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Regularized solutions for terminal problems of parabolic equations.

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REGULARIZED SOLUTIONS FOR TERMINAL PROBLEMS OF PARABOLIC EQUATIONS

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DEDICATION

To my Family, my wife Sandamala Hettigoda, my daughter Viyathma, and my son Vethum Hapuarachchi

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ABSTRACT

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Sujeewa Indika Hapuarachchi

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The heat equation with a terminal condition problem is not well-posed in the sense of Hadamard so regularization is needed. In general, partial differential equations (PDE) with terminal conditions are those in which the solution depends uniquely but not continuously on the given condition. In this dissertation, we explore how to find an approximation problem for a nonlinear heat equation which is well-posed. By using a small parameter, we construct an approximation problem and use a modified quasi-boundary value method to regularize a time dependent thermal conductivity heat equation and a quasi-boundary value method to regularize a space dependent thermal conductivity heat equation. Finally we prove, in both cases, the approximation solution converges to the original solution whenever the parameter goes to zero.

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CHAPTER 1
INTRODUCTION

A homogenous heat equation with an initial condition/boundary condition is defined as

$$\left.\begin{aligned} u_t(x,t) = ku_{xx}(x,t) \quad x \in (0,l), t > 0 \\ u(x,0) = g(x) \quad x \in (0,l) \\ u \text{ satisfies certain BC's} \end{aligned}\right\} \tag{1.1}$$

The most common boundary conditions for the heat equation are

- Dirichlet condition, i.e. $u(0,t) = u(l,t) = 0$
- Neumann condition, i.e. $u_x(0,t) = u_x(l,t) = 0$

Definition 1.1. *We say a PDE problem is well-posed if it satisfies the following properties:*

- *A solution exists,*
- *The solution is unique,*
- *The solution continuously depends on initial conditions.*

If such a problem is not well-posed, we say it is ill-posed. In general, a heat equation with an initial condition is well-posed. We can define the concrete version of (1.1) as follows:

$$\left.\begin{aligned} u_t(x,t) + kAu(x,t) = 0 \quad x \in (0,l), t > 0 \\ u(x,0) = g(x) \quad x \in (0,l) \\ u \text{ satisfies certain BC's} \end{aligned}\right\} \tag{1.2}$$

where A is an unbounded self-adjoint and positive operator on a Hilbert space. This equation is called a parabolic equation with an initial condition.

Consider again the heat equation:

$$\left.\begin{aligned} u_t(x,t) = ku_{xx}(x,t) \quad x \in (0,l), 0 \leq t < T \\ u(x,T) = g(x) \quad x \in (0,l), \\ u \text{ satisfies certain BC's} \end{aligned}\right\} \tag{1.3}$$

We call such a problem a heat equation with a terminal condition or a backward heat equation. Similarly, we can define a solid version of the parabolic equation with a terminal condition as follows:

$$\left.\begin{aligned} u_t(x,t) + kAu(x,t) = f(x,t,u) \quad x \in (0,l), 0 \leq t < T \\ u(x,T) = g(x) \quad x \in (0,l) \\ u \text{ satisfies certain BC's} \end{aligned}\right\} \tag{1.4}$$

where A is an unbounded self-adjoint and positive operator on a Hilbert space.

In this thesis we introduce a better approximation problem to regularize (1.4). In chapter 3 and 4, we consider the heat equation with time and space dependent thermal conductivity, that is $k = a(t)$ and $k = a(x)$ and source function f. For the homogenous parabolic equation with a terminal condition case, i.e $f = 0$ see in [1-6] A heat equation with a terminal condition plays a significant role in the fields of physics and engineering, especially with time and space dependent thermal conductivity. Thermal conductivity is important in material science, electronics, and other related fields. Thermal conductivity depends on time, space or both.

Thermal conductivity can be defined as the amount of heat transmitted through a material which is highly dependent on the material specific property. Therefore, materials with high thermal conductivity, such as diamond, silver, or copper, transfer heats at a higher rate across the material whereas materials with lower thermal conductivity, such as wood, transfer heat at a lower rate.

Since a nonlinear heat equation with a terminal condition is not well-posed, no solution which satisfies the heat conduction equation with final data and the boundary conditions exists. Even if a solution exists, it will not be continuously dependent on the final data and consequently calculation in numerical simulations will be very difficult. Therefore, some special regularization methods are required. The Tikhonov regularization method is one of the most commonly used methods for linear ill-posed problems. The quasi reversibility method, quasi boundary value method, and modified quasi boundary value method are other commonly used methods to regularize nonlinear ill-posed problems. Given an ill-posed problem, it is often convenient to define an approximate problem that is well-posed. Generally, we seek to ensure that a solution to the original problem, if it exists, will be appropriately close to the solution to the approximate problem.

CHAPTER 2
PRELIMINARIES

All preliminary results used in this thesis can be found in references [22, 23] and [24] and some of them will be highlighted in this section.

2.1 Hilbert Spaces

Definition 2.1 (Domain of an Unbounded Operator)**.** *Let B_1 and B_2 be Banach spaces. An unbounded linear operator from B_1 into B_2 is a linear map $A : D(A) \subset B_1 \to B_2$. The linear subspace $D(A)$ is called the domain of A.*

The operator A is bounded if $D(A) = B_1$ and if there is a $c \geq 0$ such that

$$\|Au\| \leq c\|u\| \quad \forall \quad u \in B_1$$

The norm of a bounded operator A is defined by

$$\|A\|_{\mathcal{L}(B_1,B_2)} = \sup_{u \neq 0} \frac{\|Au\|}{\|u\|}$$

Example 2.1. *Let $B_1 = B_2 = L^2(\mathbb{R})$ (Define later). Now consider one dimension Laplace operator, i.e. $Au = -u_{xx}$. Then A is an unbounded operator on $L^2(\mathbb{R})$.*

Example 2.2. *Let $B_1 = B_2 = L^2(0,1)$. Consider the derivative operator defined by $D(A) = C^1(0,1)$ and $Au = \frac{d}{dx}u$ for all $u \in C^1(0,1)$, where $C^1(0,1)$ is the collection of continuously differentiable functions over (0,1). Then A is an unbounded linear operator on $L^2(0,1)$.*

DEFINITION 2.2 (Adjoint of A). *Let B_1 and B_2 be Banach Spaces and $A : D(A) \subset B_1 \to B_2$ be a densely defined unbounded operator. We define*

$$A^* : D(A^*) \subset B_2^* \to B_1^*$$

Where, the domain of A^* is defined as:

$$D(A^*) := \{v \in B_2^* \ \mid \ \exists c \geq 0 \ s.t \ |\langle v, Av \rangle| \leq c\|u\| \quad \forall u \in D(A)\}.$$

Then the unbounded operator $A^* : D(A^*) \subset B_2^* \to B_1^*$ is called the adjoint of A. Mathematically, we say

$$\langle u, Av \rangle_{B_2^*, B_2} = \langle A^* u, v \rangle_{B_1, B_1^*}$$

for all $v \in D(A)$ and $u \in D(A^*)$.

DEFINITION 2.3. *Let H be a Hilbert space. An unbounded operator $A : D(A) \subset H \to H$ is said to be monotone, if*

$$\langle Av, v \rangle \geq 0$$

for all $v \in D(A)$ and is called maximum monotone if $R(I + A) = H$, that is, there is exists $u \in D(A)$ such that $u + Au = f$ for all $f \in H$.

Remark 1: If A is a maximal monotone, then for all $\alpha > 0$, αA is also a maximal monotone operator.

DEFINITION 2.4. *Let A be a maximal monotone operator. For every $\alpha > 0$, set*

$$J_\alpha = (I + \alpha A)^{-1} \quad \text{and} \quad A_\alpha = \frac{1}{\alpha}(I - J_\alpha)$$

J_α is called the resolvent of A and A_α is called regularization of A.

DEFINITION 2.5. • *Symmetric Operator*

A is symmetric in H if $\langle Au, v \rangle = \langle u, Av \rangle$ for $u, v \in D(A)$.

- *Self-adjoint Operator*

  A densely defined symmetric operator A on H is called self-adjoint if $D(A) = D(A^*)$ and $A = A^*$.

- *Positive Operator*

  A is positive if A is self-adjoint and $\langle Au, u\rangle \geq 0$ for $u \in D(A)$.

THEOREM 2.1. *Let A be a positive operator, then eigenvalues of A are positive.*

Proof. If λ is an eigenvalue of A, then there is a $u \in D(A)$ such that

$$Au = \lambda u$$

Since A is positive, we have $0 \leq \langle Au, u\rangle$, then

$$0 \leq \langle Au, u\rangle = \langle \lambda u, u\rangle = \lambda\langle u, u\rangle = \lambda\|u\|^2$$

so $\lambda > 0$ □

Following result is an very important for later to prove uniqueness of parabolic problems. The theorem was proven in [11].

THEOREM 2.2. *We assume that $0 < T < \infty$. Let u satisfy*

$$u \in C([0,T]; H) \cap L^2(0,T; D(A))$$

$$\left.\begin{aligned} u_t + \nu Au \in H \quad &\text{for a.e. } t, \\ |u_t + \nu Au| \leq \eta\|u\| \quad &\text{for a.e. } t, \end{aligned}\right\}$$

where ν is complex number such that $\Re(\nu) > 0$ and

$$\eta \in L^2(0,T)$$

If $u(T) = 0$, then $u(t) \equiv 0$ for $0 \leq t \leq T$.

Note: The notation $u \in L^2(0,T; D(A))$ means that for a.e. t, $u \in D(A)$ and $Au \in L^2(0,T; H)$.

2.2 L^p Spaces

Let $\Omega \in \mathbb{R}^n$ and let p be a positive real number. We denote by $L^p(\Omega)$ the class of all measurable functions f, for which

$$\int_\Omega |f|^p d\mu < \infty$$

The norm of $L^p(\Omega)$ space is, that is $\|\cdot\|_{L^p(\Omega)}$, denoted by

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f|^p d\mu\right)^{\frac{1}{p}}$$

for $1 \leq p < \infty$.

$$\|f\|_{L^\infty} = \operatorname{ess\,sup}|f|$$

is the norm of $L^\infty(\Omega)$.

The case $p = 2$ is very special because it is the $L^2(\Omega)$ space. The norm of the $L^2(\Omega)$ makes inner product space defined by

$$\langle f, g\rangle = \int_\Omega f\bar{g}d\mu$$

Also $L^2(\Omega)$ is a Hilbert space.

Theorem 2.3 (Gronwall's Inequalities)**.** *It has following two forms*

1. *Differential Form*

   Let $u \in C^1([a,\infty))$ *and* $\alpha \in C[a,\infty))$ *be such that*

   $$u'(t) \leq \alpha(t)u(t), \quad \text{for all } \; t > a$$

   Then

   $$u(t) \leq u(a)e^{\int_a^t \alpha(s)ds}$$

2. *Integral Form*

   Let $\beta, u \in C([a,\infty))$, α *is a function on* $[0,\infty)$, *and* $\alpha^-(t) = \max\{-\alpha(t), 0\} \in L^\infty_{loc}([a,\infty))$.

(a) If $\beta \geq 0$, and $u(t) \leq \alpha(t) + \int_a^t \beta(s)u(s)ds$, then

$$u(t) \leq \alpha(t) + \int_a^t \alpha(s)\beta(s)e^{\int_s^t \beta(r)dr}ds \quad \textit{for all} \quad t \geq a$$

(b) If $\beta \geq 0$, α is increasing and $u(t) \leq \alpha(t) + \int_a^t \beta(s)u(s)ds$, then

$$u(t) \leq \alpha(t)e^{\int_a^t \beta(s)ds} \quad \textit{for all} \quad t \geq a$$

In particular, if $\beta \geq 0$ and $\alpha = 0$, then $u(t) \leq 0$ for all $t \geq a$.

Theorem 2.4 (Plancherel equality). *If $f \in L^2(\mathbb{R}^n)$, then $\hat{f} \in L^2(\mathbb{R}^n)$ and $\|f\|_{L^2(\mathbb{R}^n)} = \|\hat{f}\|_{L^2(\mathbb{R}^n)}$, where $\hat{f}$ is the fourier transform of f defined by*

$$\hat{f}(\boldsymbol{\zeta}) := \frac{1}{(\sqrt{2\pi})^n} \int_{\mathbb{R}^n} f(\boldsymbol{x})e^{-i\boldsymbol{\zeta}\cdot\boldsymbol{x}}d\boldsymbol{x}$$

2.3 Sobolev Spaces

These spaces are defined over an arbitrary domain $\Omega \subset \mathbb{R}^n$ and are vector sub spaces of various spaces $L^p(\Omega)$.

Let $\Omega \subset R^n$ and vector sub space $L^p(\Omega)$. Define for any $u \in W^{m,p}(\Omega)$

$$W^{m,p}(\Omega) \equiv \{u \in L^p(\Omega) \ : \ D^\alpha u \in L^p(\Omega) \text{ for } 0 \leq |\alpha| \leq m\}$$

where $D^\alpha u$ is the weak or distributional partial derivative of u. That is for all α with $0 \leq |\alpha| \leq m$, there exists $g_\alpha \in L^p(\Omega)$ such that

$$\int_\Omega uD^\alpha\phi = (-1)^{|\alpha|}\int_\Omega g_\alpha\phi$$

for all $\phi \in C_c^\infty(\Omega)$. Here α is standard multi-index notation. That is, If $\alpha = (\alpha_1, \alpha_2 \cdots \alpha_N)$ with for all i, $\alpha_i \geq 0$ and

$$|\alpha| = \sum_{i=1}^N \alpha_i$$

and

$$D^\alpha \phi = \frac{\partial^{|\alpha|}\phi}{\partial^{\alpha_1}x_1 \partial^{\alpha_2}x_2 \cdots \partial^{\alpha_N}x_N}.$$

We set $D^\alpha u = g_\alpha$.

Definition 2.6. *If $u \in W^{m,p}(\Omega)$, we define its morn to be*

$$\|u\|_{W^{m,p}(\Omega)} = \left(\sum_{0 \le |\alpha| \le m} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}} \quad (0 \le p < \infty)$$

and

$$\|u\|_{W^{m,\infty}(\Omega)} = \sum_{0 \le |\alpha| \le m} \text{ess sup}_\Omega |D^\alpha u| \quad (p = \infty)$$

Definition 2.7. *We denote by $W_0^{m,p}(\Omega)$ the closure of $C_c^\infty(\Omega)$ in $W^{m,p}(\Omega)$.*

- *If $p = 2$ we write $W^{m,2}(\Omega) = H^m(\Omega)$ for $k = 0, 1, 2, \cdots$.*
- *Also we write $W_0^{m,2}(\Omega) = H_0^m(\Omega)$*

We can easily prove that $H^2(\Omega)$ is a Hilbert space and $H^0(\Omega) = L^2(\Omega)$.

We consider an elliptic operator having the divergence from

$$Lu = -\sum_{i,j=1}^{n} \left(a^{ij} u_{x_i}\right)_{x_j}$$

where $a^{ij} \in C^\infty(\overline{\Omega})$ and $i, j = 1, 2, \cdots, n$. Suppose usual uniform ellipticity condition to hold, and usual suppose

$$a^{ij} = a^{ji} \quad i, j = 1, 2, \cdots, n$$

Also suppose L is symmetric and associated bilinear form satisfies $B[u, v] = B[v, u]$ for $u, v \in H_0^1(\Omega)$.

Theorem 2.5. *eigenvalues of symmetric elliptic operator*

1. *Each eigenvalue of L is real.*

2. If we repeat each eigenvalue according to its (finite) multiplicity, we have

$$\sum = \{\lambda_k\}_{k=1}^{\infty}$$

where

$$0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_2 \leq \cdots$$

and

$$\lambda_k \to \infty \quad as \quad k \to \infty$$

Theorem 2.6. $-\Delta$ *is a self-adjoint, positive and unbounded operator in* $L^2(\Omega)$ *and* $H^2(\Omega)$.

Definition 2.8. *The space*

$$L^p(0, T : B)$$

where B *is a real Banach space, consists of all measurable function* $u : [0, T] \to B$ *with*

- *for* $1 \leq p < \infty$,

$$\|u\|_{L^p(0,T,B)} := \left(\int_0^T \|u(t)\|_B^p dt \right)^{\frac{1}{p}} < \infty$$

- *If* $p = \infty$,

$$\|u\|_{L^\infty(0,T,B)} := \text{ess sup}_{0 \leq t \leq T} \|u(t)\|_B < \infty$$

CHAPTER 3
NONLINEAR PARABOLIC EQUATIONS I

3.1 Introduction

Consider the parabolic equation defined on a domain $\Omega \subset \mathbb{R}^n$ and unknown function u is defined in $\Omega \times [0, T]$. In order to find a solution, existence and uniqueness, we need initial/terminal and boundary conditions. The example of nonlinear parabolic equations with constant coefficients is the heat equation, i.e

$$u_t(\boldsymbol{x}, t) - \Delta u(\boldsymbol{x}, t) = f(\boldsymbol{x}, t, u)$$

where, f is a source function. We consider this problem with an initial or a terminal condition, then we say, the problem is a forward or a backward heat equation respectively. In this section we consider a backward heat equation and a better approximation problem to regularize it. One of the main assumptions of the source function is it satisfies either the global Lipschitz continuity or the local Lipschitz continuity with respect to the variable u. That is

$$\|f(\boldsymbol{x}, t, u) - f(\boldsymbol{x}, t, v)\| \leq k\|u - v\|$$

for some constant k. The global Lipschitz continuity is the most common assumption for the source function, see in [7-10, 14, 16-18].

3.2 Nonlinear Parabolic Equation

Consider the concrete version of a nonlinear parabolic equation with a terminal condition. Let H be a Hilbert space. We consider the nonlinear parabolic equation with a terminal condition problem of finding an unknown function $u : [0,T] \to H$ such that

$$\left.\begin{aligned} u_t(t) + Au(t) &= f(t,u) \quad 0 \le t < T \\ u(T) &= g \end{aligned}\right\} \tag{3.1}$$

where f is a source function, $g \in H$ and A is a self-adjoint, unbounded operator on dense space $D(A)$ of H. Then the integral form of equation (3.1) (if it exists) can represent as

$$u(t) = \sum_{n=1}^{\infty} \left(e^{(T-t)\lambda_n} \langle g, \phi_n \rangle - \int_t^T e^{(s-t)\lambda_n} \langle f(s, u(s), \phi_n \rangle ds \right) \phi_n$$

We can see that instability due to the fast growth of $e^{(T-t)\lambda_n}$ and $e^{(s-t)\lambda_n}$ as $n \to \infty$. Hence, regularization methods are necessary to make numerical computation possible.

That is, We need to replace the terms $e^{(T-t)\lambda_n}$ and $e^{(s-t)\lambda_n}$ by better terms. For given $\epsilon > 0$, we need to replace $e^{(T-t)\lambda_n}$ and $e^{(s-t)\lambda_n}$ by $L(\epsilon, T, t, \lambda_n)$ and $S(\epsilon, t, s, \lambda_n)$ such that $|L(\epsilon, T, t, \lambda_n)| \le D_\epsilon$, $|S(\epsilon, t, s, \lambda_n)| \le D_\epsilon$ and $\lim_{\epsilon \to 0^+} L(\epsilon, T, t, \lambda_n) = e^{(T-t)\lambda_n}$, $\lim_{\epsilon \to 0^+} S(\epsilon, t, s, \lambda_n) = e^{(s-t)\lambda_n}$.

