)}(\varphi)$  with (B6) and taking the partial trace of the result over the field modes; we find this gives the mixed state

$$\frac{1}{2} \left( |1, 0\rangle\langle 1, 0| + C |1, 0\rangle\langle 0, 1| + C^* |0, 1\rangle\langle 1, 0| + |0, 1\rangle\langle 0, 1| \right)_{AB} \quad (\text{B7})$$

with probability  $1/2\pi$  where

$$C = \int_{2\pi} \int_{2\pi} P_\varphi(\theta, \phi) e^{-i(\theta-\phi)} d\theta d\phi. \quad (\text{B8})$$

Here  $P_\varphi(\theta, \phi)$  represents the resolution of the phase-difference measurement,

$$P_\varphi(\theta, \phi) = \int_{2\pi} P_A(\theta - \theta') P_B(\phi - \varphi - \theta') \frac{d\theta'}{2\pi} \quad (\text{B9})$$

where  $P_Z(\theta)$  is the canonical phase distribution of the state  $|c(0)\rangle_Z$  for a mode at site  $Z \in \{A, B\}$ , i.e.  $P_Z(\theta) = |\sum_n c_n e^{-in\theta}|^2 / 2\pi$ .

## APPENDIX C

We derive here the optimum conditions for maximizing the entanglement of formation (33) using the Heisenberg-Robertson uncertainty relations for particle number and phase operators. We note that the commutator of the

number-difference operator with the cosine of the phase difference operator can be written as

$$[\hat{N}_A - \hat{N}_B, \cos(\hat{\phi}_A - \hat{\phi}_B)] = [\hat{N}_A, \cos(\hat{\phi}_A - \hat{\phi}_B)] - [\hat{N}_B, \cos(\hat{\phi}_A - \hat{\phi}_B)] \quad (\text{C1})$$

where  $\hat{\phi}_Z$  and  $\hat{N}_Z$  are the Pegg-Barnett phase operator[12] and particle number operator, respectively, for site  $Z \in \{A, B\}$ , and

$$\cos(\hat{\phi}_A - \hat{\phi}_B) = \frac{e^{i(\hat{\phi}_A - \hat{\phi}_B)} + e^{-i(\hat{\phi}_A - \hat{\phi}_B)}}{2}. \quad (\text{C2})$$

It is not difficult to show, using the results and methods in Ref. [18] (see, in particular p. 32), that

$$\begin{aligned} \langle \Phi | [\hat{N}_A, \cos(\hat{\phi}_A - \hat{\phi}_B)] | \Phi \rangle &= -\langle \Phi | [\hat{N}_B, \cos(\hat{\phi}_A - \hat{\phi}_B)] | \Phi \rangle \\ &= -i \langle \Phi | \sin(\hat{\phi}_A - \hat{\phi}_B) | \Phi \rangle \end{aligned} \quad (\text{C3})$$

where

$$\sin(\hat{\phi}_A - \hat{\phi}_B) = \frac{e^{i(\hat{\phi}_A - \hat{\phi}_B)} - e^{-i(\hat{\phi}_A - \hat{\phi}_B)}}{2i} \quad (\text{C4})$$

and so

$$\langle \Phi | [\hat{N}_A - \hat{N}_B, \cos(\hat{\phi}_A - \hat{\phi}_B)] | \Phi \rangle = -2i \langle \Phi | \sin(\hat{\phi}_A - \hat{\phi}_B) | \Phi \rangle \quad (\text{C5})$$

where  $|\Phi\rangle$  is a physical state.[12, 18] Similarly

$$\langle \Phi | [\hat{N}_A - \hat{N}_B, \sin(\hat{\phi}_A - \hat{\phi}_B)] | \Phi \rangle = 2i \langle \Phi | \cos(\hat{\phi}_A - \hat{\phi}_B) | \Phi \rangle. \quad (\text{C6})$$