There are many different kinds of regularization methods for a linear heat equation with a terminal condition. One method is called quasi reversibility. It was introduced by Lattes and Lions [1] for the solution of non well-posed problems. They approximated homogeneous backward heat problem by the equation,

$$\left.\begin{aligned} u_t^\epsilon(t) - Au^\epsilon(t) - \epsilon A^* A u^\epsilon(t) &= 0 \quad 0 \le t < T \\ u^\epsilon(x,T) &= g(x) \end{aligned}\right\}$$

where A is a positive, self-adjoint operator, and A^* is adjoint of operator A. This method gives the stability magnitude is of order $e^{c/\epsilon}$. Here stability magnitude is so large for small ϵ. In [2] Miller introduced a method is called stabilized quasi reversibility and he approximated the problem with

$$\left.\begin{array}{rcl} u_t(t) - f(A)u(t) & = & 0 \quad 0 \leq t \leq T \\ u(x,T) & = & g(x) \end{array}\right\}$$

He showed that the stability magnitude of the method was of order c/ϵ. This stability magnitude is smaller than stability magnitude of method of quasi reversibility for small ϵ. In [3], Showalter approximated homogenous case of 3.1 with

$$\left.\begin{array}{rcl} u_t^\epsilon(t) - \epsilon A u_t^\epsilon(t) + Au^\epsilon(t) & = & 0 \quad 0 < t < T \\ u^\epsilon(x,T) & = & g(x) \end{array}\right\}$$

Also Showalter [4] introduced a more general problem in a different way. He approximated the problem

$$\left.\begin{array}{rcl} u(t) + Au_t(t) - Bu(t) & = & 0 \quad 0 < t < T \\ u(x,0) & = & g(x) \end{array}\right\}$$

with

$$\left.\begin{array}{rcl} u_t^\epsilon(t) + Au^\epsilon(t) - B_\epsilon u^\epsilon(t) & = & 0 \quad 0 < t < T \\ u^\epsilon(x,0) + \epsilon u^\epsilon(x,T) & = & g(x) \end{array}\right\}$$

Here A and B denote self-adjoint, non-negative, unbounded operators on a Hilbert space and their resolvents are commute and B_ϵ is the Yosida approximation of B. He calls this the quasi-boundary value method. Also he shows it gives a better approximation than many other quasi reversibility type methods. Later, G.W. Clark and S.F. Oppenheimer [5] applied this quasi-boundary value method for backward heat equation approximated with

$$\left.\begin{array}{rcl} u_t(t) - Au^\epsilon(t) & = & 0 \quad 0 < t < T \\ \epsilon u^\epsilon(x,0) + u^\epsilon(x,T) & = & g(x) \end{array}\right\}$$

This method has a huge advantage. The foremost advantage of this method is we do not need to consider a forward case. More importantly, the error introduced by small changes in the final value g is not exponential, but of the order $1/\epsilon$. Later in [6] M. Denche, K. Bessila applied quasi-boundary value method approximated with

$$\left.\begin{array}{rcl} u_t^\epsilon(t) - Au^\epsilon(t) & = & 0 \quad 0 < t < T \\ -\epsilon u_t^\epsilon(x,0) + u^\epsilon(x,T) & = & g(x) \end{array}\right\}$$

The mix boundary condition in [5] and [6] are very important boundary conditions to solve the nonlinear parabolic equation with a terminal condition.

If $H = L^2(0,l)$ for $l > 0$, $A = -\Delta$, and $f(t,u) = u\|u\|^2_{L^2(0,l)}$, problem (3.1) is given by:

$$\left.\begin{array}{rcl} u_t(x,t) - \Delta u(x,t) & = & u(x,t)\|u(\cdot,t)\|^2_{L^2(0,l)}, \quad (x,t) \in (0,l) \times (0,1) \\ u(0,t) = u(l,t) & = & 0, \quad t \in (0,1) \\ u(x,1) & = & g(x), \quad x \in (0,l) \end{array}\right\} \tag{3.2}$$

We call (3.2) a semilinear heat equation with cubic type nonlinearity. It has many applications in computational neuroscience and occurs in neurophysiological modeling of large nerve cell systems in mathematical biology in [19].

The source function $f(t,u) = u\|u\|^2_{L^2(0,l)}$ satisfies the following properties:

i For each $p > 0$, there exists a constant $K_p >$ such that $f : \mathbb{R} \times H \to H$ satisfies a local Lipschitz condition

$$\|f(t,u) - f(t,v)\| \leq K_p \|u - v\|$$

for every $u, v \in H$ such that $\|u\|, \|v\| \leq p$.

ii There exists a constant $L \geq 0$ such that

$$\langle f(t,u) - f(t,v), u - v\rangle + L\|u - v\|^2 \geq 0$$

iii $f(t,0) = 0$ for $t \in [0,1]$

The last two properties are additional conditions for the source function. N. H. Tuan, D. D. Trong in [13], they have assumed the above properties satisfy the source function f in (3.1).

Let A admit an orthogonal eigenbasis $\{\phi_k\}$ on H and corresponding eigenvalues $\{\lambda_k\}$ of A. Consider the approximation problem for (3.1) is

$$\left.\begin{array}{rcl} u_t^\epsilon(t) + A_\epsilon u^\epsilon(t) &=& f(t, u^\epsilon) \quad 0 \le t < 1 \\ u^\epsilon(1) &=& g \end{array}\right\} \tag{3.3}$$

For $u \in H$ having the expansion

$$u = \sum_{k=1}^{n} \langle u, \phi_k \rangle \phi_k$$

as defined

$$A_\epsilon(u) = \sum_{k=1}^{n} \ln^+ \left(\frac{1}{\epsilon\lambda_k + e^{-\lambda_k}} \right) \langle u, \phi_k \rangle \phi_k$$

where $\ln^+(x) = \max\{\ln(x), 0\}$. Then the solution of (3.3) converges to the solution of (3.1), see [13]

The disadvantage of the above problem is the source function needed to satisfy the above three conditions, but not all source functions satisfy these three conditions. For example $f(u) = au - bu^3 (b > 0)$ of the Ginzburg-Landau equation. Because of that in [15], D.D.Trong, B.T. Duy, M.N. Minh, they have introduced another condition for f that satisfies as follows, i.e. assume $K_M < \infty$, where

$$K_M := \sup \left\{ \left| \frac{f(\boldsymbol{x}, t, u) - f(\boldsymbol{x}, t, v)}{u - v} \right| \ : \ |u|, |v| \le M, \ \ u \neq v, \ \ (\boldsymbol{x}, t) \in \mathbb{R}^n \times [0, T] \right\}$$

It is clear that K_M is a non-decreasing function of M and

$$\|f(\boldsymbol{x}, t, u) - f(\boldsymbol{x}, t, v)\| \le K_M \|u - v\|$$

for every $M > 0$, $|u|, |v| \le M$ and $(\boldsymbol{x}, t) \in \mathbb{R}^n \times [0, T]$. So f is a local Lipschitz with respect to the variable u. Suppose $\lim_{M\to\infty} K_M = \infty$. To construct the regularization for (3.3), they approximate the function f such that

$$f_M(\boldsymbol{x}, t, u) = \begin{cases} f(\boldsymbol{x}, t, M) & \text{for} \quad u > M \\ f(\boldsymbol{x}, t, u) & \text{for} \quad -M \leq u(\boldsymbol{x}, t) \leq M \\ f(x, t, -M) & \text{for} \quad u < -M \end{cases} \tag{3.4}$$

for $M > 0$.

see also [12, 20] for the local Lipschitz continuous source functions. In [8], Trong, Quan consider a backward heat equation with time-dependent thermal conductivity and 1-dimension space.

Here we consider the time dependent thermal conductivity in the n-dimension space and $H = L^2(\mathbb{R}^n)$.

Let T be a positive number and $f : \mathbb{R}^n \times [0, T] \times \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function such that $f(x, t, 0) = 0$ and $f(x, 0, u) = 0$. Now consider the following parabolic equation:

$$\left.\begin{array}{rcl} u_t(\boldsymbol{x}, t) - a(t)\Delta u(\boldsymbol{x}, t) & = & f(\boldsymbol{x}, t, u(x, t)) \quad 0 \leq t < T, \quad x \in \mathbb{R}^n \\ u(\boldsymbol{x}, T) & = & g(\boldsymbol{x}) \quad \boldsymbol{x} \in \mathbb{R}^n \end{array}\right\} \tag{3.5}$$

where $0 < \delta \leq a(t) \in C([0, T], \mathbb{R})$, δ is a constant, is an increasing function, and is called thermal conductivity and $g \in L^2(\mathbb{R}^n)$. We need to find the solution $u(\boldsymbol{x}, t)$ such that $u : \mathbb{R}^n \times [0, T] \to \mathbb{R}$. The solution representation of (3.5) gives by the $n-$dimension Fourier transform form;

$$\hat{u}(\boldsymbol{\zeta}, t) = e^{|\boldsymbol{\zeta}|^2(\lambda(T)-\lambda(t))}\hat{g}(\boldsymbol{\zeta}) - \int_t^T e^{|\boldsymbol{\zeta}|^2(\lambda(s)-\lambda(t))}\hat{f}_u(\boldsymbol{\zeta}, s)ds \tag{3.6}$$

where

$$\hat{u}(\boldsymbol{\zeta}, t) = \frac{1}{(\sqrt{2\pi})^n}\int_{\mathbb{R}^n} u(\boldsymbol{x}, t)e^{-i\boldsymbol{\zeta}\cdot\boldsymbol{x}}d\boldsymbol{x}$$

and

$$\lambda(t) = \int_0^t a(s)ds$$

Here we define $\boldsymbol{x} = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$, $\boldsymbol{\zeta} = (\zeta_1, \zeta_2, \cdots, \zeta_n) \in \mathbb{R}^n$, $\boldsymbol{x} \cdot \boldsymbol{y}$ is the scalar product of $\boldsymbol{x}$ and $\boldsymbol{y}$ and $f(\boldsymbol{x}, t, u(\boldsymbol{x}, t)) = f_u(\boldsymbol{x}, t)$. This problem is called the backward heat equation with a time dependent variable coefficient and such a problem is not well-posed because of Hadamard. That is, there is no solution or even it has a unique solution on $[0, T]$ it does not depend continuously on the final value of g.

EXAMPLE 3.1. *Suppose $n = 1$. If u is the solution of (3.5) with $u(x, T) = g(x)$, where $g \in L^2(\mathbb{R})$ such that $f(x, t, u) = \frac{u}{1+u^2}(e^t - 1)$. Clearly $f(x, t, 0) = 0$ and $f(x, 0, u) = 0$. Also f is a Lipschitz continuous function, because, for any $u, v \in L^2(\mathbb{R})$*

$$\begin{aligned} f(x,t,u) - f(x,t,v) &= (e^t - 1)\left(\frac{u}{1+u^2} - \frac{v}{1+v^2}\right) \\ &= (e^t - 1)\left(\frac{u + uv^2 - v - vu^2}{(1+u^2)(1+v^2)}\right) \\ &= (e^t - 1)\left(\frac{(u-v)(1-uv)}{(1+u^2)(1+v^2)}\right) \\ &= (e^t - 1)(u - v)\left(\frac{1}{(1+u^2)(1+v^2)} - \frac{uv}{(1+u^2)(1+v^2)})\right) \end{aligned}$$

then

$$\begin{aligned} |f(x,t,u) - f(x,t,v)|^2 &= (e^t - 1)^2 |u - v|^2 \left|\frac{1}{(1+u^2)(1+v^2)} - \frac{uv}{(1+u^2)(1+v^2)}\right|^2 \\ &\leq (e^t - 1)^2 |u - v|^2 \left|\frac{1}{(1+u^2)(1+v^2)} + \frac{uv}{(1+u^2)(1+v^2)}\right|^2 \\ &\leq (e^t - 1)^2 |u - v|^2 \left|1 + \frac{uv}{(1+u^2)(1+v^2)}\right|^2 \\ &\leq (e^t - 1)^2 |u - v|^2 \left(1 + \frac{1}{4}\right)^2 \\ &= \frac{25}{16}(e^t - 1)^2 |u - v|^2 \\ &= \frac{25}{16}(e^T - 1)^2 |u - v|^2 \end{aligned}$$

That is,

$$\|f(\cdot,t,u) - f(\cdot,t,v)\|_{L^2(\mathbb{R})} \leq k\|u(\cdot,t) - v(\cdot,t)\|_{L^2(\mathbb{R})}$$

where $k = \frac{5}{4}(e^T - 1)$. *Hence* f *is a Lipschitz continuous function.*

Now we regularize the following approximation problem with (3.5). Here we use a modified quasi-boundary value method to regularize. A quasi boundary value method is the most common method to regularize nonlinear parabolic equations, see in [8,9, 10, 15,16,17].

3.3 Approximation Problem

Since system (3.5) imposes us to consider regularization, we need to develop a better approximation problem for (3.5). Now, we consider the following approximation problem:

$$\left.\begin{array}{rcl} u_t^\epsilon - a(t)\Delta u^\epsilon &=& H(\boldsymbol{x},t,u^\epsilon) \quad 0 \leq t < T \\ -\epsilon u_t^\epsilon(0) + u^\epsilon(T) &=& g^\epsilon(\boldsymbol{x}) \end{array}\right\} \tag{3.7}$$

where

$$\hat{H_{u^\epsilon}}(\boldsymbol{\zeta},t) = \frac{e^{-|\boldsymbol{\zeta}|^2\lambda(t)}}{\epsilon^{\lambda(t)/\lambda(T)} + e^{-|\boldsymbol{\zeta}|^2\lambda(t)}}\hat{f_{u^\epsilon}}(\boldsymbol{\zeta},t)$$

and

$$\hat{g^\epsilon}(\boldsymbol{\zeta}) = \hat{g}(\boldsymbol{\zeta}) - \int_0^T \frac{\epsilon|\boldsymbol{\zeta}|^2 a(0)}{\epsilon^{\lambda(s)/\lambda(T)} + e^{-|\boldsymbol{\zeta}|^2\lambda(s)}}\hat{f_{u^\epsilon}}(\boldsymbol{\zeta},s)ds$$

Then the solution representation of system (3.7) is given by:

$$\hat{u^\epsilon}(\boldsymbol{\zeta},t) = \frac{e^{-|\boldsymbol{\zeta}|^2\lambda(t)}}{\epsilon|\boldsymbol{\zeta}|^2 a(0) + e^{-|\boldsymbol{\zeta}|^2\lambda(T)}}\hat{g}(\boldsymbol{\zeta}) - \int_t^T \frac{e^{-|\boldsymbol{\zeta}|^2\lambda(t)}}{\epsilon^{\lambda(s)/\lambda(T)} + e^{-|\boldsymbol{\zeta}|^2\lambda(s)}}\hat{f_{u^\epsilon}}(\boldsymbol{\zeta},s)ds \tag{3.8}$$

or

$$\begin{aligned} u^\epsilon(\boldsymbol{x},t) = \int_{\mathbb{R}^n} e^{i\boldsymbol{x}\cdot\boldsymbol{\zeta}} \Bigg(&\frac{e^{-|\boldsymbol{\zeta}|^2\lambda(t)}}{\epsilon|\boldsymbol{\zeta}|^2 a(0) + e^{-|\boldsymbol{\zeta}|^2\lambda(T)}}\hat{g}(\boldsymbol{\zeta}) \\ &- \int_t^T \frac{e^{-|\boldsymbol{\zeta}|^2\lambda(t)}}{\epsilon^{\lambda(s)/\lambda(T)} + e^{-|\boldsymbol{\zeta}|^2\lambda(s)}}\hat{f_{u^\epsilon}}(\boldsymbol{\zeta},s)ds\Bigg) d\boldsymbol{\zeta} \end{aligned} \tag{3.9}$$

Now we have the approximation problem and its solution representation. Our main goal is to show this approximation problem is well-posed and its solution converges to the solution of (3.5) whenever ϵ approaches zero. Before that we want to prove (3.8) is a solution representation of the system (3.7). To prove this, first consider (3.7) and differentiate it with respect to t. Then we have

$$\hat{u_t^\epsilon}(\boldsymbol{\zeta},t) = -\frac{|\boldsymbol{\zeta}|^2 a(t)e^{-|\boldsymbol{\zeta}|^2\lambda(t)}}{\epsilon|\boldsymbol{\zeta}|^2 a(0)+e^{-|\boldsymbol{\zeta}|^2\lambda(T)}}\hat{g}(\boldsymbol{\zeta}) \quad + \int_t^T \frac{|\boldsymbol{\zeta}|^2 a(t)e^{-|\boldsymbol{\zeta}|^2\lambda(t)}}{\epsilon^{\lambda(s)/\lambda(T)}+e^{-|\boldsymbol{\zeta}|^2\lambda(s)}}\hat{f_{u^\epsilon}}(\boldsymbol{\zeta},s)ds + \frac{e^{-|\boldsymbol{\zeta}|^2\lambda(t)}}{\epsilon^{\lambda(t)/\lambda(T)}+e^{-|\boldsymbol{\zeta}|^2\lambda(t)}}\hat{f_{u^\epsilon}}(\boldsymbol{\zeta},t) \tag{3.10}$$

Now consider $\hat{u_t^\epsilon}(\boldsymbol{\zeta},t)+a(t)|\boldsymbol{\zeta}|^2\hat{u}(\boldsymbol{\zeta},t)$, then we have

$$\hat{u_t^\epsilon}(\boldsymbol{\zeta},t)+a(t)|\boldsymbol{\zeta}|^2\hat{u}(\boldsymbol{\zeta},t) = \frac{e^{-|\boldsymbol{\zeta}|^2\lambda(t)}}{\epsilon^{\lambda(t)/\lambda(T)}+e^{-|\boldsymbol{\zeta}|^2\lambda(t)}}\hat{f_{u^\epsilon}}(\boldsymbol{\zeta},t)$$

and

$$\hat{u_t^\epsilon}(\boldsymbol{\zeta},t)+a(t)|\boldsymbol{\zeta}|^2\hat{u}(\boldsymbol{\zeta},t) = \hat{H}(\boldsymbol{\zeta},t)$$

Now take the inverse Fourier transform and get $u_t^\epsilon - a(t)\Delta u^\epsilon = H(\boldsymbol{x},t,u)$. Also consider the quasi-boundary condition, equation (3.10) gives

$$\hat{u_t^\epsilon}(\boldsymbol{\zeta},0) = -\frac{|\boldsymbol{\zeta}|^2 a(t)e^{-|\boldsymbol{\zeta}|^2\lambda(0)}}{\epsilon|\boldsymbol{\zeta}|^2 a(0)+e^{-|\boldsymbol{\zeta}|^2\lambda(T)}}\hat{g}(\boldsymbol{\zeta}) \quad + \int_0^T \frac{|\boldsymbol{\zeta}|^2 a(0)e^{-|\boldsymbol{\zeta}|^2\lambda(0)}}{\epsilon^{\lambda(s)/\lambda(T)}+e^{-|\boldsymbol{\zeta}|^2\lambda(s)}}\hat{f_{u^\epsilon}}(\boldsymbol{\zeta},s)ds + \frac{e^{-|\boldsymbol{\zeta}|^2\lambda(0)}}{\epsilon^{\lambda(0)/\lambda(T)}+e^{-|\boldsymbol{\zeta}|^2\lambda(0)}}\hat{f_{u^\epsilon}}(\boldsymbol{\zeta},0)$$

since

$$\lambda(t) = \int_0^t a(s)ds$$

implies $\lambda(0)=0$. Also $f(\boldsymbol{x},0,u^\epsilon) = \hat{f_{u^\epsilon}}(\boldsymbol{\zeta},0) = 0$ implies

$$\hat{u_t^\epsilon}(\boldsymbol{\zeta},0) = -\frac{|\boldsymbol{\zeta}|^2 a(t)}{\epsilon|\boldsymbol{\zeta}|^2 a(0)+e^{-|\boldsymbol{\zeta}|^2\lambda(T)}}\hat{g}(\boldsymbol{\zeta}) \quad + \int_0^T \frac{|\boldsymbol{\zeta}|^2 a(0)}{\epsilon^{\lambda(s)/\lambda(T)}+e^{-|\boldsymbol{\zeta}|^2\lambda(s)}}\hat{f_{u^\epsilon}}(\boldsymbol{\zeta},s)ds$$

and by (3.8) gives

$$\hat{u^\epsilon}(\boldsymbol{\zeta},T) = \frac{e^{-|\boldsymbol{\zeta}|^2\lambda(T)}}{\epsilon|\boldsymbol{\zeta}|^2 a(0)+e^{-|\boldsymbol{\zeta}|^2\lambda(T)}}\hat{g}(\boldsymbol{\zeta}) \tag{3.11}$$

then

$$\hat{u^\epsilon}(\boldsymbol{\zeta},T)-\epsilon\hat{u_t^\epsilon}(\boldsymbol{\zeta},0)=\frac{e^{-|\boldsymbol{\zeta}|^2\lambda(T)}}{\epsilon|\boldsymbol{\zeta}|^2a(0)+e^{-|\boldsymbol{\zeta}|^2\lambda(T)}}\hat{g}(\boldsymbol{\zeta})+\frac{\epsilon|\boldsymbol{\zeta}|^2a(t)}{\epsilon|\boldsymbol{\zeta}|^2a(0)+e^{|\boldsymbol{\zeta}|^2\lambda(T)}}\hat{g}(\boldsymbol{\zeta})$$
$$+\int_0^T\frac{\epsilon|\boldsymbol{\zeta}|^2a(0)}{\epsilon^{\lambda(s)/\lambda(T)}+e^{-|\boldsymbol{\zeta}|^2\lambda(s)}}\hat{f_{u^\epsilon}}(\boldsymbol{\zeta},s)ds$$

that is

$$\hat{u^\epsilon}(\boldsymbol{\zeta},T)-\epsilon\hat{u_t^\epsilon}(\boldsymbol{\zeta},0)=\hat{g}(\boldsymbol{\zeta})+\int_0^T\frac{\epsilon|\boldsymbol{\zeta}|^2a(0)}{\epsilon^{\lambda(s)/\lambda(T)}+e^{-|\boldsymbol{\zeta}|^2\lambda(s)}}\hat{f_{u^\epsilon}}(\boldsymbol{\zeta},s)ds$$
$$=\hat{g^\epsilon}(\zeta)$$

Now taking the inverse Fourier transform, we have $-\epsilon u_t^\epsilon(0)+u^\epsilon(T)=g^\epsilon(\boldsymbol{x})$. Hence equation (3.8) is a solution representation of the system (3.7).

Next we need to show the system (3.7) has the solution representation as equation (3.8) or (3.9). To show this, take the Fourier transform for equation (3.7), then we have

$$\left.\begin{array}{rcll}\frac{d\hat{u^\epsilon}}{dt}+a(t)|\boldsymbol{\zeta}|^2\hat{u^\epsilon} & = & \hat{H}(\boldsymbol{\zeta},t,u^\epsilon) & 0\le t<T\\ -\epsilon\hat{u_t^\epsilon}(0)+\hat{u^\epsilon}(T) & = & \hat{g^\epsilon}(\boldsymbol{\zeta}) & \end{array}\right\}\tag{3.12}$$

The above equation we rewrite as

$$\frac{d}{ds}\hat{u^\epsilon}(s)e^{|\boldsymbol{\zeta}|^2\lambda(s)}=\frac{e^{|\boldsymbol{\zeta}|^2\lambda(s)}e^{-|\boldsymbol{\zeta}|^2\lambda(s)}}{\epsilon^{\lambda(s)/\lambda(T)}+e^{-|\boldsymbol{\zeta}|^2\lambda(s)}}\hat{f_{u^\epsilon}}(\boldsymbol{\zeta},s)=\frac{1}{\epsilon^{\lambda(s)/\lambda(T)}+e^{-|\boldsymbol{\zeta}|^2\lambda(s)}}\hat{f_{u^\epsilon}}(\boldsymbol{\zeta},s)$$

Integrating both sides, we have

$$\hat{u^\epsilon}(t)=e^{|\boldsymbol{\zeta}|^2(\lambda(T)-\lambda(t))}\hat{u^\epsilon}(T)-\int_t^T\frac{e^{-|\boldsymbol{\zeta}|^2\lambda(t)}}{\epsilon^{\lambda(s)/\lambda(T)}+e^{-|\boldsymbol{\zeta}|^2\lambda(s)}}\hat{f_{u^\epsilon}}(\boldsymbol{\zeta},s)ds$$

To find the $\hat{u^\epsilon}(T)$ we can apply the given condition

$$-\epsilon\hat{u_t^\epsilon}(0)=-\epsilon|\boldsymbol{\zeta}|^2a(0)e^{|\boldsymbol{\zeta}|^2\lambda(T)}\hat{u^\epsilon}(T)-\int_t^T\frac{\epsilon|\boldsymbol{\zeta}|^2a(0)}{\epsilon^{\lambda(s)/\lambda(T)}+e^{-|\boldsymbol{\zeta}|^2\lambda(s)}}\hat{f_{u^\epsilon}}(\boldsymbol{\zeta},s)ds$$

then

$$\hat{u^\epsilon}(T)-\epsilon\hat{u_t^\epsilon}(0)=\hat{u^\epsilon}(T)+\epsilon|\boldsymbol{\zeta}|^2a(0)e^{|\boldsymbol{\zeta}|^2\lambda(T)}\hat{u^\epsilon}(T)$$
$$-\int_t^T\frac{\epsilon|\boldsymbol{\zeta}|^2a(0)}{\epsilon^{\lambda(s)/\lambda(T)}+e^{-|\boldsymbol{\zeta}|^2\lambda(s)}}\hat{f_{u^\epsilon}}(\boldsymbol{\zeta},s)ds$$
$$=\hat{u^\epsilon}(T)e^{|\boldsymbol{\zeta}|^2\lambda(T)}(\epsilon|\boldsymbol{\zeta}|^2a(0)+e^{-|\boldsymbol{\zeta}|^2\lambda(T)})$$
$$-\int_t^T\frac{\epsilon|\boldsymbol{\zeta}|^2a(0)}{\epsilon^{\lambda(s)/\lambda(T)}+e^{-|\boldsymbol{\zeta}|^2\lambda(s)}}\hat{f_{u^\epsilon}}(\boldsymbol{\zeta},s)ds$$

since

$$\hat{u^\epsilon}(T) - \epsilon \hat{u_t^\epsilon}(0) = \hat{g^\epsilon}(\boldsymbol{\zeta}) = \hat{g}(\boldsymbol{\zeta}) - \int_0^T \frac{\epsilon|\boldsymbol{\zeta}|^2 a(0)}{\epsilon^{\lambda(s)/\lambda(T)} + e^{-|\boldsymbol{\zeta}|^2\lambda(s)}} \hat{f_{u^\epsilon}}(\boldsymbol{\zeta}, s)ds$$

implies

$$\hat{u^\epsilon}(T) = \frac{e^{|\boldsymbol{\zeta}|^2\lambda(T)}\hat{g}(\boldsymbol{\zeta})}{\epsilon|\zeta|^2 a(0) + e^{-|\boldsymbol{\zeta}|^2\lambda(T)}}.$$

therefore,

$$\hat{u^\epsilon}(t) = \frac{e^{-|\boldsymbol{\zeta}|^2\lambda(t)}\hat{g}(\boldsymbol{\zeta})}{\epsilon|\boldsymbol{\zeta}|^2 a(0) + e^{-|\boldsymbol{\zeta}|^2\lambda(T)}} - \int_t^T \frac{e^{-|\boldsymbol{\zeta}|^2\lambda(t)}}{\epsilon^{\lambda(s)/\lambda(T)} + e^{-|\boldsymbol{\zeta}|^2\lambda(s)}} \hat{f_{u^\epsilon}}(\boldsymbol{\zeta}, s)ds$$

This is the solution representation of (3.7) and taking the inverse Fourier transform we have (3.9).

3.4 Existence, Uniqueness and Stability of the Problem

In this section, we discuss the uniqueness and stability of the approximation problem (3.7). First, we discuss the existence and uniqueness results of solution of (3.7). The existence and uniqueness of the problem (3.5) gives the following theorem.

Theorem 3.1. *If $a(t) \geq \delta > 0$ for all $t \in [0, T]$, where δ is constant, and $g \in L^2(\mathbb{R}^n)$ and $f : \mathbb{R}^n \times [0, T] \times \mathbb{R}$ be a Lipschitz continuous function, then system (3.5) has, at most, one solution*

$$u \in C([0, T]; L^2(\mathbb{R}^n)) \cap L^2(0, T; H^2(\mathbb{R}^n)) \cap C^1((0, T); L^2(\mathbb{R}^n)).$$

Proof. Suppose system (3.5) has two solutions u and v such that

$$\left.\begin{aligned} u_t(\boldsymbol{x}, t) - a(t)\Delta u(\boldsymbol{x}, t) &= f(\boldsymbol{x}, t, u(\boldsymbol{x}, t)) \quad 0 \leq t < T \\ u(\boldsymbol{x}, T) &= g(\boldsymbol{x}) \end{aligned}\right\}$$

and

$$\left.\begin{aligned} v_t(\boldsymbol{x}, t) - a(t)\Delta v(\boldsymbol{x}, t) &= f(\boldsymbol{x}, t, v(\boldsymbol{x}, t)) \quad 0 \leq t < T \\ v(\boldsymbol{x}, T) &= g(\boldsymbol{x}) \end{aligned}\right\}$$

Define $w(\boldsymbol{x},t) = u(\boldsymbol{x},t) - v(\boldsymbol{x},t)$, then $w_t(\boldsymbol{x},t) = u_t(\boldsymbol{x},t) - v_t(\boldsymbol{x},t)$ and $\Delta w(\boldsymbol{x},t) = \Delta u(\boldsymbol{x},t) - \Delta v(\boldsymbol{x},t)$. Also $w(\boldsymbol{x},T) = 0$. Now we have

$$\left.\begin{array}{rcl} w_t(\boldsymbol{x},t) - a(t)\Delta w(\boldsymbol{x},t) &=& f(\boldsymbol{x},t,u(\boldsymbol{x},t)) - f(\boldsymbol{x},t,v(\boldsymbol{x},t)) \quad 0 \le t < T \\ w(\boldsymbol{x},T) &=& 0 \end{array}\right\}$$

Then

$$\begin{aligned} |w_t(\boldsymbol{x},t) - a(t)\Delta w(\boldsymbol{x},t)| &= |f(\boldsymbol{x},t,u(\boldsymbol{x},t)) - f(\boldsymbol{x},t,v(\boldsymbol{x},t))| \\ &\le k|u(\boldsymbol{x},t) - v(\boldsymbol{x},t)| \\ &= k|w(\boldsymbol{x},t)| \end{aligned}$$

By theorem 1.1 in [11], we have $w(\boldsymbol{x},t) = 0$ and system (3.5) has a unique solution. Now we assume (3.5) has a unique solution in $L^2(\mathbb{R}^n)$ with given condition $g \in L^2(\mathbb{R}^n)$. □

THEOREM 3.2. *Put*

$$\begin{aligned} B(u^\epsilon)(\boldsymbol{x},t) = &\frac{1}{(\sqrt{2\pi})^n}\int_{\mathbb{R}^n} \frac{e^{ix\cdot\boldsymbol{\zeta}}e^{-|\boldsymbol{\zeta}|^2\lambda(t)}}{\epsilon|\boldsymbol{\zeta}|^2 a(0) + e^{-|\boldsymbol{\zeta}|^2\lambda(T)}}\hat{g}(\boldsymbol{\zeta})d\boldsymbol{\zeta} \\ &- \frac{1}{(\sqrt{2\pi})^n}\int_{\mathbb{R}^n}\int_t^T \frac{e^{i\boldsymbol{x}\cdot\boldsymbol{\zeta}}e^{-|\boldsymbol{\zeta}|^2\lambda(t)}}{\epsilon^{\frac{\lambda(s)}{\lambda(T)}} + e^{-|\boldsymbol{\zeta}|^2\lambda(s)}}\hat{f_{u^\epsilon}}(\boldsymbol{\zeta},s)dsd\boldsymbol{\zeta} \end{aligned} \tag{3.13}$$

Then for $u, v \in C([0,T]; L^2(\mathbb{R}^n))$ and $n \ge 1$ we have

$$\|B^n(u)(\cdot,t) - B^n(v)(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} \le \left(\frac{k}{\epsilon}\right)^{2n}\frac{(T-t)^n C^n}{n!}|||u-v|||^2 \tag{3.14}$$

where $C = \max\{T,1\}$, and $|||\cdot|||$ is the sup *norm in $C([0,T]; L^2(\mathbb{R}^n))$. Also this uniqueness of the solution representation implies that the solution of system (3.7) has a unique solution in $C([0,T]; \mathbb{R}^n)$.*

Proof. We can prove this result by Induction method. Since $B(u)(x,t)$ as (3.13), then the Fourier transform of (3.13) gives

$$\begin{aligned} \hat{B}(u^\epsilon)(\boldsymbol{\zeta},t) = &\frac{e^{-|\boldsymbol{\zeta}|^2\lambda(t)}}{\epsilon|\boldsymbol{\zeta}|^2 a(0) + e^{-|\boldsymbol{\zeta}|^2\lambda(T)}}\hat{g}(\boldsymbol{\zeta})d\boldsymbol{\zeta} \\ &- \int_t^T \frac{e^{-|\boldsymbol{\zeta}|^2\lambda(t)}}{\epsilon^{\frac{\lambda(s)}{\lambda(T)}} + e^{-|\boldsymbol{\zeta}|^2\lambda(s)}}\hat{f_{u^\epsilon}}(\boldsymbol{\zeta},s)dsd\boldsymbol{\zeta} \end{aligned} \tag{3.15}$$

and we know

$$\|B(u^\epsilon)(\cdot,t) - B(v^\epsilon)(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} = \|\hat{B}(u^\epsilon)(\cdot,t) - \hat{B}(v^\epsilon)(\cdot,t)\|^2_{L^2(\mathbb{R}^n)}$$

then for $n = 1$, we have

$$\begin{aligned}
\|B(u^\epsilon)(\cdot,t) - B(v^\epsilon)(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} &= \|\hat{B}(u^\epsilon)(\cdot,t) - \hat{B}(v^\epsilon)(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} \\
&= \int_{\mathbb{R}^n} \left| \int_t^T \frac{e^{-|\boldsymbol{\zeta}|^2\lambda(t)}}{\epsilon^{\frac{\lambda(s)}{\lambda(T)}} + e^{-|\boldsymbol{\zeta}|^2\lambda(s)}} \hat{f_{v^\epsilon}}(\boldsymbol{\zeta},s)ds \right. \\
&\left. - \int_t^T \frac{e^{-|\boldsymbol{\zeta}|^2\lambda(t)}}{\epsilon^{\frac{\lambda(s)}{\lambda(T)}} + e^{-|\boldsymbol{\zeta}|^2\lambda(s)}} \hat{f_{u^\epsilon}}(\boldsymbol{\zeta},s)ds \right|^2 d\boldsymbol{\zeta} \\
&= \int_{\mathbb{R}^n} \left| \int_t^T \frac{e^{-|\boldsymbol{\zeta}|^2\lambda(t)}}{\epsilon^{\frac{\lambda(s)}{\lambda(T)}} + e^{-|\boldsymbol{\zeta}|^2\lambda(s)}} (\hat{f_{u^\epsilon}}(\boldsymbol{\zeta},s) - \hat{f_{v^\epsilon}}(\boldsymbol{\zeta},s))ds \right|^2 d\boldsymbol{\zeta} \\
&\leq \int_{\mathbb{R}^n} \left(\int_t^T \left(\frac{e^{-|\boldsymbol{\zeta}|^2\lambda(t)}}{\epsilon^{\frac{\lambda(s)}{\lambda(T)}} + e^{-|\boldsymbol{\zeta}|^2\lambda(s)}} \right)^2 ds \right. \\
&\left. \int_t^T (\hat{f_{u^\epsilon}}(\boldsymbol{\zeta},s) - \hat{f_{v^\epsilon}}(\boldsymbol{\zeta},s))^2 ds \right) d\boldsymbol{\zeta}
\end{aligned}$$

Since $\frac{e^{-|\boldsymbol{\zeta}|^2\lambda(t)}}{\epsilon^{\frac{\lambda(s)}{\lambda(T)}} + e^{-|\boldsymbol{\zeta}|^2\lambda(s)}} \leq \frac{1}{\epsilon}$, we have

$$\begin{aligned}
\|B(u^\epsilon)(\cdot,t) - B(v^\epsilon)(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} &\leq \int_{\mathbb{R}^n} \left(\int_t^T \frac{1}{\epsilon^2} ds \int_t^T (\hat{f_{u^\epsilon}}(\boldsymbol{\zeta},s) - \hat{f_{v^\epsilon}}(\boldsymbol{\zeta},s))^2 ds \right) d\boldsymbol{\zeta} \\
&= \int_{\mathbb{R}^n} \left(\frac{(T-t)}{\epsilon^2} \int_t^T (\hat{f_{u^\epsilon}}(\boldsymbol{\zeta},s) - \hat{f_{v^\epsilon}}(\boldsymbol{\zeta},s))^2 ds \right) d\boldsymbol{\zeta} \\
&= \frac{(T-t)}{\epsilon^2} \int_{\mathbb{R}^n} \int_t^T (\hat{f_{u^\epsilon}}(\boldsymbol{\zeta},s) - \hat{f_{v^\epsilon}}(\boldsymbol{\zeta},s))^2 ds d\boldsymbol{\zeta} \\
&= \frac{(T-t)}{\epsilon^2} \int_t^T \int_{\mathbb{R}^n} (\hat{f_{u^\epsilon}}(\boldsymbol{\zeta},s) - \hat{f_{v^\epsilon}}(\boldsymbol{\zeta},s))^2 d\boldsymbol{\zeta} ds \\
&= \frac{(T-t)}{\epsilon^2} \int_t^T \|\hat{f_{u^\epsilon}}(\cdot,s) - \hat{f_{v^\epsilon}}(\cdot,s)\|^2_{L^2(\mathbb{R}^n)} ds \\
&\leq \frac{(T-t)}{\epsilon^2} \int_t^T \|f_{u^\epsilon}(\cdot,s) - f_{v^\epsilon}(\cdot,s)\|^2_{L^2(\mathbb{R}^n)} ds
\end{aligned}$$

By the Lipschitz continuity of f we have a constant k such that

$$\|f_{u^\epsilon}(\cdot,s) - f_{v^\epsilon}(\cdot,s)\|_{L^2(\mathbb{R}^n)} \leq k\|u^\epsilon(\cdot,s) - v^\epsilon(\cdot,s)\|_{L^2(\mathbb{R}^n)}$$

Therefore,

$$
\begin{aligned}
\|B(u^\epsilon)(\cdot,t) - B(v^\epsilon)(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} &\le k^2 \frac{(T-t)}{\epsilon^2} \int_t^T \|u^\epsilon(\cdot,s) - v^\epsilon(\cdot,s)\|^2_{L^2(\mathbb{R}^n)} ds \\
&= Ck^2 \frac{(T-t)}{\epsilon^2} |||u^\epsilon - v^\epsilon|||^2
\end{aligned}
$$

Now suppose (3.14) holds for any $n \ge 1$. Next we need to prove (3.14) holds for $n+1$, that is

$$
\begin{aligned}
&\|B^{n+1}(u^\epsilon)(\cdot,t) - B^{n+1}(v^\epsilon)(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} \\
&= \|\hat{B}^{n+1}(u^\epsilon)(\cdot,t) - \hat{B}^{n+1}(v^\epsilon)(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} \\
&\le \int_{\mathbb{R}^n} \left(\int_t^T \left(\frac{e^{-|\boldsymbol{\zeta}|^2\lambda(t)}}{\epsilon^{\frac{\lambda(s)}{\lambda(T)}} + e^{-|\boldsymbol{\zeta}|^2\lambda(s)}} \right)^2 ds \right. \\
&\quad \left. \int_t^T (\hat{f}(B^n(u^\epsilon)(\boldsymbol{\zeta},s) - \hat{f}(B^n(v^\epsilon)(\boldsymbol{\zeta},s))))^2 ds \right) d\boldsymbol{\zeta} \\
&\le \frac{T-t}{\epsilon^2} \int_{\mathbb{R}^n} \int_t^T (\hat{f}(B^n(u^\epsilon)(\boldsymbol{\zeta},s) - \hat{f}(B^n(v^\epsilon)(\boldsymbol{\zeta},s))))^2 ds d\boldsymbol{\zeta} \\
&\le \frac{T-t}{\epsilon^2} \int_t^T \int_{\mathbb{R}^n} (\hat{f}(B^n(u^\epsilon))(\boldsymbol{\zeta},s) - \hat{f}(B^n(v^\epsilon)(\boldsymbol{\zeta},s)))^2 d\boldsymbol{\zeta} ds \\
&= \frac{T-t}{\epsilon^2} \int_t^T \|\hat{f}(B^n(u^\epsilon))(\cdot,s) - \hat{f}(B^n(v^\epsilon))(\cdot,s)\|^2_{L^2(\mathbb{R}^n)} ds \\
&= \frac{T-t}{\epsilon^2} \int_t^T \|f(B^n(u^\epsilon))(\cdot,s) - f(B^n(v^\epsilon))(\cdot,s)\|^2_{L^2(\mathbb{R}^n)} ds \\
&\le k^2 \frac{T-t}{\epsilon^2} \int_t^T \|B^n(u^\epsilon)(\cdot,s) - B^n(v^\epsilon)(\cdot,s)\|^2_{L^2(\mathbb{R}^n)} ds \\
&\le k^2 \frac{T-t}{\epsilon^2} \int_t^T \left(\frac{k}{\epsilon}\right)^{2n} \frac{(T-s)^n C^n}{n!} |||u-v|||^2 ds \\
&\le C^n \left(\frac{k}{\epsilon}\right)^{2n+2} |||u-v|||^2 \int_t^T \frac{(T-s)^n}{n!} ds \\
&\le C^{(n+1)} \left(\frac{k}{\epsilon}\right)^{2(n+1)} |||u-v|||^2 \frac{(T-t)^{(n+1)}}{(n+1)n!} \\
&\le C^{(n+1)} \left(\frac{k}{\epsilon}\right)^{2(n+1)} \frac{(T-t)^{(n+1)}}{(n+1)!} |||u-v|||^2
\end{aligned}
$$

This is the proof of equation (3.14).