Hence, from Robertson's uncertainty relation[19] we find

$$\langle \Delta^2(\hat{N}_A - \hat{N}_B) \rangle \langle \Delta^2 \cos(\hat{\phi}_A - \hat{\phi}_B) \rangle \geq |\langle \sin(\hat{\phi}_A - \hat{\phi}_B) \rangle|^2, \quad (C7)$$

$$\langle \Delta^2(\hat{N}_A - \hat{N}_B) \rangle \langle \Delta^2 \sin(\hat{\phi}_A - \hat{\phi}_B) \rangle \geq |\langle \cos(\hat{\phi}_A - \hat{\phi}_B) \rangle|^2 \quad (C8)$$

for physical states, where  $\langle \Delta^2 \hat{Q} \rangle = \langle \hat{Q}^2 \rangle - \langle \hat{Q} \rangle^2$  is the variance in  $\hat{Q}$ . Adding these inequalities and using

$$\begin{aligned} & \langle \Delta^2 \cos(\hat{\phi}_A - \hat{\phi}_B) \rangle + \langle \Delta^2 \sin(\hat{\phi}_A - \hat{\phi}_B) \rangle \\ &= 1 - \langle \cos(\hat{\phi}_A - \hat{\phi}_B) \rangle^2 - \langle \sin(\hat{\phi}_A - \hat{\phi}_B) \rangle^2 \\ &= 1 - |\langle e^{i(\hat{\phi}_A - \hat{\phi}_B)} \rangle|^2 \end{aligned} \quad (C9)$$

gives

$$\langle \Delta^2(\hat{N}_A - \hat{N}_B) \rangle (1 - |C|^2) \geq |C|^2 \quad (C10)$$

where, from (31),

$$|C|^2 = |\langle e^{i(\hat{\phi}_A - \hat{\phi}_B)} \rangle|^2. \quad (C11)$$

Rearranging (C10) and using  $\langle \Delta^2(\hat{N}_A - \hat{N}_B) \rangle = \langle \Delta^2 \hat{N}_A \rangle + \langle \Delta^2 \hat{N}_B \rangle$  for uncorrelated fields gives

$$|C|^2 \leq \frac{\langle \Delta^2 \hat{N}_A \rangle + \langle \Delta^2 \hat{N}_B \rangle}{1 + \langle \Delta^2 \hat{N}_A \rangle + \langle \Delta^2 \hat{N}_B \rangle}. \quad (C12)$$

In a similar way, we derive the Heisenberg-Robertson uncertainty relations:

$$\langle \Delta^2 \hat{N}_Z \rangle \langle \Delta^2 \cos(\hat{\phi}_A - \hat{\phi}_B) \rangle \geq \frac{1}{4} |\langle \sin(\hat{\phi}_A - \hat{\phi}_B) \rangle|^2,$$

$$(C13)$$

$$\langle \Delta^2 \hat{N}_Z \rangle \langle \Delta^2 \sin(\hat{\phi}_A - \hat{\phi}_B) \rangle \geq \frac{1}{4} |\langle \cos(\hat{\phi}_A - \hat{\phi}_B) \rangle|^2, \quad (C14)$$

using the separate commutators of  $\hat{N}_A$  and  $\hat{N}_B$  with  $\cos(\hat{\phi}_A - \hat{\phi}_B)$  and  $\sin(\hat{\phi}_A - \hat{\phi}_B)$ , and then using (C13) and (C14) in place of (C7) and (C8) we find that

$$|C|^2 \leq \frac{4 \langle \Delta^2 \hat{N}_Z \rangle}{1 + 4 \langle \Delta^2 \hat{N}_Z \rangle} \quad (C15)$$

for site  $Z \in \{A, B\}$ .

We want to derive the conditions for the optimum situation where the variance in the number of particles transported is the minimum for a given value of  $E_F$ . Consider the case where Alice transports particles to Bob. The entanglement of formation  $E_F$  given by (33) increases monotonically with  $|C|^2$ . Comparing (C12) and (C15), we see that the optimum situation occurs when  $\langle \Delta^2 \hat{N}_B \rangle \geq 3 \langle \Delta^2 \hat{N}_A \rangle$  for which the bound on  $|C|^2$  is given by (C15) with  $Z$  being the transported mode (i.e. A in this case).

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