Now consider

$$
a_n = C^n \left(\frac{k}{\epsilon}\right)^{2n} \frac{(T-t)^n}{n!}
$$

Then the series $\sum_{n\geq 1} a_n$ converges by the ratio test because,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0 < 1.$$

Since the series converges, there exists $n_0 \in \mathbb{N}$ such that $a_n < 1$ for all $n > n_0$. Hence, the Banach fixed point theorem implies there exists an unique solution $u^\epsilon \in C([0,T];\mathbb{R}^n)$, such that $B^{n_0}(u^\epsilon) = u^\epsilon$. Then

$$B(B^{n_0}(u^\epsilon)) = B(u^\epsilon) \Rightarrow B^{n_0+1}(u^\epsilon) = B(u^\epsilon) \Rightarrow B^{n_0}(B(u^\epsilon)) = B(u^\epsilon)$$

Then B^{n_0} has a another fixed point, but uniqueness implies $B(u^\epsilon) = u^\epsilon$. This implies the existence of solution representation (3.13). Also this representation implies the solution of system (3.7) is unique. If u^ϵ and v^ϵ are two solutions of (3.7), then (3.7) has solution representations

$$\hat{u^\epsilon}(\boldsymbol{\zeta},t) = \frac{e^{-|\boldsymbol{\zeta}|^2\lambda(t)}}{\epsilon|\boldsymbol{\zeta}|^2 a(0) + e^{-|\boldsymbol{\zeta}|^2\lambda(T)}}\hat{g}(\boldsymbol{\zeta}) - \int_t^T \frac{e^{-|\boldsymbol{\zeta}|^2\lambda(t)}}{\epsilon^{\frac{\lambda(s)}{\lambda(T)}} + e^{-|\boldsymbol{\zeta}|^2\lambda(s)}}\hat{f_{u^\epsilon}}(\boldsymbol{\zeta},s)ds$$

and

$$\hat{v^\epsilon}(\boldsymbol{\zeta},t) = \frac{e^{-|\boldsymbol{\zeta}|^2\lambda(t)}}{\epsilon|\boldsymbol{\zeta}|^2 a(0) + e^{-|\boldsymbol{\zeta}|^2\lambda(T)}}\hat{g}(\boldsymbol{\zeta}) - \int_t^T \frac{e^{-|\boldsymbol{\zeta}|^2\lambda(t)}}{\epsilon^{\frac{\lambda(s)}{\lambda(T)}} + e^{-|\boldsymbol{\zeta}|^2\lambda(s)}}\hat{f_{v^\epsilon}}(\boldsymbol{\zeta},s)ds$$

then

$$\begin{aligned}|\hat{u^\epsilon}(\boldsymbol{\zeta},t) - \hat{v^\epsilon}(\boldsymbol{\zeta},t)|^2 &= \left|\int_t^T \frac{e^{-|\boldsymbol{\zeta}|^2\lambda(t)}}{\epsilon^{\frac{\lambda(s)}{\lambda(T)}} + e^{-|\boldsymbol{\zeta}|^2\lambda(s)}}(\hat{f_{u^\epsilon}}(\boldsymbol{\zeta},s) - \hat{f_{v^\epsilon}}(\boldsymbol{\zeta},s))ds\right|^2 \\ &\leq (T-t)\int_t^T \epsilon^{\frac{\lambda(t)}{\lambda(T)} - \frac{\lambda(s)}{\lambda(T)}}\left|\hat{f_{u^\epsilon}}(\boldsymbol{\zeta},s) - \hat{f_{v^\epsilon}}(\boldsymbol{\zeta},s)\right|^2 ds\end{aligned}$$

since

$$\|\hat{u^\epsilon}(\cdot,t) - \hat{v^\epsilon}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\hat{u^\epsilon}(\boldsymbol{\zeta},t) - \hat{v^\epsilon}(\boldsymbol{\zeta},t)|^2 d\boldsymbol{\zeta}$$

and

$$\|\hat{u^\epsilon}(\cdot,t) - \hat{v^\epsilon}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} = \|u^\epsilon(\cdot,t) - v^\epsilon(\cdot,t)\|^2_{L^2(\mathbb{R}^n)}$$

$$
\begin{aligned}
\|\hat{u^\epsilon}(\cdot,t)-\hat{v^\epsilon}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} &\le (T-t)\int_{\mathbb{R}^n}\int_t^T \epsilon^{\frac{\lambda(t)}{\lambda(T)}-\frac{\lambda(s)}{\lambda(T)}}\left|\hat{f_{u^\epsilon}}(\boldsymbol{\zeta},s)-\hat{f_{v^\epsilon}}(\boldsymbol{\zeta},s)\right|^2 ds d\boldsymbol{\zeta} \\
&= (T-t)\int_t^T \epsilon^{\frac{\lambda(t)}{\lambda(T)}-\frac{\lambda(s)}{\lambda(T)}}\int_{\mathbb{R}^n}\left|\hat{f_{u^\epsilon}}(\boldsymbol{\zeta},s)-\hat{f_{v^\epsilon}}(\boldsymbol{\zeta},s)\right|^2 d\boldsymbol{\zeta} ds \\
&= (T-t)\int_t^T \epsilon^{\frac{\lambda(t)}{\lambda(T)}-\frac{\lambda(s)}{\lambda(T)}}\|\hat{f_{u^\epsilon}}(\cdot,s)-\hat{f_{v^\epsilon}}(\cdot,s)\|^2_{L^2(\mathbb{R}^n)} ds \\
&\le k^2(T-t)\int_t^T \epsilon^{\frac{\lambda(t)}{\lambda(T)}-\frac{\lambda(s)}{\lambda(T)}}\|\hat{u^\epsilon}(\cdot,s)-\hat{v^\epsilon}(\cdot,s)\|^2_{L^2(\mathbb{R}^n)} ds
\end{aligned}
$$

Then, we have

$$
\epsilon^{-\frac{\lambda(t)}{\lambda(T)}}\|\hat{u^\epsilon}(\cdot,t)-\hat{v^\epsilon}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} \le k^2(T-t)\int_t^T \epsilon^{-\frac{\lambda(s)}{\lambda(T)}}\|\hat{u^\epsilon}(\cdot,s)-\hat{v^\epsilon}(\cdot,s)\|^2_{L^2(\mathbb{R}^n)} ds
$$

Gronwall's Identity implies

$$
\epsilon^{-\frac{\lambda(t)}{\lambda(T)}}\|\hat{u^\epsilon}(\cdot,t)-\hat{v^\epsilon}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} \le 0 \Rightarrow \|\hat{u^\epsilon}(\cdot,t)-\hat{v^\epsilon}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} = 0
$$

that is,

$$
\|u^\epsilon(\cdot,t)-v^\epsilon(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} = 0 \Rightarrow u^\epsilon(t) = v^\epsilon(t)
$$

for all $x \in \mathbb{R}^n$. Therefore, system (3.7) has a unique solution in $C([0,T];\mathbb{R}^n)$. □

Next we need to discuss the stability of the problem. That is, the solution of system (3.7) continuously depends on the given condition.

THEOREM 3.3. *Suppose $u^\epsilon(\boldsymbol{x},t)$ and $v^\epsilon(\boldsymbol{x},t)$ are two solutions of (3.7) with given conditions $\theta(\boldsymbol{x})$ and $\phi(\boldsymbol{x})$ respectively. Then*

$$
\|u^\epsilon(\cdot,t)-v^\epsilon(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} \le K_1\|F_\theta(\cdot,t)-G_\phi(\cdot,t)\|^2_{L^2(\mathbb{R}^n)}
$$

where

$$
\hat{F_\theta}(\boldsymbol{\zeta},t) = |\boldsymbol{\zeta}|^{2\left(\frac{\lambda(t)}{\lambda(T)}-1\right)}\hat{\theta}(\boldsymbol{\zeta}), \qquad \hat{G_\phi}(\boldsymbol{\zeta},t) = |\boldsymbol{\zeta}|^{2\left(\frac{\lambda(t)}{\lambda(T)}-1\right)}\hat{\phi}(\boldsymbol{\zeta})
$$

and

$$
K_1 = 2\epsilon^{2\left(\frac{\lambda(t)}{\lambda(T)}-1\right)}e^{2k^2(T-t)}a(0)^{2\left(\frac{\lambda(t)}{\lambda(T)}-1\right)}
$$

Proof. Consider $\hat{u^\epsilon}(\boldsymbol{\zeta},t)-\hat{v^\epsilon}(\boldsymbol{\zeta},t)$

$$\begin{aligned}\hat{u^\epsilon}(\boldsymbol{\zeta},t)-\hat{v^\epsilon}(\boldsymbol{\zeta},t)= \; & \frac{e^{-|\boldsymbol{\zeta}|^2\lambda(t)}}{\epsilon|\boldsymbol{\zeta}|^2a(0)+e^{-|\boldsymbol{\zeta}|^2\lambda(T)}}\hat{\theta}(\boldsymbol{\zeta})-\int_t^T\frac{e^{-|\boldsymbol{\zeta}|^2\lambda(t)}}{\epsilon^{\frac{\lambda(s)}{\lambda(T)}}+e^{-|\boldsymbol{\zeta}|^2\lambda(s)}}\hat{f_{u^\epsilon}}(\boldsymbol{\zeta},s)ds\\ & -\frac{e^{-|\boldsymbol{\zeta}|^2\lambda(t)}}{\epsilon|\boldsymbol{\zeta}|^2a(0)+e^{-|\boldsymbol{\zeta}|^2\lambda(T)}}\hat{\phi}(\boldsymbol{\zeta})+\int_t^T\frac{e^{-|\boldsymbol{\zeta}|^2\lambda(t)}}{\epsilon^{\frac{\lambda(s)}{\lambda(T)}}+e^{-|\boldsymbol{\zeta}|^2\lambda(s)}}\hat{f_{v^\epsilon}}(\boldsymbol{\zeta},s)ds\end{aligned}$$

then

$$\begin{aligned}|\hat{u^\epsilon}(\boldsymbol{\zeta},t)-\hat{v^\epsilon}(\boldsymbol{\zeta},t)|^2= \; & \Bigg|\frac{e^{-|\boldsymbol{\zeta}|^2\lambda(t)}}{\epsilon|\boldsymbol{\zeta}|^2a(0)+e^{-|\boldsymbol{\zeta}|^2\lambda(T)}}(\hat{\theta}(\boldsymbol{\zeta})-\hat{\phi}(\boldsymbol{\zeta}))\\ & +\int_t^T\frac{e^{-|\boldsymbol{\zeta}|^2\lambda(t)}}{\epsilon^{\frac{\lambda(s)}{\lambda(T)}}+e^{-|\boldsymbol{\zeta}|^2\lambda(s)}}(\hat{f_{u^\epsilon}}(\boldsymbol{\zeta},s)-\hat{f_{v^\epsilon}}(\boldsymbol{\zeta},s))ds\Bigg|^2\end{aligned}$$

since $(a+b)^2\le 2a^2+2b^2$, we have

$$\begin{aligned}|\hat{u^\epsilon}(\boldsymbol{\zeta},t)-\hat{v^\epsilon}(\boldsymbol{\zeta},t)|^2 & \le 2\left|\frac{e^{-|\boldsymbol{\zeta}|^2\lambda(t)}}{\epsilon|\boldsymbol{\zeta}|^2a(0)+e^{-|\boldsymbol{\zeta}|^2\lambda(T)}}(\hat{\theta}(\boldsymbol{\zeta})-\hat{\phi}(\boldsymbol{\zeta}))\right|^2\\ & +2\left|\int_t^T\frac{e^{-|\boldsymbol{\zeta}|^2\lambda(t)}}{\epsilon^{\frac{\lambda(s)}{\lambda(T)}}+e^{-|\boldsymbol{\zeta}|^2\lambda(s)}}(\hat{f_{u^\epsilon}}(\boldsymbol{\zeta},s)-\hat{f_{v^\epsilon}}(\boldsymbol{\zeta},s))ds\right|^2\\ & \le 2\left(\frac{e^{-|\boldsymbol{\zeta}|^2\lambda(t)}}{\epsilon|\boldsymbol{\zeta}|^2a(0)+e^{-|\boldsymbol{\zeta}|^2\lambda(T)}}\right)^2|(\hat{\theta}(\zeta)-\hat{\phi}(\boldsymbol{\zeta}))|^2.\\ & +2\int_t^T\left(\frac{e^{-|\boldsymbol{\zeta}|^2\lambda(t)}}{\epsilon^{\frac{\lambda(s)}{\lambda(T)}}+e^{-|\boldsymbol{\zeta}|^2\lambda(s)}}\right)^2|(\hat{f_{u^\epsilon}}(\boldsymbol{\zeta},s)-\hat{f_{v^\epsilon}}(\boldsymbol{\zeta},s))|^2ds\end{aligned}$$

therefore,

$$\begin{aligned}\|\hat{u^\epsilon}(\cdot,t)-\hat{v^\epsilon}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)}\le \; & 2\int_{L^2(\mathbb{R}^n)}\left(\frac{e^{-|\boldsymbol{\zeta}|^2\lambda(t)}}{\epsilon|\boldsymbol{\zeta}|^2a(0)+e^{-|\boldsymbol{\zeta}|^2\lambda(T)}}\right)^2|(\hat{\theta}(\boldsymbol{\zeta})-\hat{\phi}(\boldsymbol{\zeta}))|^2d\boldsymbol{\zeta}\\ & +2\int_{L^2(\mathbb{R}^n)}\int_t^T\left(\frac{e^{-|\boldsymbol{\zeta}|^2\lambda(t)}}{\epsilon^{\frac{\lambda(s)}{\lambda(T)}}+e^{-|\boldsymbol{\zeta}|^2\lambda(s)}}\right)^2\\ & \times|(\hat{f_{u^\epsilon}}(\boldsymbol{\zeta},s)-\hat{f_{v^\epsilon}}(\boldsymbol{\zeta},s))|^2dsd\boldsymbol{\zeta}\end{aligned}$$

Note: For $0<t\le T$, we have

$$\frac{e^{-p^2t}}{\alpha+e^{-p^2T}}\le\alpha^{\frac{t}{T}-1}$$

therefore, we have the following estimates

$$\frac{e^{-|\boldsymbol{\zeta}|^2\lambda(t)}}{\epsilon|\boldsymbol{\zeta}|^2a(0)+e^{-|\boldsymbol{\zeta}|^2\lambda(T)}} \leq \left(\epsilon|\boldsymbol{\zeta}|^2a(0)\right)^{\left(\frac{\lambda(t)}{\lambda(T)}-1\right)}$$

and

$$\frac{e^{-|\boldsymbol{\zeta}|^2\lambda(t)}}{\epsilon^{\frac{\lambda(s)}{\lambda(T)}}+e^{-|\boldsymbol{\zeta}|^2\lambda(s)}} \leq \left(\epsilon^{\frac{\lambda(s)}{\lambda(T)}}\right)^{\left(\frac{\lambda(t)}{\lambda(s)}-1\right)} = \epsilon^{\frac{\lambda(t)}{\lambda(T)}-\frac{\lambda(s)}{\lambda(T)}}$$

then, we have

$$\begin{aligned}\|\hat{u^\epsilon}(\cdot,t)-\hat{v^\epsilon}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} &\leq 2\int_{L^2(\mathbb{R}^n)}\left((\epsilon|\boldsymbol{\zeta}|^2a(0))^{\left(\frac{\lambda(t)}{\lambda(T)}-1\right)}\right)^2|(\hat{\theta}(\boldsymbol{\zeta})-\hat{\phi}(\boldsymbol{\zeta}))|^2d\boldsymbol{\zeta}\\ &+2\int_{L^2(\mathbb{R}^n)}\int_t^T\left(\epsilon^{\frac{\lambda(t)}{\lambda(T)}-\frac{\lambda(s)}{\lambda(T)}}\right)^2|(\hat{f_{u^\epsilon}}(\boldsymbol{\zeta},s)-\hat{f_{v^\epsilon}}(\boldsymbol{\zeta},s))|^2dsd\boldsymbol{\zeta}\end{aligned}$$

$$\begin{aligned}&\leq 2(\epsilon a(0))^{2\left(\frac{\lambda(t)}{\lambda(T)}-1\right)}\int_{L^2(\mathbb{R}^n)}|\boldsymbol{\zeta}|^{2^{\left(\frac{\lambda(t)}{\lambda(T)}-1\right)}}|(\hat{\theta}(\boldsymbol{\zeta})-\hat{\phi}(\boldsymbol{\zeta}))|^2d\boldsymbol{\zeta}\\ &+2\epsilon^{2\frac{\lambda(t)}{\lambda(T)}}\int_t^T\epsilon^{-2\frac{\lambda(s)}{\lambda(T)}}\int_{L^2(\mathbb{R}^n)}|(\hat{f_{u^\epsilon}}(\boldsymbol{\zeta},s)-\hat{f_{v^\epsilon}}(\boldsymbol{\zeta},s))|^2d\boldsymbol{\zeta}ds\end{aligned}$$

This implies

$$\begin{aligned}\epsilon^{-2\frac{\lambda(t)}{\lambda(T)}}\|\hat{u^\epsilon}(\cdot,t)-\hat{v^\epsilon}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} &\leq 2\epsilon^{-2}(a(0))^{2\left(\frac{\lambda(t)}{\lambda(T)}-1\right)}\int_{L^2(\mathbb{R}^n)}|\hat{F}_\theta(\boldsymbol{\zeta},t)-\hat{G}_\phi(\boldsymbol{\zeta},t)|^2d\boldsymbol{\zeta}\\ &+2\epsilon^{2\frac{\lambda(t)}{\lambda(T)}}\int_t^T\epsilon^{-2\frac{\lambda(s)}{\lambda(T)}}\int_{L^2(\mathbb{R}^n)}|(\hat{f_{u^\epsilon}}(\boldsymbol{\zeta},s)-\hat{f_{v^\epsilon}}(\boldsymbol{\zeta},s))|^2d\boldsymbol{\zeta}ds\\ &=2\epsilon^{-2}(a(0))^{2\left(\frac{\lambda(t)}{\lambda(T)}-1\right)}\|\hat{F}_\theta(\cdot,t)-\hat{G}_\phi(\cdot,t)\|^2_{L^2(\mathbb{R}^n)}\\ &+2\int_t^T\epsilon^{-2\frac{\lambda(s)}{\lambda(T)}}\|(\hat{f_{u^\epsilon}}(\cdot,s)-\hat{f_{v^\epsilon}}(\cdot,s))\|^2_{L^2(\mathbb{R}^n)}ds\end{aligned}$$

since $\|\hat{f}\|^2_{L(\mathbb{R}^n)}=\|f\|^2_{L(\mathbb{R}^n)}$, we have

$$\begin{aligned}\epsilon^{-2\frac{\lambda(t)}{\lambda(T)}}\|u^\epsilon(\cdot,t)-v^\epsilon(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} &\leq 2\epsilon^{-2}(a(0))^{2\left(\frac{\lambda(t)}{\lambda(T)}-1\right)}\|F_\theta(\cdot,t)-G_\phi(\cdot,t))\|^2_{L^2(\mathbb{R}^n)}\\ &+2\int_t^T\epsilon^{-2\frac{\lambda(s)}{\lambda(T)}}\|(f_{u^\epsilon}(\cdot,s)-f_{v^\epsilon}(\cdot,s))\|^2_{L^2(\mathbb{R}^n)}ds\\ &\leq 2\epsilon^{-2}(a(0))^{2\left(\frac{\lambda(t)}{\lambda(T)}-1\right)}\|F_\theta(\cdot,t)-G_\phi(\cdot,t))\|^2_{L^2(\mathbb{R}^n)}\\ &+2k^2\int_t^T\epsilon^{-2\frac{\lambda(s)}{\lambda(T)}}\|u^\epsilon(\cdot,s)-v^\epsilon(\cdot,s))\|^2_{L^2(\mathbb{R}^n)}ds\end{aligned}$$

Gronwall's Identity implies

$$\epsilon^{-2\frac{\lambda(t)}{\lambda(T)}}\|u^\epsilon(\cdot,t)-v^\epsilon(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} \le 2\epsilon^{-2}e^{2k^2(T-t)}(a(0))^{2\left(\frac{\lambda(t)}{\lambda(T)}-1\right)}\|F_\theta(\cdot,t)-G_\phi(\cdot,t)\|^2_{L^2(\mathbb{R}^n)}$$

$$\|u^\epsilon(\cdot,t)-v^\epsilon(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} \le 2\epsilon^{2\frac{\lambda(t)}{\lambda(T)}-2}e^{2k^2(T-t)}(a(0))^{2\left(\frac{\lambda(t)}{\lambda(T)}-1\right)}\|F_\theta(\cdot,t)-G_\phi(\cdot,t)\|^2_{L^2(\mathbb{R}^n)}$$

This completes the proof. □

3.5 Convergent Result of Approximation Problem

In this section we discuss convergent results of the approximation problem.

Theorem 3.4. *If $u^\epsilon(\boldsymbol{x},t)\in L^2(\mathbb{R}^n)$ and $P\in L^2(\mathbb{R}^n)$, where $\hat{P}(\boldsymbol{\zeta})=|\boldsymbol{\zeta}|^2e^{|\boldsymbol{\zeta}|^2\lambda(T)}\hat{g}(\boldsymbol{\zeta})$. Then $u^\epsilon(\boldsymbol{x},T)\to g(\boldsymbol{x})$ as $\epsilon\to 0$.*

Proof. Consider $|\hat{u^\epsilon}(\boldsymbol{\zeta},T)-\hat{g}(\boldsymbol{\zeta})|$

$$\begin{aligned}|\hat{u^\epsilon}(\boldsymbol{\zeta},T)-\hat{g}(\boldsymbol{\zeta})|^2 &= \left|\frac{e^{-|\boldsymbol{\zeta}|^2\lambda(T)}}{\epsilon|\boldsymbol{\zeta}|^2a(0)+e^{-|\boldsymbol{\zeta}|^2\lambda(T)}}\hat{g}(\boldsymbol{\zeta})-\hat{g}(\boldsymbol{\zeta})\right|^2\\ &= \left|\frac{\epsilon|\boldsymbol{\zeta}|^2a(0)}{\epsilon|\boldsymbol{\zeta}|^2a(0)+e^{-|\boldsymbol{\zeta}|^2\lambda(T)}}\hat{g}(\boldsymbol{\zeta})\right|^2\\ &= \left(\frac{\epsilon|\boldsymbol{\zeta}|^2a(0)}{\epsilon|\boldsymbol{\zeta}|^2a(0)+e^{-|\boldsymbol{\zeta}|^2\lambda(T)}}\right)^2|\hat{g}(\boldsymbol{\zeta})|^2\end{aligned}$$

therefore

$$\begin{aligned}\|\hat{u^\epsilon}(\boldsymbol{\zeta},T)-\hat{g}(\boldsymbol{\zeta})\|^2_{L^2(\mathbb{R}^n)} &= \int_{L^2(\mathbb{R}^n)}\left(\frac{\epsilon|\boldsymbol{\zeta}|^2a(0)}{\epsilon|\boldsymbol{\zeta}|^2a(0)+e^{-|\boldsymbol{\zeta}|^2\lambda(T)}}\right)^2|\hat{g}(\boldsymbol{\zeta})|^2d\boldsymbol{\zeta}\\ &\le \epsilon^2a(0)^2\int_{L^2(\mathbb{R}^n)}|\boldsymbol{\zeta}|^4e^{2|\boldsymbol{\zeta}|^2\lambda(T)}|\hat{g}(\boldsymbol{\zeta})|^2d\boldsymbol{\zeta}\\ &\le \epsilon^2a(0)^2\int_{L^2(\mathbb{R}^n)}|\hat{P}(\boldsymbol{\zeta})|^2d\boldsymbol{\zeta}\\ &\le \epsilon^2a(0)^2\|\hat{P}(\boldsymbol{\zeta})\|^2_{L^2(\mathbb{R}^n)}\end{aligned}$$

hence

$$\|u^\epsilon(\boldsymbol{x},T)-g(\boldsymbol{x})\|^2_{L^2(\mathbb{R}^n)} \le \epsilon^2a(0)^2\|P(\boldsymbol{x})\|^2_{L^2(\mathbb{R}^n)}$$

and $u^\epsilon(\boldsymbol{x},T) \to g(\boldsymbol{x})$ as $\epsilon \to 0$. Therefore, the given condition of the approximation problem (3.7) converges to $g(\boldsymbol{x})$. $\square$

THEOREM 3.5. *If $q \in L^2(\mathbb{R}^n)$ and $p \in L^2(0,T;L^2(\mathbb{R}^n))$, where $\hat{q}(\boldsymbol{\zeta},t) = |\boldsymbol{\zeta}|^{2\frac{\lambda(t)}{\lambda(T)}}|\hat{g}(\boldsymbol{\zeta})|$ and $\hat{p}(\boldsymbol{\zeta},t) = e^{|\boldsymbol{\zeta}|^2\lambda(t)}|\hat{f}_u(\boldsymbol{\zeta},t)|$.*
Then for all $0 < t \le T$

$$\|u^\epsilon(\cdot,t) - u(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} \le K_2 \epsilon^{2\frac{\lambda(t)}{\lambda(T)}}$$

where

$$K_2 = \left(3a(0)^{2\frac{\lambda(t)}{\lambda(T)}}\|q(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + 3T\|p\|_{L^2(0,T;L^2(\mathbb{R}^n))}\right)e^{3k^2T}$$

That is $u^\epsilon \to u$ as $\epsilon \to 0$.

Proof. Consider

$$\begin{aligned}
u^\epsilon(\boldsymbol{\zeta},t) - u(\boldsymbol{\zeta},t) &= \frac{\epsilon|\boldsymbol{\zeta}|^2 a(0) e^{|\boldsymbol{\zeta}|^2(\lambda(T)-\lambda(t))}}{\epsilon|\boldsymbol{\zeta}|^2 a(0) + e^{-|\boldsymbol{\zeta}|^2\lambda(T)}}\hat{g}(\boldsymbol{\zeta}) - \int_t^T e^{|\boldsymbol{\zeta}|^2(\lambda(s)-\lambda(t))}\hat{f}_u(\boldsymbol{\zeta},s)ds \\
&+ \int_t^T \frac{e^{-|\boldsymbol{\zeta}|^2\lambda(t)}}{\epsilon^{\frac{\lambda(s)}{\lambda(T)}} + e^{-|\boldsymbol{\zeta}|^2\lambda(s)}}\hat{f_{u^\epsilon}}(\boldsymbol{\zeta},s)ds \\
&= \frac{\epsilon|\boldsymbol{\zeta}|^2 a(0) e^{|\boldsymbol{\zeta}|^2(\lambda(T)-\lambda(t))}}{\epsilon|\boldsymbol{\zeta}|^2 a(0) + e^{-|\boldsymbol{\zeta}|^2\lambda(T)}}\hat{g}(\boldsymbol{\zeta}) - \int_t^T \frac{\epsilon^{\frac{\lambda(s)}{\lambda(T)}} e^{|\boldsymbol{\zeta}|^2(\lambda(s)-\lambda(t))}}{\epsilon^{\frac{\lambda(s)}{\lambda(T)}} + e^{-|\boldsymbol{\zeta}|^2\lambda(s)}}\hat{f}_u(\boldsymbol{\zeta},s)ds \\
&+ \int_t^T \frac{e^{-|\boldsymbol{\zeta}|^2\lambda(t)}}{\epsilon^{\frac{\lambda(s)}{\lambda(T)}} + e^{-|\boldsymbol{\zeta}|^2\lambda(s)}}(\hat{f_{u^\epsilon}}(\boldsymbol{\zeta},s) - \hat{f}_u(\boldsymbol{\zeta},s))ds
\end{aligned}$$

then

$$\begin{aligned}
|u^\epsilon(\boldsymbol{\zeta},t) - u(\boldsymbol{\zeta},t)| &\le \frac{\epsilon|\boldsymbol{\zeta}|^2 a(0) e^{|\boldsymbol{\zeta}|^2(\lambda(T)-\lambda(t))}}{\epsilon|\boldsymbol{\zeta}|^2 a(0) + e^{-|\boldsymbol{\zeta}|^2\lambda(T)}}|\hat{g}(\boldsymbol{\zeta})| + \int_t^T \frac{\epsilon^{\frac{\lambda(s)}{\lambda(T)}} e^{|\boldsymbol{\zeta}|^2(\lambda(s)-\lambda(t))}}{\epsilon^{\frac{\lambda(s)}{\lambda(T)}} + e^{-|\boldsymbol{\zeta}|^2\lambda(s)}}|\hat{f}_u(\boldsymbol{\zeta},s)|ds \\
&+ \int_t^T \frac{e^{-|\boldsymbol{\zeta}|^2\lambda(t)}}{\epsilon^{\frac{\lambda(s)}{\lambda(T)}} + e^{-|\boldsymbol{\zeta}|^2\lambda(s)}}|\hat{f_{u^\epsilon}}(\boldsymbol{\zeta},s) - \hat{f}_u(\boldsymbol{\zeta},s)|ds \\
&\le \epsilon^{\frac{\lambda(t)}{\lambda(T)}}|\boldsymbol{\zeta}|^{2\frac{\lambda(t)}{\lambda(T)}}a(0)^{\frac{\lambda(t)}{\lambda(T)}}|\hat{g}(\boldsymbol{\zeta})| + \int_t^T \epsilon^{\frac{\lambda(t)}{\lambda(T)}} e^{|\boldsymbol{\zeta}|^2\lambda(s)}|\hat{f}_u(\boldsymbol{\zeta},s)|ds \\
&+ \int_t^T \epsilon^{\frac{\lambda(t)}{\lambda(T)} - \frac{\lambda(s)}{\lambda(T)}}|\hat{f_{u^\epsilon}}(\boldsymbol{\zeta},s) - \hat{f}_u(\boldsymbol{\zeta},s)|ds \\
&= \epsilon^{\frac{\lambda(t)}{\lambda(T)}}|\boldsymbol{\zeta}|^{2\frac{\lambda(t)}{\lambda(T)}}a(0)^{\frac{\lambda(t)}{\lambda(T)}}|\hat{g}(\boldsymbol{\zeta})| + \epsilon^{\frac{\lambda(t)}{\lambda(T)}}\int_t^T e^{|\boldsymbol{\zeta}|^2\lambda(s)}|\hat{f}_u(\boldsymbol{\zeta},s)|ds \\
&+ \epsilon^{\frac{\lambda(t)}{\lambda(T)}}\int_t^T \epsilon^{-\frac{\lambda(s)}{\lambda(T)}}|\hat{f_{u^\epsilon}}(\boldsymbol{\zeta},s) - \hat{f}_u(\boldsymbol{\zeta},s)|ds
\end{aligned}$$

since

$$\|u^\epsilon(\cdot,t)-u(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} = \int_{L^2(\mathbb{R}^n)} |u^\epsilon(\boldsymbol{\zeta},t)-u(\boldsymbol{\zeta},t)|^2 d\boldsymbol{\zeta}$$

and

$$(a+b+c)^2 \le 3a^2+3b^2+3c^2$$

and we have

$$\begin{aligned}
\|u^\epsilon(\cdot,t)-u(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} &= \int_{L^2(\mathbb{R}^n)} \epsilon^{-2\frac{\lambda(t)}{\lambda(T)}} |u^\epsilon(\boldsymbol{\zeta},t)-u(\boldsymbol{\zeta},t)|^2 d\boldsymbol{\zeta} \\
&\le \int_{L^2(\mathbb{R}^n)} \Big(\epsilon^{\frac{\lambda(t)}{\lambda(T)}} |\boldsymbol{\zeta}|^{2\frac{\lambda(t)}{\lambda(T)}} a(0)^{\frac{\lambda(t)}{\lambda(T)}} |\hat{g}(\boldsymbol{\zeta})| \\
&+ \epsilon^{\frac{\lambda(t)}{\lambda(T)}} \int_t^T e^{|\boldsymbol{\zeta}|^2\lambda(s)} |\hat{f}_u(\boldsymbol{\zeta},s)| ds \\
&+ \epsilon^{\frac{\lambda(t)}{\lambda(T)}} \int_t^T \epsilon^{-\frac{\lambda(s)}{\lambda(T)}} |\hat{f_{u^\epsilon}}(\boldsymbol{\zeta},s)-\hat{f}_u(\boldsymbol{\zeta},s)| ds \Big)^2 d\boldsymbol{\zeta} \\
&\le 3\int_{L^2(\mathbb{R}^n)} \epsilon^{2\frac{\lambda(t)}{\lambda(T)}} |\boldsymbol{\zeta}|^{4\frac{\lambda(t)}{\lambda(T)}} a(0)^{2\frac{\lambda(t)}{\lambda(T)}} |\hat{g}(\boldsymbol{\zeta})|^2 d\boldsymbol{\zeta} \\
&+ 3\epsilon^{2\frac{\lambda(t)}{\lambda(T)}} \int_{L^2(\mathbb{R}^n)} \left(\int_t^T e^{|\boldsymbol{\zeta}|^2\lambda(s)} |\hat{f}_u(\boldsymbol{\zeta},s)| ds \right)^2 d\boldsymbol{\zeta} \\
&+ 3\epsilon^{2\frac{\lambda(t)}{\lambda(T)}} \int_{L^2(\mathbb{R}^n)} \left(\int_t^T \epsilon^{-\frac{\lambda(s)}{\lambda(T)}} |\hat{f_{u^\epsilon}}(\boldsymbol{\zeta},s)-\hat{f}_u(\boldsymbol{\zeta},s)| ds \right)^2 d\boldsymbol{\zeta} \\
&\le 3\epsilon^{2\frac{\lambda(t)}{\lambda(T)}} a(0)^{2\frac{\lambda(t)}{\lambda(T)}} \int_{L^2(\mathbb{R}^n)} |\boldsymbol{\zeta}|^{4\frac{\lambda(t)}{\lambda(T)}} |\hat{g}(\boldsymbol{\zeta})|^2 d\boldsymbol{\zeta} \\
&+ 3\epsilon^{2\frac{\lambda(t)}{\lambda(T)}} \int_{L^2(\mathbb{R}^n)} \left(\int_t^T e^{|\boldsymbol{\zeta}|^2\lambda(s)} |\hat{f}_u(\boldsymbol{\zeta},s)| ds \right)^2 d\boldsymbol{\zeta} \\
&+ 3T\epsilon^{2\frac{\lambda(t)}{\lambda(T)}} \int_{L^2(\mathbb{R}^n)} \int_t^T \epsilon^{-2\frac{\lambda(s)}{\lambda(T)}} |\hat{f_{u^\epsilon}}(\boldsymbol{\zeta},s)-\hat{f}_u(\boldsymbol{\zeta},s)|^2 ds d\boldsymbol{\zeta} \\
&= 3\epsilon^{2\frac{\lambda(t)}{\lambda(T)}} a(0)^{2\frac{\lambda(t)}{\lambda(T)}} \int_{L^2(\mathbb{R}^n)} |\hat{q}(\boldsymbol{\zeta},t)|^2 d\boldsymbol{\zeta} \\
&+ 3T\epsilon^{2\frac{\lambda(t)}{\lambda(T)}} \int_{L^2(\mathbb{R}^n)} \int_t^T e^{2|\boldsymbol{\zeta}|^2\lambda(s)} |\hat{f}_u(\boldsymbol{\zeta},s)|^2 ds d\boldsymbol{\zeta} \\
&+ 3T\epsilon^{2\frac{\lambda(t)}{\lambda(T)}} \int_t^T \epsilon^{-2\frac{\lambda(s)}{\lambda(T)}} \int_{L^2(\mathbb{R}^n)} |\hat{f_{u^\epsilon}}(\boldsymbol{\zeta},s)-\hat{f}_u(\boldsymbol{\zeta},s)|^2 d\boldsymbol{\zeta} ds
\end{aligned}$$

$$
\begin{aligned}
&= 3\epsilon^{2\frac{\lambda(t)}{\lambda(T)}} a(0)^{2\frac{\lambda(t)}{\lambda(T)}} \|\hat{q}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} \\
&+ 3T\epsilon^{2\frac{\lambda(t)}{\lambda(T)}} \int_t^T \int_{L^2(\mathbb{R}^n)} e^{2|\boldsymbol{\zeta}|^2\lambda(s)} |\hat{f}_u(\boldsymbol{\zeta},s)|^2 d\boldsymbol{\zeta} ds \\
&+ 3T\epsilon^{2\frac{\lambda(t)}{\lambda(T)}} \int_t^T \epsilon^{-2\frac{\lambda(s)}{\lambda(T)}} \|\hat{f_{u^\epsilon}}(\cdot,s) - \hat{f}_u(\cdot,s)\|^2_{L^2(\mathbb{R}^n)} ds \\
&= 3\epsilon^{2\frac{\lambda(t)}{\lambda(T)}} a(0)^{2\frac{\lambda(t)}{\lambda(T)}} \|q(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} \\
&+ 3T\epsilon^{2\frac{\lambda(t)}{\lambda(T)}} \int_t^T \int_{L^2(\mathbb{R}^n)} |\hat{p}(\boldsymbol{\zeta},s)|^2 d\boldsymbol{\zeta} ds \\
&+ 3T\epsilon^{2\frac{\lambda(t)}{\lambda(T)}} \int_t^T \epsilon^{-2\frac{\lambda(s)}{\lambda(T)}} \|f_{u^\epsilon}(\cdot,s) - f_u(\cdot,s)\|^2_{L^2(\mathbb{R}^n)} ds
\end{aligned}
$$

therefore,

$$
\begin{aligned}
\epsilon^{-2\frac{\lambda(t)}{\lambda(T)}} \|u^\epsilon(\cdot,t) - u(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} &\le 3a(0)^{2\frac{\lambda(t)}{\lambda(T)}} \|q(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + 3T\int_0^T \|\hat{p}(\cdot,s)\|^2_{L^2(\mathbb{R}^n)} ds \\
&+ 3\int_t^T \epsilon^{-2\frac{\lambda(s)}{\lambda(T)}} \|f_{u^\epsilon}(\cdot,s) - f_u(\cdot,s)\|^2_{L^2(\mathbb{R}^n)} ds \\
&\le 3a(0)^{2\frac{\lambda(t)}{\lambda(T)}} \|q(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + 3T\int_0^T \|p(\cdot,s)\|^2_{L^2(\mathbb{R}^n)} ds \\
&+ 3k^2\int_t^T \epsilon^{-2\frac{\lambda(s)}{\lambda(T)}} \|u^\epsilon(\cdot,s) - u(\cdot,s)\|^2_{L^2(\mathbb{R}^n)} ds \\
&\le 3a(0)^{2\frac{\lambda(t)}{\lambda(T)}} \|q(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + 3T\|p\|_{L^2(0,T;L^2(\mathbb{R}^n))} \\
&+ 3k^2\int_t^T \epsilon^{-2\frac{\lambda(s)}{\lambda(T)}} \|u^\epsilon(\cdot,s) - u(\cdot,s)\|^2_{L^2(\mathbb{R}^n)} ds
\end{aligned}
$$

Gronwall's Identity implies

$$
\begin{aligned}
\epsilon^{-2\frac{\lambda(t)}{\lambda(T)}} \|u^\epsilon(\cdot,t) - u(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} \le \Big(& 3a(0)^{2\frac{\lambda(t)}{\lambda(T)}} \|q(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} \\
&+ 3T\|p\|_{L^2(0,T;L^2(\mathbb{R}^n))}\Big) e^{3k^2T}
\end{aligned}
$$

and

$$
\begin{aligned}
\|u^\epsilon(\cdot,t) - u(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} \le \Big(& 3a(0)^{2\frac{\lambda(t)}{\lambda(T)}} \|q(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} \\
&+ 3T\|p\|_{L^2(0,T;L^2(\mathbb{R}^n))}\Big) e^{3k^2T} \epsilon^{2\frac{\lambda(t)}{\lambda(T)}}
\end{aligned}
$$

This implies the solution of problem (3.7) converges to the solution of problem (3.5) when ϵ approaches zero for all $0 < t \le T$. □

CHAPTER 4
NONLINEAR PARABOLIC EQUATION II

4.1 Second Order PDE Operator

In this section we discuss second order PDE problems, especially the space dependent thermal conductivity heat equation with a terminal condition. Before we discuss this problem, we consider the general form of parabolic equations.
Let $\Omega \subset \mathbb{R}^n$ be opened and bounded, then for any fixed time $T > 0$, consider the terminal value problem

$$\left.\begin{aligned} u_t(\boldsymbol{x},t) + Lu(\boldsymbol{x},t) &= f(\boldsymbol{x},t,u) \quad \boldsymbol{x} \in \Omega \text{ and } 0 \le t < T \\ u(\boldsymbol{x},T) &= g(\boldsymbol{x}) \quad \boldsymbol{x} \in \Omega \end{aligned}\right\} \tag{4.1}$$

where $f : \Omega \times [0,T] \to \mathbb{R}$ and $g : \Omega \to \mathbb{R}$ are given functions. u is an unknown function such that $u : \overline{\Omega} \times [0,T] \to \mathbb{R}$. The operator L is a second order partial differential operator having two different forms as shown by the following definition [22].

Definition 4.1. *Define L as a second order partial differential operator having either the divergence form*

$$Lu = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a^{ij}(x,t)\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{n} b^i(x,t)\frac{\partial u}{\partial x_i} + c(x,t)u \tag{4.2}$$

or else the non-divergence form

$$Lu = -\sum_{i,j=1}^{n} a^{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b^i(x,t)\frac{\partial u}{\partial x_i} + c(x,t)u \tag{4.3}$$

where $a^{i,j}, b^i$ and c are given constants for $i,j = 1,2,\cdots,n$.

Definition 4.2. *The operator L is elliptic (uniformly) if there exists a constant $\theta > 0$ such that*

$$\sum_{i,j=1}^{n} a^{ij}(\boldsymbol{x}, t)\zeta_i\zeta_j \geq \theta|\boldsymbol{\zeta}|^2$$

for a.e $\boldsymbol{x} \in \Omega$ and all $\boldsymbol{\zeta} \in \mathbb{R}^n$, where $\Omega \subset \mathbb{R}^n$ is opened and bounded.

Definition 4.3. *The partial differential operator $\frac{\partial}{\partial t} + L$ is parabolic, if there exists a constant $\alpha > 0$ such that*

$$\sum_{i,j=1}^{n} a^{ij}(\boldsymbol{x}, t)\beta_i\beta_j \geq \alpha|\beta|^2$$

for all $(\boldsymbol{x}, t) \in \Omega, \beta \in \mathbb{R}^n$

Example 4.1. *Choose $a^{i,j}(x,t) = \begin{cases} 1 & if \quad j = j \\ 0 & if \quad i \neq j \end{cases}$ and $b^i = c = f = 0$, then $L = -\Delta$ and equation (4.1) becomes the heat equation.*

Suppose

$$a^{ij}(x) = \begin{cases} a(x) & \text{for } i = j \\ 0 & \text{otherwise} \end{cases}$$

and $b^i \equiv c \equiv 0$, then $L = -a(x)\Delta$. then equation (4.1) becomes

$$\left.\begin{aligned} u_t(x,t) - a(x)\Delta u(x,t) &= f(x, t, u, u_{x_1}, \cdots, u_{x_n}, u_{x_1x_1}, \cdots, u_{x_nx_n}) \quad 0 \leq t < T \\ u(x,T) &= g(x) \end{aligned}\right\} \tag{4.4}$$

This is called the space dependent nonlinear heat equation. Especially the term $a(x)$ is called space dependent thermal conductivity of a heat equation.

4.2 Space Dependent Nonlinear Equation

Let T be a positive number. Suppose $\boldsymbol{x} = (x_1, x_2 \cdots x_n) \in \mathbb{R}^n$ and $f : \mathbb{R}^n \times [0,T] \times \mathbb{R}^{2n+1} \to \mathbb{R}$ be a continuous function which satisfies

$$\|f(\cdot, t, u, v, w) - f(\cdot, t, u', v', w')\|_{L^2(\mathbb{R}^n)} \leq K\|u(\cdot, t) - u'(\cdot, t)\|_{H^2(\mathbb{R}^n)} \tag{4.5}$$

where K is a constant, $(x, t, u, v, w), (x, t, u', v', w') \in \mathbb{R}^n \times [0, T] \times \mathbb{R}^{2n+1}$, such that $f \equiv 0$ whenever $u \equiv 0$. Since $L^2(\mathbb{R}^n)$ and $H^2(\mathbb{R}^n)$ are Hilbert spaces, consider the following inverse problem:

$$\left.\begin{array}{rcl} u_t(\boldsymbol{x}, t) - a(\boldsymbol{x})\Delta u(\boldsymbol{x}, t) &=& f(\boldsymbol{x}, t, u, u_{x_1}, \cdots, u_{x_n}, u_{x_1x_1}, \cdots, u_{x_nx_n}) \quad 0 \leq t < T \\ u(\boldsymbol{x}, T) &=& g(\boldsymbol{x}) \quad \boldsymbol{x} \in \mathbb{R}^n \end{array}\right\} \tag{4.6}$$

where $a(\boldsymbol{x})$ is given continuous function such that there exists $q, r > 0$ satisfying

$$0 < q \leq a(x) \leq r$$

Also $(\boldsymbol{x}, t, u, u_{x_1}, \cdots, u_{x_n}, u_{x_1x_1}, \cdots, u_{x_nx_n}) \in \mathbb{R}^n \times [0, T] \times \mathbb{R}^{2n+1}$ and $g \in L^2(\mathbb{R}^n)$. Here $a(\boldsymbol{x})$ is called space dependent thermal conductivity of a heat equation. We need to find a solution $u \in H^2(\mathbb{R}^n)$ such that $u : \mathbb{R}^n \times [0, T] \to \mathbb{R}$. Suppose $a(\boldsymbol{x})$ has the following properties, see in [21]. That is

$$\lim_{\boldsymbol{x} \to \infty} a(\boldsymbol{x}) = c$$

where c is a constant, $\boldsymbol{x} \to \infty \Rightarrow x_i \to \infty$ for each $i = 1, 2, \cdots, n$, and there is a function $b(\boldsymbol{x})$ such that

$$b(\boldsymbol{x}) = a(\boldsymbol{x}) - c$$

this implies

$$|b(x)| \leq 2r$$

Using the above transformation, we rewrite equation (4.6) as follows:

$$\left.\begin{array}{rcl} u_t(\boldsymbol{x}, t) - c\Delta u(\boldsymbol{x}, t) &=& F(\boldsymbol{x}, t, u, u_{x_1} \cdots u_{x_n}, u_{x_1x_1} \cdots u_{x_nx_n}) \quad 0 \leq t < T \\ u(\boldsymbol{x}, T) &=& g(\boldsymbol{x}) \quad \boldsymbol{x} \in \mathbb{R}^n \end{array}\right\} \tag{4.7}$$

where $F(\boldsymbol{x}, t, u, u_{x_1} \cdots u_{x_n}, u_{x_1x_1} \cdots u_{x_nx_n}) = f(\boldsymbol{x}, t, u, u_{x_1} \cdots u_{x_n}, u_{x_1x_1} \cdots u_{x_nx_n}) + b(x)\Delta u$.

The $n-$dimension Fourier transform form for given function u is

$$\hat{u}(\boldsymbol{\zeta},t) = \frac{1}{(\sqrt{2\pi})^n}\int_{\mathbb{R}^n} u(\boldsymbol{x},t)e^{-i\boldsymbol{\zeta}\cdot\boldsymbol{x}}d\boldsymbol{x}.$$

where $\boldsymbol{\zeta} = (\zeta_1, \zeta_2 \cdots, \zeta_n) \in \mathbb{R}^n$ and $|\boldsymbol{\zeta}|^2 = \zeta_1^2 + \zeta_2^2 + \cdots + \zeta_n^2$.

For convenience we use $F_u(\boldsymbol{x},t) = F(\boldsymbol{x},t,u,u_{x_1}\cdots u_{x_n}, u_{x_1x_1}\cdots u_{x_nx_n})$.

Then the solution representation of (4.7) is

$$\hat{u}(\boldsymbol{\zeta},t) = e^{c|\boldsymbol{\zeta}|^2(T-t)}\hat{g}(\boldsymbol{\zeta}) - \int_t^T e^{c|\boldsymbol{\zeta}|^2(s-t)}\hat{F}_u(\boldsymbol{\zeta},s)ds$$

or

$$u(\boldsymbol{x},t) = \frac{1}{(\sqrt{2\pi})^n}\int_{\mathbb{R}^n} e^{c|\boldsymbol{\zeta}|^2(T-t)}\hat{g}(\boldsymbol{\zeta})e^{i\boldsymbol{\zeta}\cdot\boldsymbol{x}}d\boldsymbol{\zeta} - \frac{1}{(\sqrt{2\pi})^n}\int_{\mathbb{R}^n}\int_t^T e^{c|\boldsymbol{\zeta}|^2(s-t)}\hat{F}_u(\boldsymbol{\zeta},s)dse^{i\boldsymbol{\zeta}\cdot\boldsymbol{x}}d\boldsymbol{\zeta}$$

The existence and uniqueness of the problem (4.7) gives by the theorem (3.1). Now we assume (4.7) has a unique solution in $H^2(\mathbb{R}^n)$ with given condition $g \in L^2(\mathbb{R}^n)$.

Definition 4.4. *For $u \in H^2(\mathbb{R}^n)$, define*

$$\|u\|^2_{H^2(\mathbb{R}^n)} = \|u\|^2_{L^2(\mathbb{R}^n)} + \sum_{i=1}^n \|u_{x_i}\|^2_{L^2(\mathbb{R}^n)} + \sum_{i=1}^n \|u_{x_ix_i}\|^2_{L^2(\mathbb{R}^n)}$$

with the above definition, we have a very important result and given by the following lemma,

Lemma 4.1. *For $u \in H^2(\mathbb{R}^n)$, we have*

$$\|u(\cdot,t)\|^2_{H^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n}(1 + |\boldsymbol{\zeta}|^2 + |\boldsymbol{\zeta}|^4)|\hat{u}(\boldsymbol{\zeta},t)|^2 d\boldsymbol{\zeta}$$

where $\hat{u}$ is Fourier transform of u

Proof. Since $\|u\|_{L^2(\mathbb{R}^n)} = \|\hat{u}\|_{L^2(\mathbb{R}^n)}$, we have

$$\begin{aligned}\|u\|^2_{H^2(\mathbb{R}^n)} &= \|\hat{u}\|^2_{L^2(\mathbb{R}^n)} + \sum_{i=1}^n \|\hat{u}_{x_i}\|^2_{L^2(\mathbb{R}^n)} + \sum_{i=1}^n \|\hat{u}_{x_ix_i}\|^2_{L^2(\mathbb{R}^n)} \\ &= \|\hat{u}\|^2_{L^2(\mathbb{R}^n)} + \sum_{i=1}^n \|i\zeta_i\hat{u}\|^2_{L^2(\mathbb{R}^n)} + \sum_{i=1}^n \|-\zeta_i^2\hat{u}\|^2_{L^2(\mathbb{R}^n)}\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^n} |\hat{u}(\boldsymbol{\zeta})|^2 d\boldsymbol{\zeta} + \sum_{i=1}^{n} \int_{\mathbb{R}^n} |i\zeta_i \hat{u}(\boldsymbol{\zeta})|^2 d\boldsymbol{\zeta} + \sum_{i=1}^{n} \int_{\mathbb{R}^n} |-\zeta_i^2 \hat{u}(\boldsymbol{\zeta})|^2 d\boldsymbol{\zeta} \\
&= \int_{\mathbb{R}^n} |\hat{u}(\boldsymbol{\zeta})|^2 d\boldsymbol{\zeta} + \sum_{i=1}^{n} \int_{\mathbb{R}^n} |\zeta_i^2||\hat{u}(\boldsymbol{\zeta})|^2 d\boldsymbol{\zeta} + \sum_{i=1}^{n} \int_{\mathbb{R}^n} |\zeta_i^4||\hat{u}(\boldsymbol{\zeta})|^2 d\boldsymbol{\zeta} \\
&= \int_{\mathbb{R}^n} |\hat{u}(\boldsymbol{\zeta})|^2 d\boldsymbol{\zeta} + \int_{\mathbb{R}^n} \left(\sum_{i=1}^{n} |\zeta_i^2| \right) |\hat{u}(\boldsymbol{\zeta})|^2 d\boldsymbol{\zeta} + \int_{\mathbb{R}^n} \left(\sum_{i=1}^{n} |\zeta_i^4| \right) |\hat{u}(\boldsymbol{\zeta})|^2 d\boldsymbol{\zeta} \\
&= \int_{\mathbb{R}^n} |\hat{u}(\boldsymbol{\zeta})|^2 d\boldsymbol{\zeta} + \int_{\mathbb{R}^n} |\boldsymbol{\zeta}|^2 |\hat{u}(\boldsymbol{\zeta})|^2 d\boldsymbol{\zeta} + \int_{\mathbb{R}^n} |\boldsymbol{\zeta}|^4 |\hat{u}(\boldsymbol{\zeta})|^2 d\boldsymbol{\zeta} \\
&= \int_{\mathbb{R}^n} (1 + |\boldsymbol{\zeta}|^2 + |\boldsymbol{\zeta}|^4) |\hat{u}(\boldsymbol{\zeta}, t)|^2 d\boldsymbol{\zeta}
\end{aligned}
$$

□

Since

$$\sum_{i=1}^{n} \|u_{x_i x_i}\|_{L^2(\mathbb{R}^n)}^2 \leq \|u\|_{H^2(\mathbb{R}^n)}^2$$

Then by easy calculation, we can prove

$$\|F(\cdot, t, u, v, w) - F(\cdot, t, u', v', w')\|_{L^2(\mathbb{R}^n)} \leq k \|u(\cdot, t) - u'(\cdot, t)\|_{H^2(\mathbb{R}^n)}$$

where $k = \sqrt{8r^2 n + 2K}$. For $\epsilon > 0$, consider the following approximation problem:

$$\left.\begin{aligned} u_t^\epsilon(\boldsymbol{x}, t) - c\Delta u^\epsilon(\boldsymbol{x}, t) &= S_{u^\epsilon}(\boldsymbol{x}, t) \quad 0 \leq t < T \\ \epsilon u^\epsilon(0) + u^\epsilon(\boldsymbol{x}, T) &= g^\epsilon(\boldsymbol{x}) \quad \boldsymbol{x} \in \mathbb{R}^n \end{aligned}\right\} \tag{4.8}$$

where $S_{u^\epsilon}(\boldsymbol{x}, t) = S(\boldsymbol{x}, t, u^\epsilon, u_{x_1}^\epsilon \cdots u_{x_n}^\epsilon, u_{x_1 x_1}^\epsilon \cdots u_{x_n x_n}^\epsilon)$,

$$S_{u^\epsilon}(\boldsymbol{x}, t) = \frac{1}{(\sqrt{2\pi})^n} \int_{\mathbb{R}^n} \frac{e^{-c|\boldsymbol{\zeta}|^2 t}}{\epsilon^{\frac{t}{T}} + e^{-c|\boldsymbol{\zeta}|^2 t}} \hat{F}_{u^\epsilon}(\boldsymbol{\zeta}, t) \mathcal{C}_{\Omega_\epsilon}^\epsilon(\boldsymbol{\zeta}) e^{i\boldsymbol{\zeta}\cdot\boldsymbol{x}} d\boldsymbol{\zeta}$$

and

$$
\begin{aligned}
g^\epsilon(\boldsymbol{x}) = &\frac{1}{(\sqrt{2\pi})^n} \int_{\mathbb{R}^n} \hat{g}(\boldsymbol{\zeta}) \mathcal{C}_{\Omega_\epsilon}^\epsilon(\boldsymbol{\zeta}) e^{i\boldsymbol{\zeta}\cdot\boldsymbol{x}} d\boldsymbol{\zeta} \\
&- \frac{1}{(\sqrt{2\pi})^n} \int_{\mathbb{R}^n} \int_0^T \frac{\epsilon}{\epsilon^{\frac{s}{T}} + e^{-c|\boldsymbol{\zeta}|^2 s}} \hat{F}_{u^\epsilon}(\boldsymbol{\zeta}, s) \mathcal{C}_{\Omega_\epsilon}^\epsilon(\boldsymbol{\zeta}) ds \;\; e^{i\boldsymbol{\zeta}\cdot\boldsymbol{x}} d\boldsymbol{\zeta}
\end{aligned}
$$

where $\mathcal{C}^\epsilon : \Omega_\epsilon \subset \mathbb{R}^n \to \mathbb{R}$ continuous and $C^\epsilon \equiv 0$ on Ω_ϵ^c such that $\mathcal{C}^\epsilon(\boldsymbol{x}) \to 1$ as $\epsilon \to 0$ and $\Omega_\epsilon \subset \mathbb{R}^n$ is a closed, $0 \in \Omega_\epsilon$ and symmetric region about at $\boldsymbol{x} = 0$ such that $\Omega_\epsilon \to \mathbb{R}^n$ as $\epsilon \to 0$. The integral form of (4.8) is given by

$$\begin{aligned} u^\epsilon(\boldsymbol{x},t) &= \frac{1}{(\sqrt{2\pi})^n}\int_{\mathbb{R}^n} e^{c|\boldsymbol{\zeta}|^2(T-t)}\hat{g}(\boldsymbol{\zeta})\mathcal{C}^\epsilon_{\Omega_\epsilon}(\boldsymbol{\zeta})e^{i\boldsymbol{\zeta}\cdot\boldsymbol{x}}d\boldsymbol{\zeta} \\ &- \frac{1}{(\sqrt{2\pi})^n}\int_{\mathbb{R}^n}\int_t^T \frac{e^{-c|\boldsymbol{\zeta}|^2 t}}{\epsilon^{\frac{s}{T}} + e^{-c|\boldsymbol{\zeta}|^2 s}}\hat{F}_{u^\epsilon}(\boldsymbol{\zeta},s)\mathcal{C}^\epsilon_{\Omega_\epsilon}(\boldsymbol{\zeta})ds \;\; e^{i\boldsymbol{\zeta}\cdot\boldsymbol{x}}d\boldsymbol{\zeta}\end{aligned}$$

4.3 Approximation Results

As we showed in chapter three, here we need to show the solution of (4.7) converges to the solution of (4.6) whenever the parameter goes to zero. First, we have the following result.

Corollary 4.1. *If $u^\epsilon(T) \in L^2(\mathbb{R}^n)$, then $u^\epsilon(T) \to g$ whenever $\epsilon \to 0$*

Proof. Consider $\hat{u}^\epsilon(\boldsymbol{\zeta},T) - \hat{g}(\boldsymbol{\zeta})$, then

$$\begin{aligned}\hat{u}^\epsilon(\boldsymbol{\zeta},T) - \hat{g}(\boldsymbol{\zeta}) &= \hat{g}(\boldsymbol{\zeta})\mathcal{C}^\epsilon_{\Omega_\epsilon}(\boldsymbol{\zeta}) - \hat{g}(\boldsymbol{\zeta}) \\ &= \hat{g}(\boldsymbol{\zeta})(\mathcal{C}^\epsilon_{\Omega_\epsilon}(\boldsymbol{\zeta}) - 1)\end{aligned}$$

consider the absolute value for both sides, we have

$$\begin{aligned}|\hat{u}^\epsilon(\boldsymbol{\zeta},T) - \hat{g}(\boldsymbol{\zeta})|^2 &= |\hat{g}(\boldsymbol{\zeta})\mathcal{C}^\epsilon_{\Omega_\epsilon}(\boldsymbol{\zeta}) - \hat{g}(\boldsymbol{\zeta})|^2 \\ &= |\hat{g}(\boldsymbol{\zeta})|^2|(\mathcal{C}^\epsilon_{\Omega_\epsilon}(\boldsymbol{\zeta}) - 1)|^2\end{aligned}$$

now consider L^2 norm

$$\begin{aligned}\|\hat{u}^\epsilon(\cdot,T) - \hat{g}(\cdot)\|^2_{L^2(\mathbb{R}^n)} &= \int_{\mathbb{R}^n}|\hat{g}(\boldsymbol{\zeta})|^2|(\mathcal{C}^\epsilon_{\Omega_\epsilon}(\boldsymbol{\zeta}) - 1)|^2 d\boldsymbol{\zeta} \\ &= \int_{\Omega_\epsilon}|\hat{g}(\boldsymbol{\zeta})|^2|(\mathcal{C}^\epsilon_{\Omega_\epsilon}(\boldsymbol{\zeta}) - 1)|^2 d\boldsymbol{\zeta} + \int_{\Omega^c_\epsilon}|\hat{g}(\boldsymbol{\zeta})|^2 d\boldsymbol{\zeta} \\ &\le \int_{\mathbb{R}^n}|\hat{g}(\boldsymbol{\zeta})|^2|(\mathcal{C}^\epsilon_{\Omega_\epsilon}(\boldsymbol{\zeta}) - 1)|^2 d\boldsymbol{\zeta} + \int_{\Omega^c_\epsilon}|\hat{g}(\boldsymbol{\zeta})|^2 d\boldsymbol{\zeta}\end{aligned}$$

Now taking $\epsilon \to 0$, the above inequality approaches zero. □

Consider the following example

EXAMPLE 4.2. *Consider* $n = 1$, $0 < \epsilon < 1$ $\Omega_\epsilon = [-\frac{1}{\epsilon}, \frac{1}{\epsilon}]$ *and* $C^\epsilon(x) = 1 - \epsilon e^{-x^2}$ *Clearly* $\Omega_\epsilon \to \mathbb{R}$ *and* $C^\epsilon(x) \to 1$ *as* $\epsilon \to 0$. *then*

$$\begin{aligned}
\|\hat{u}^\epsilon(\cdot, T) - \hat{g}(\cdot)\|^2_{L^2(\mathbb{R})} &\leq \int_{\mathbb{R}} |\hat{g}(\boldsymbol{\zeta})|^2 |(1 - \epsilon e^{-x^2} - 1)|^2 d\boldsymbol{\zeta} + \int_{\Omega_\epsilon^c} |\hat{g}(\boldsymbol{\zeta})|^2 d\boldsymbol{\zeta} \\
&= \epsilon^2 \int_{\mathbb{R}} |\hat{g}(\boldsymbol{\zeta})|^2 e^{-2x^2} d\boldsymbol{\zeta} + \int_{\Omega_\epsilon^c} |\hat{g}(\boldsymbol{\zeta})|^2 d\boldsymbol{\zeta} \\
&\leq \epsilon^2 \int_{\mathbb{R}} |\hat{g}(\boldsymbol{\zeta})|^2 d\boldsymbol{\zeta} + \int_{\Omega_\epsilon^c} |\hat{g}(\boldsymbol{\zeta})|^2 d\boldsymbol{\zeta}
\end{aligned}$$

when $\epsilon \to 0$, *then* Ω_ϵ^c *becomes empty set, i.e.* $\int_{\Omega_\epsilon^c} |\hat{g}(\boldsymbol{\zeta})|^2 d\boldsymbol{\zeta} \to 0$, *hence*

$$\begin{aligned}
&\leq \epsilon^2 \int_{\mathbb{R}} |\hat{g}(\boldsymbol{\zeta})|^2 d\boldsymbol{\zeta} \\
&= \epsilon^2 \|g\|^2_{L^2(\mathbb{R})}
\end{aligned}$$

this implies $u^\epsilon(x, T) \to g(x)$ *whenever* $\epsilon \to 0$.

COROLLARY 4.2. *Suppose* $u^\epsilon \in H^2(\mathbb{R}^n)$ *and* $v^\epsilon \in H^2(\mathbb{R}^n)$ *are two solutions of (4.8) with the terminal values* ϕ *and* ψ *respectively, then*

$$\|u^\epsilon(\cdot, t) - v^\epsilon(\cdot, t)\|^2_{H^2(\mathbb{R}^n)} \leq 2R_\epsilon e^{2k^2 R_\epsilon (T-t)^2} e^{2c|a_\epsilon|^2(T-t)} \|\hat{\psi} - \hat{\psi}\|^2_{L^2(\mathbb{R}^n)}$$

where $R_\epsilon = \max\{1 + |\boldsymbol{\zeta}|^2 + |\boldsymbol{\zeta}|^4 : \boldsymbol{\zeta} \in \Omega_\epsilon\}$ *and* $a_\epsilon \in \Omega_\epsilon$ *such that* $e^{2c|\boldsymbol{\zeta}|^2(T-t)}$ *has the maximum at* $\boldsymbol{\zeta} = a_\epsilon$.

Proof. Let

$$\begin{aligned}
u^\epsilon(\boldsymbol{x}, t) = &\frac{1}{(\sqrt{2\pi})^n} \int_{\mathbb{R}^n} e^{c|\boldsymbol{\zeta}|^2(T-t)} \hat{\phi}(\boldsymbol{\zeta}) \mathcal{C}^\epsilon_{\Omega_\epsilon}(\boldsymbol{\zeta}) e^{i\boldsymbol{\zeta}\cdot\boldsymbol{x}} d\boldsymbol{\zeta} \\
&- \frac{1}{(\sqrt{2\pi})^n} \int_{\mathbb{R}^n} \int_t^T \frac{e^{-c|\boldsymbol{\zeta}|^2 t}}{\epsilon^{\frac{s}{T}} + e^{-c|\boldsymbol{\zeta}|^2 s}} \hat{F}_{u^\epsilon}(\boldsymbol{\zeta}, t) \mathcal{C}^\epsilon_{\Omega_\epsilon}(\boldsymbol{\zeta}) ds \quad e^{i\boldsymbol{\zeta}\cdot\boldsymbol{x}} d\boldsymbol{\zeta}
\end{aligned}$$

and

$$\begin{aligned}
v^\epsilon(\boldsymbol{x}, t) = &\frac{1}{(\sqrt{2\pi})^n} \int_{\mathbb{R}^n} e^{c|\boldsymbol{\zeta}|^2(T-t)} \hat{\psi}(\boldsymbol{\zeta}) \mathcal{C}^\epsilon_{\Omega_\epsilon}(\boldsymbol{\zeta}) e^{i\boldsymbol{\zeta}\cdot\boldsymbol{x}} d\boldsymbol{\zeta} \\
&- \frac{1}{(\sqrt{2\pi})^n} \int_{\mathbb{R}^n} \int_t^T \frac{e^{-c|\boldsymbol{\zeta}|^2 t}}{\epsilon^{\frac{s}{T}} + e^{-c|\boldsymbol{\zeta}|^2 s}} \hat{F}_{v^\epsilon}(\boldsymbol{\zeta}, t) \mathcal{C}^\epsilon_{\Omega_\epsilon}(\boldsymbol{\zeta}) ds \quad e^{i\boldsymbol{\zeta}\cdot\boldsymbol{x}} d\boldsymbol{\zeta}
\end{aligned}$$

since

$$\|u^\epsilon(\cdot,t)-v^\epsilon(\cdot,t)\|^2_{H^2(\mathbb{R}^n)} = \|u^\epsilon(\cdot,t)-v^\epsilon(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + \sum_{i=1}^n \|u^\epsilon_{x_i}(\cdot,t)-v^\epsilon_{x_i}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + \sum_{i=1}^n \|u^\epsilon_{x_ix_i}(\cdot,t)-v^\epsilon_{x_ix_i}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)}$$

and $\|u\|_{L^2(\mathbb{R}^n)} = \|\hat{u}\|_{L^2(\mathbb{R}^n)}$, we have

$$\|u^\epsilon(\cdot,t)-v^\epsilon(\cdot,t)\|^2_{H^2(\mathbb{R}^n)} = \|\hat{u}^\epsilon(\cdot,t)-\hat{v}^\epsilon(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + \sum_{i=1}^n \|\hat{u}^\epsilon_{x_i}(\cdot,t)-\hat{v}^\epsilon_{x_i}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + \sum_{i=1}^n \|\hat{u}^\epsilon_{x_ix_i}(\cdot,t)-\hat{v}^\epsilon_{x_ix_i}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)}$$

consider

$$|\hat{u}^\epsilon(\boldsymbol{x},t)-\hat{v}^\epsilon(\boldsymbol{x},t)|^2 = \left| e^{c|\boldsymbol{\zeta}|^2(T-t)}\mathcal{C}^\epsilon_{\Omega_\epsilon}(\boldsymbol{\zeta})(\hat{\psi}(\boldsymbol{\zeta})-\hat{\psi}(\boldsymbol{\zeta})) - \int_t^T \frac{e^{-c|\boldsymbol{\zeta}|^2 t}}{\epsilon^{\frac{s}{T}}+e^{-c|\boldsymbol{\zeta}|^2 s}}\mathcal{C}^\epsilon_{\Omega_\epsilon}(\boldsymbol{\zeta})(\hat{F}_{u^\epsilon}(\boldsymbol{\zeta},t)-\hat{F}_{v^\epsilon}(\boldsymbol{\zeta},t))ds \right|^2$$

also

$$|\hat{u}^\epsilon_{x_i}(\boldsymbol{x},t)-\hat{v}^\epsilon_{x_i}(\boldsymbol{x},t)|^2 = \zeta_i^2\left| e^{c|\boldsymbol{\zeta}|^2(T-t)}\mathcal{C}^\epsilon_{\Omega_\epsilon}(\boldsymbol{\zeta})(\hat{\psi}(\boldsymbol{\zeta})-\hat{\psi}(\boldsymbol{\zeta})) - \int_t^T \frac{e^{-c|\boldsymbol{\zeta}|^2 t}}{\epsilon^{\frac{s}{T}}+e^{-c|\boldsymbol{\zeta}|^2 s}}\mathcal{C}^\epsilon_{\Omega_\epsilon}(\boldsymbol{\zeta})(\hat{F}_{u^\epsilon}(\boldsymbol{\zeta},t)-\hat{F}_{v^\epsilon}(\boldsymbol{\zeta},t))ds \right|^2$$

$$\sum_{i=1}^n|\hat{u}^\epsilon_{x_i}(\boldsymbol{x},t)-\hat{v}^\epsilon_{x_i}(\boldsymbol{x},t)|^2 = |\boldsymbol{\zeta}|^2\left| e^{c|\boldsymbol{\zeta}|^2(T-t)}\mathcal{C}^\epsilon_{\Omega_\epsilon}(\boldsymbol{\zeta})(\hat{\psi}(\boldsymbol{\zeta})-\hat{\psi}(\boldsymbol{\zeta})) - \int_t^T \frac{e^{-c|\boldsymbol{\zeta}|^2 t}}{\epsilon^{\frac{s}{T}}+e^{-c|\boldsymbol{\zeta}|^2 s}}\mathcal{C}^\epsilon_{\Omega_\epsilon}(\boldsymbol{\zeta})(\hat{F}_{u^\epsilon}(\boldsymbol{\zeta},t)-\hat{F}_{v^\epsilon}(\boldsymbol{\zeta},t))ds \right|^2$$

and

$$\sum_{i=1}^n|\hat{u}^\epsilon_{x_ix_i}(\boldsymbol{x},t)-\hat{v}^\epsilon_{x_ix_i}(\boldsymbol{x},t)|^2 = |\boldsymbol{\zeta}|^4\left| e^{c|\boldsymbol{\zeta}|^2(T-t)}\mathcal{C}^\epsilon_{\Omega_\epsilon}(\boldsymbol{\zeta})(\hat{\psi}(\boldsymbol{\zeta})-\hat{\psi}(\boldsymbol{\zeta})) - \int_t^T \frac{e^{-c|\boldsymbol{\zeta}|^2 t}}{\epsilon^{\frac{s}{T}}+e^{-c|\boldsymbol{\zeta}|^2 s}}\mathcal{C}^\epsilon_{\Omega_\epsilon}(\boldsymbol{\zeta})(\hat{F}_{u^\epsilon}(\boldsymbol{\zeta},t)-\hat{F}_{v^\epsilon}(\boldsymbol{\zeta},t))ds \right|^2.$$

therefore

$$\|u^\epsilon(\cdot,t)-v^\epsilon(\cdot,t)\|^2_{H^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n}(1+|\boldsymbol{\zeta}|^2+|\boldsymbol{\zeta}|^4)\left| e^{c|\boldsymbol{\zeta}|^2(T-t)}\mathcal{C}^\epsilon_{\Omega_\epsilon}(\boldsymbol{\zeta})(\hat{\psi}(\boldsymbol{\zeta})-\hat{\psi}(\boldsymbol{\zeta})) - \int_t^T \frac{e^{-c|\boldsymbol{\zeta}|^2 t}}{\epsilon^{\frac{s}{T}}+e^{-c|\boldsymbol{\zeta}|^2 s}}\mathcal{C}^\epsilon_{\Omega_\epsilon}(\boldsymbol{\zeta})(\hat{F}_{u^\epsilon}(\boldsymbol{\zeta},t)-\hat{F}_{v^\epsilon}(\boldsymbol{\zeta},t))ds \right|^2 d\boldsymbol{\zeta}$$

Let $h(\boldsymbol{\zeta}) = 1 + |\boldsymbol{\zeta}|^2 + |\boldsymbol{\zeta}|^4$, then

$$\leq 2\int_{\mathbb{R}^n} h(\boldsymbol{\zeta}) \left(e^{c|\boldsymbol{\zeta}|^2(T-t)}\right)^2 (\mathcal{C}^\epsilon)^2_{\Omega_\epsilon}(\boldsymbol{\zeta}) |(\hat{\psi}(\boldsymbol{\zeta}) - \hat{\psi}(\boldsymbol{\zeta}))|^2 d\boldsymbol{\zeta}$$
$$+ 2(T-t)\int_{\mathbb{R}^n}\int_t^T h(\boldsymbol{\zeta}) \left(\frac{e^{-c|\boldsymbol{\zeta}|^2 t}}{\epsilon^{\frac{s}{T}} + e^{-c|\boldsymbol{\zeta}|^2 s}}\right)^2 \mathcal{C}^\epsilon_{\Omega_\epsilon}(\boldsymbol{\zeta}) |(\hat{F}_{u^\epsilon}(\boldsymbol{\zeta}, t) - \hat{F}_{v^\epsilon}(\boldsymbol{\zeta}, t))|^2 ds d\boldsymbol{\zeta}$$
$$= 2\int_{\Omega_\epsilon} h(\boldsymbol{\zeta}) e^{2c|\boldsymbol{\zeta}|^2(T-t)} |\hat{\psi}(\boldsymbol{\zeta}) - \hat{\psi}(\boldsymbol{\zeta})|^2 d\boldsymbol{\zeta}$$
$$+ 2(T-t)\int_{\Omega_\epsilon}\int_t^T h(\boldsymbol{\zeta}) \left(\frac{e^{-c|\boldsymbol{\zeta}|^2 t}}{\epsilon^{\frac{s}{T}} + e^{-c|\boldsymbol{\zeta}|^2 s}}\right)^2 |\hat{F}_{u^\epsilon}(\boldsymbol{\zeta}, t) - \hat{F}_{v^\epsilon}(\boldsymbol{\zeta}, t)|^2 ds d\boldsymbol{\zeta}$$

since $R_\epsilon = \max\{h(\boldsymbol{\zeta}), \boldsymbol{\zeta} \in \Omega_\epsilon\}$, and $a_\epsilon \in \Omega_\epsilon$ such that $e^{2c|\boldsymbol{\zeta}|^2(T-t)}$ has the maximum at $\boldsymbol{\zeta} = a_\epsilon$, then

$$\leq 2R_\epsilon \int_{\Omega_\epsilon} e^{2c|\boldsymbol{\zeta}|^2(T-t)} |\hat{\psi}(\boldsymbol{\zeta}) - \hat{\psi}(\boldsymbol{\zeta})|^2 d\boldsymbol{\zeta}$$
$$+ 2R_\epsilon(T-t)\int_{\Omega_\epsilon}\int_t^T e^{2c|\boldsymbol{\zeta}|^2(s-t)} |\hat{F}_{u^\epsilon}(\boldsymbol{\zeta}, t) - \hat{F}_{v^\epsilon}(\boldsymbol{\zeta}, t)|^2 ds d\boldsymbol{\zeta}$$
$$\leq 2R_\epsilon e^{2c|a_\epsilon|^2(T-t)} \int_{\mathbb{R}^n} |\hat{\psi}(\boldsymbol{\zeta}) - \hat{\psi}(\boldsymbol{\zeta})|^2 d\boldsymbol{\zeta}$$
$$+ 2R_\epsilon(T-t)\int_t^T e^{2c|a_\epsilon|^2(s-t)} \int_{\mathbb{R}^n} |\hat{F}_{u^\epsilon}(\boldsymbol{\zeta}, t) - \hat{F}_{v^\epsilon}(\boldsymbol{\zeta}, t)|^2 d\boldsymbol{\zeta} ds$$
$$= 2R_\epsilon e^{-2c|a_\epsilon|^2 t} e^{2c|a_\epsilon|^2 T} \|\psi - \psi\|^2_{L^2(\mathbb{R}^n)}$$
$$+ 2R_\epsilon e^{-2c|a_\epsilon|^2 t} \int_t^T e^{2c|a_\epsilon|^2 s} \|F_{u^\epsilon}(\cdot, s) - F_{v^\epsilon}(\cdot, s)\|^2_{L^2(\mathbb{R}^n)} ds$$
$$\leq 2R_\epsilon e^{-2c|a_\epsilon|^2 t} e^{2c|a_\epsilon|^2 T} \|\hat{\psi} - \hat{\psi}\|^2_{L^2(\mathbb{R}^n)}$$
$$+ 2k^2 R_\epsilon e^{-2c|a_\epsilon|^2 t} \int_t^T e^{2c|a_\epsilon|^2 s} \|u^\epsilon(\cdot, s) - v^\epsilon(\cdot, s)\|^2_{H^2(\mathbb{R}^n)} ds$$

therefore, we have

$$\|u^\epsilon(\cdot, t) - v^\epsilon(\cdot, t)\|^2_{H^2(\mathbb{R}^n)} \leq 2R_\epsilon e^{-2c|a_\epsilon|^2 t} e^{2c|a_\epsilon|^2 T} \|\hat{\psi} - \hat{\psi}\|^2_{L^2(\mathbb{R}^n)}$$
$$+ 2k^2 R_\epsilon e^{-2c|a_\epsilon|^2 t} \int_t^T e^{2c|a_\epsilon|^2 s} \|u^\epsilon(\cdot, s) - v^\epsilon(\cdot, s)\|^2_{H^2(\mathbb{R}^n)} ds$$

and

$$e^{2c|a_\epsilon|^2 t} \|u^\epsilon(\cdot, t) - v^\epsilon(\cdot, t)\|^2_{H^2(\mathbb{R}^n)} \leq 2R_\epsilon e^{2c|a_\epsilon|^2 T} \|\hat{\psi} - \hat{\psi}\|^2_{L^2(\mathbb{R}^n)}$$
$$+ 2k^2 R_\epsilon(T-t) \int_t^T e^{2c|a_\epsilon|^2 s} \|u^\epsilon(\cdot, s) - v^\epsilon(\cdot, s)\|^2_{H^2(\mathbb{R}^n)} ds$$

Gronwall's Inequality implies

$$e^{2c|a_\epsilon|^2 t}\|u^\epsilon(\cdot,t)-v^\epsilon(\cdot,t)\|^2_{H^2(\mathbb{R}^n)} \le 2R_\epsilon e^{2c|a_\epsilon|^2 T} e^{2k^2 R_\epsilon (T-t)^2}\|\hat{\psi}-\hat{\psi}\|^2_{L^2(\mathbb{R}^n)}$$

and

$$\|u^\epsilon(\cdot,t)-v^\epsilon(\cdot,t)\|^2_{H^2(\mathbb{R}^n)} \le 2R_\epsilon e^{2k^2 R_\epsilon (T-t)^2} e^{2c|a_\epsilon|^2 (T-t)}\|\hat{\psi}-\hat{\psi}\|^2_{L^2(\mathbb{R}^n)}$$

□

This corollary implies that the approximation problem continuously depends on the given initial value.

Theorem 4.1. *Define for* $u^\epsilon \in H^2(\mathbb{R}^n)$

$$\begin{aligned} G(u^\epsilon(\boldsymbol{x},t)) &= \frac{1}{(\sqrt{2\pi})^n}\int_{\mathbb{R}^n} e^{c|\boldsymbol{\zeta}|^2(T-t)}\hat{\phi}(\boldsymbol{\zeta})\mathcal{C}^\epsilon_{\Omega_\epsilon}(\boldsymbol{\zeta})e^{i\boldsymbol{\zeta}\cdot\boldsymbol{x}}d\boldsymbol{\zeta} \\ &\quad - \frac{1}{(\sqrt{2\pi})^n}\int_{\mathbb{R}^n}\int_t^T \frac{e^{-c|\boldsymbol{\zeta}|^2 t}}{\epsilon^{\frac{s}{T}}+e^{-c|\boldsymbol{\zeta}|^2 s}}\hat{F}_{u^\epsilon}(\boldsymbol{\zeta},t)\mathcal{C}^\epsilon_{\Omega_\epsilon}(\boldsymbol{\zeta})ds \quad e^{i\boldsymbol{\zeta}\cdot\boldsymbol{x}}d\boldsymbol{\zeta} \end{aligned}$$

Then for $n \in \mathbb{N}$ *we have*

$$\|G^n(u^\epsilon(\cdot,t)) - G^n(v^\epsilon(\cdot,t))\|^2_{H^2(\mathbb{R}^n)} \le \frac{R^n_\epsilon M^{2n}k^{2n}T^n(T-t)^n}{n!}e^{2ncb_\epsilon T}|||u^\epsilon - v^\epsilon|||^2 \quad (4.9)$$

where $R_\epsilon = \max\{h(\boldsymbol{\zeta}) : \boldsymbol{\zeta}\in\Omega_\epsilon\}$, $b_\epsilon = \max|\boldsymbol{\zeta}|^2$ *on* Ω_ϵ, $|\mathcal{C}_\epsilon(\boldsymbol{\zeta})| \le M$ *for all* $\boldsymbol{\zeta}\in\Omega_\epsilon$, *and* $|||.|||$ *is the supremum norm in* $C([0,T],H^2(\mathbb{R}^n))$.
Also there exists an $u^\epsilon \in H^2(\mathbb{R}^n)$ *such that* $G(u^\epsilon(\boldsymbol{x},t)) = u^\epsilon(\boldsymbol{x},t)$

Proof. Proof by induction. Suppose $n=1$, then

$$\begin{aligned} G(u^\epsilon(\boldsymbol{x},t)) &= \frac{1}{(\sqrt{2\pi})^n}\int_{\mathbb{R}^n} e^{c|\boldsymbol{\zeta}|^2(T-t)}\hat{\phi}(\boldsymbol{\zeta})\mathcal{C}^\epsilon_{\Omega_\epsilon}(\boldsymbol{\zeta})e^{i\boldsymbol{\zeta}\cdot\boldsymbol{x}}d\boldsymbol{\zeta} \\ &\quad - \frac{1}{(\sqrt{2\pi})^n}\int_{\mathbb{R}^n}\int_t^T \frac{e^{-c|\boldsymbol{\zeta}|^2 t}}{\epsilon^{\frac{s}{T}}+e^{-c|\boldsymbol{\zeta}|^2 s}}\hat{F}_{u^\epsilon}(\boldsymbol{\zeta},t)\mathcal{C}^\epsilon_{\Omega_\epsilon}(\boldsymbol{\zeta})ds \quad e^{i\boldsymbol{\zeta}\cdot\boldsymbol{x}}d\boldsymbol{\zeta} \end{aligned}$$

For any $u^\epsilon, v^\epsilon \in H^2(\mathbb{R}^n) \times [0,T]$, consider $\|G(u^\epsilon(\cdot,t)) - G(v^\epsilon(\cdot,t))\|^2_{H^2(\mathbb{R}^n)}$, then

$$
\begin{aligned}
&= \int_{\mathbb{R}^n} h(\boldsymbol{\zeta}) \left| \int_t^T \frac{e^{-c|\boldsymbol{\zeta}|^2 t}}{\epsilon^{\frac{s}{T}} + e^{-c|\boldsymbol{\zeta}|^2 s}} (\hat{F}_{u^\epsilon}(\boldsymbol{\zeta},s) - \hat{F}_{v^\epsilon}(\boldsymbol{\zeta},s)) \mathcal{C}^\epsilon_{\Omega_\epsilon}(\boldsymbol{\zeta}) ds \right|^2 d\boldsymbol{\zeta} \\
&\leq (T-t) \int_{\Omega_\epsilon} h(\boldsymbol{\zeta}) \int_t^T \left(\frac{e^{-c|\boldsymbol{\zeta}|^2 t}}{\epsilon^{\frac{s}{T}} + e^{-c|\boldsymbol{\zeta}|^2 s}} \right)^2 |\hat{F}_{u^\epsilon}(\boldsymbol{\zeta},s) - \hat{F}_{v^\epsilon}(\boldsymbol{\zeta},s)|^2 |\mathcal{C}^\epsilon_{\Omega_\epsilon}|^2 ds d\boldsymbol{\zeta} \\
&\leq R_\epsilon M^2 (T-t) \int_t^T \int_{\Omega_\epsilon} e^{2c|\boldsymbol{\zeta}|^2(s-t)} |\hat{F}_{u^\epsilon}(\boldsymbol{\zeta},s) - \hat{F}_{v^\epsilon}(\boldsymbol{\zeta},s)|^2 d\boldsymbol{\zeta} ds
\end{aligned}
$$

since $b_\epsilon = \max\{|\boldsymbol{\zeta}|^2 \mid \boldsymbol{\zeta} \in \Omega_\epsilon\}$, then

$$
\begin{aligned}
&\leq R_\epsilon M^2 (T-t) \int_t^T \int_{\Omega_\epsilon} e^{2cb_\epsilon(s-t)} |\hat{F}_{u^\epsilon}(\boldsymbol{\zeta},s) - \hat{F}_{v^\epsilon}(\boldsymbol{\zeta},s)|^2 d\boldsymbol{\zeta} ds \\
&\leq R_\epsilon M^2 (T-t) e^{2cb_\epsilon T} \int_t^T \int_{\mathbb{R}^n} |\hat{F}_{u^\epsilon}(\boldsymbol{\zeta},s) - \hat{F}_{v^\epsilon}(\boldsymbol{\zeta},s)|^2 d\boldsymbol{\zeta} ds \\
&\leq R_\epsilon M^2 (T-t) e^{2cb_\epsilon T} \int_t^T \|\hat{F}_{u^\epsilon}(\cdot,s) - \hat{F}_{v^\epsilon}(\cdot,s)\|^2_{L^2(\mathbb{R}^n)} ds \\
&= R_\epsilon M^2 (T-t) e^{2cb_\epsilon T} \int_t^T \|F_{u^\epsilon}(\cdot,s) - F_{v^\epsilon}(\cdot,s)\|^2_{L^2(\mathbb{R}^n)} ds \\
&\leq R_\epsilon M^2 k^2 (T-t) e^{2cb_\epsilon T} \int_t^T \|u^\epsilon(\cdot,s) - v^\epsilon(\cdot,s)\|^2_{H^2(\mathbb{R}^n)} ds \\
&\leq R_\epsilon M^2 k^2 (T-t) e^{2cb_\epsilon T} \int_t^T \|u^\epsilon(\cdot,s) - v^\epsilon(\cdot,s)\|^2_{H^2(\mathbb{R}^n)} ds \\
&\leq R_\epsilon M^2 k^2 T (T-t) e^{2cb_\epsilon T} |||u^\epsilon - v^\epsilon|||^2
\end{aligned}
$$

This implies (4.9) is true for $n = 1$. Now suppose inequality (4.9) is true when $n = p$, that is,

$$
\|G^p(u^\epsilon(\cdot,t)) - G^p(v^\epsilon(\cdot,t))\|^2_{H^2(\mathbb{R}^n)} \leq \frac{R^p_\epsilon M^{2p} k^{2p} T^p (T-t)^p}{p!} e^{2pcb_\epsilon T} |||u^\epsilon - v^\epsilon|||^2
$$

We need to show (4.9) is true when $n = p+1$. Consider $\|G^{p+1}(u^\epsilon(\cdot,t)) - G^{p+1}(v^\epsilon(\cdot,t))\|^2_{H^2(\mathbb{R}^n)}$, then

$$
\begin{aligned}
&= \|G(G^p(u^\epsilon(\cdot,t))) - G(G^p(v^\epsilon(\cdot,t)))\|^2_{\cdot} H^2(\mathbb{R}^n) \\
&= \int_{\mathbb{R}^n} h(\boldsymbol{\zeta}) \left| \int_t^T \frac{e^{-c|\boldsymbol{\zeta}|^2 t}}{\epsilon^{\frac{s}{T}} + e^{-c|\boldsymbol{\zeta}|^2 s}} (\hat{F}_{G^p(u^\epsilon)}(\boldsymbol{\zeta},s) - \hat{F}_{G^p(v^\epsilon)}(\boldsymbol{\zeta},s)) \mathcal{C}^\epsilon_{\Omega_\epsilon}(\boldsymbol{\zeta}) ds \right|^2 d\boldsymbol{\zeta}
\end{aligned}
$$

$$\leq (T-t)\int_{\Omega_\epsilon} h(\boldsymbol{\zeta})\int_t^T \left(\frac{e^{-c|\boldsymbol{\zeta}|^2 t}}{\epsilon^{\frac{s}{T}}+e^{-c|\boldsymbol{\zeta}|^2 s}}\right)^2 |\hat{F}_{G^p(u^\epsilon)}(\boldsymbol{\zeta},s)-\hat{F}_{G^p(v^\epsilon)}(\boldsymbol{\zeta},s)|^2|\mathcal{C}^\epsilon_{\Omega_\epsilon}(\boldsymbol{\zeta})|^2 ds d\boldsymbol{\zeta}$$
$$\leq M^2(T-t)\int_{\Omega_\epsilon} h(\boldsymbol{\zeta})\int_t^T e^{c|\boldsymbol{\zeta}|^2(s-t)}|\hat{F}_{G^p(u^\epsilon)}(\boldsymbol{\zeta},s)-\hat{F}_{G^p(v^\epsilon)}(\boldsymbol{\zeta},s)|^2 ds d\boldsymbol{\zeta}$$
$$\leq R_\epsilon M^2k^2(T-t)e^{2cb_\epsilon T}\int_t^T \|G^p(u^\epsilon)(\cdot,s)-G^p(v^\epsilon)(\cdot,s)\|^2_{L^2(\mathbb{R}^n)}ds$$

by the hypothesis, we have

$$\begin{aligned}
&\leq R_\epsilon M^2k^2(T-t)e^{2cb_\epsilon T}\int_t^T \frac{R_\epsilon^p K^{2p}T^p(T-s)^p}{p!}e^{2pcb_\epsilon T}|||u^\epsilon-v^\epsilon|||^2 ds\\
&= \frac{R_\epsilon^{p+1}M^{2(p+1)}k^{2(p+1)}TT^p e^{2(p+1)cb_\epsilon T}}{p!}|||u^\epsilon-v^\epsilon|||^2_{H^2(\mathbb{R}^n)}\int_t^T (T-s)^p ds\\
&= \frac{R_\epsilon^{p+1}M^{2(p+1)}k^{2(p+1)}T^{p+1} e^{2(p+1)cb_\epsilon T}}{p!}|||u^\epsilon-v^\epsilon|||^2_{H^2(\mathbb{R}^n)}\frac{(T-t)^{p+1}}{p+1}\\
&= \frac{R_\epsilon^{p+1}M^{2(p+1)}k^{2(p+1)}T^{p+1}(T-t)^{p+1} e^{2(p+1)cb_\epsilon T}}{(p+1)!}|||u^\epsilon-v^\epsilon|||^2_{H^2(\mathbb{R}^n)}
\end{aligned}$$

therefore (4.9) is true for all $n\in\mathbb{N}$. That is, for all $n\in\mathbb{N}$ we have

$$\|G^n(u^\epsilon(\cdot,t))-G^n(v^\epsilon(\cdot,t))\|^2_{H^2(\mathbb{R}^n)}\leq \frac{R_\epsilon^n M^{2n}k^{2n}T^n(T-t)^n}{n!}e^{2ncb_\epsilon T}|||u^\epsilon-v^\epsilon|||^2$$

Let $s_n=\frac{R_\epsilon^n M^{2n}k^{2n}T^n(T-t)^n}{n!}e^{2ncb_\epsilon T}$, then $\sum_{n\in\mathbb{N}} s_n = e^{R_\epsilon M^2k^2T(T-t)L_\epsilon}$ where $L_\epsilon = e^{2cb_\epsilon T}$. That is s_n is convergent sequence. Then there exists $n_0\in\mathbb{N}$ such that $s_n<1$ for all $n>n_0$. The Banach fixed point theorem implies there exists a unique solution $u^\epsilon\in H^2(\mathbb{R}^n)$ such that $G^{n_0}(u^\epsilon)=u^\epsilon$ and this implies $G(u^\epsilon)=u^\epsilon$.

Now suppose u^ϵ and v^ϵ are two solutions of the approximation problem (4.5), then

$$\begin{aligned}
u^\epsilon(\boldsymbol{x},t) &= \frac{1}{(\sqrt{2\pi})^n}\int_{\mathbb{R}^n}\frac{e^{-c|\boldsymbol{\zeta}|^2 t}\hat{\phi}(\boldsymbol{\zeta})}{\epsilon+e^{-c|\boldsymbol{\zeta}|^2T}}\mathcal{C}^\epsilon_{\Omega_\epsilon}(\boldsymbol{\zeta})e^{i\boldsymbol{\zeta}\cdot\boldsymbol{x}}d\boldsymbol{\zeta}\\
&\quad -\frac{1}{(\sqrt{2\pi})^n}\int_{\mathbb{R}^n}\int_t^T\frac{e^{-c|\boldsymbol{\zeta}|^2 t}}{\epsilon^{\frac{s}{T}}+e^{-c|\boldsymbol{\zeta}|^2 s}}\hat{F}_{u^\epsilon}(\boldsymbol{\zeta},t)\mathcal{C}^\epsilon_{\Omega_\epsilon}(\boldsymbol{\zeta})ds \quad e^{t\boldsymbol{\zeta}\cdot\boldsymbol{x}}d\boldsymbol{\zeta}\\
v^\epsilon(\boldsymbol{x},t) &= \frac{1}{(\sqrt{2\pi})^n}\int_{\mathbb{R}^n}\frac{e^{-c|\boldsymbol{\zeta}|^2 t}\hat{\phi}(\boldsymbol{\zeta})}{\epsilon+e^{-c|\boldsymbol{\zeta}|^2T}}\mathcal{C}^\epsilon_{\Omega_\epsilon}(\boldsymbol{\zeta})e^{i\boldsymbol{\zeta}\cdot\boldsymbol{x}}d\boldsymbol{\zeta}\\
&\quad -\frac{1}{(\sqrt{2\pi})^n}\int_{\mathbb{R}^n}\int_t^T\frac{e^{-c|\boldsymbol{\zeta}|^2 t}}{\epsilon^{\frac{s}{T}}+e^{-c|\boldsymbol{\zeta}|^2 s}}\hat{F}_{v^\epsilon}(\boldsymbol{\zeta},t)\mathcal{C}^\epsilon_{\Omega_\epsilon}(\boldsymbol{\zeta})ds \quad e^{t\boldsymbol{\zeta}\cdot\boldsymbol{x}}d\boldsymbol{\zeta}
\end{aligned}$$

Consider $\|u^\epsilon(\cdot,t)-v^\epsilon(\cdot,t)\|_{H^2(\mathbb{R}^n)}$, then

$$
\begin{aligned}
&= \int_{\mathbb{R}^n} h(\boldsymbol{\zeta}) \left| \int_t^T \frac{e^{-c|\boldsymbol{\zeta}|^2 t}}{\epsilon^{\frac{s}{T}} + e^{-c|\boldsymbol{\zeta}|^2 s}} (\hat{F}_{u^\epsilon}(\boldsymbol{\zeta},t) - \hat{F}_{v^\epsilon}(\boldsymbol{\zeta},t)) \mathcal{C}^\epsilon_{\Omega_\epsilon}(\boldsymbol{\zeta}) ds \right|^2 d\boldsymbol{\zeta} \\
&\leq (T-t) \int_{\mathbb{R}^n} h(\boldsymbol{\zeta}) \int_t^T \left(\frac{e^{-c|\boldsymbol{\zeta}|^2 t}}{\epsilon^{\frac{s}{T}} + e^{-c|\boldsymbol{\zeta}|^2 s}} \right)^2 |\hat{F}_{u^\epsilon}(\boldsymbol{\zeta},t) - \hat{F}_{v^\epsilon}(\boldsymbol{\zeta},t)|^2 |\mathcal{C}^\epsilon_{\Omega_\epsilon}(\boldsymbol{\zeta})|^2 ds d\boldsymbol{\zeta} \\
&\leq M^2 (T-t) \int_{\Omega_\epsilon} h(\boldsymbol{\zeta}) \int_t^T \left(\frac{e^{-c|\boldsymbol{\zeta}|^2 t}}{\epsilon^{\frac{s}{T}} + e^{-c|\boldsymbol{\zeta}|^2 s}} \right)^2 |\hat{F}_{u^\epsilon}(\boldsymbol{\zeta},t) - \hat{F}_{v^\epsilon}(\boldsymbol{\zeta},t)|^2 ds d\boldsymbol{\zeta} \\
&\leq R_\epsilon M^2 (T-t) \int_{\Omega_\epsilon} \int_t^T e^{\frac{t}{T} - \frac{s}{T}} |\hat{F}_{u^\epsilon}(\boldsymbol{\zeta},t) - \hat{F}_{v^\epsilon}(\boldsymbol{\zeta},t)|^2 ds d\boldsymbol{\zeta} \\
&\leq R_\epsilon M^2 (T-t) e^{\frac{t}{T}} \int_t^T \int_{\mathbb{R}^n} e^{-\frac{s}{T}} |\hat{F}_{u^\epsilon}(\boldsymbol{\zeta},t) - \hat{F}_{v^\epsilon}(\boldsymbol{\zeta},t)|^2 ds d\boldsymbol{\zeta} \\
&\leq R_\epsilon M^2 (T-t) e^{\frac{t}{T}} \int_t^T e^{-\frac{s}{T}} \int_{\mathbb{R}^n} |\hat{F}_{u^\epsilon}(\boldsymbol{\zeta},t) - \hat{F}_{v^\epsilon}(\boldsymbol{\zeta},t)|^2 d\boldsymbol{\zeta} ds \\
&\leq R_\epsilon M^2 k^2 (T-t) e^{\frac{t}{T}} \int_t^T e^{-\frac{s}{T}} \|u^\epsilon(\cdot,s) - v^\epsilon(\cdot,s)\|^2_{H^2(\mathbb{R}^n)} ds
\end{aligned}
$$

that is

$$
\|u^\epsilon(\cdot,t) - v^\epsilon(\cdot,t)\|^2_{H^2(\mathbb{R}^n)} \leq R_\epsilon M^2 k^2 (T-t) e^{\frac{t}{T}} \int_t^T e^{-\frac{s}{T}} \|u^\epsilon(\cdot,s) - v^\epsilon(\cdot,s)\|^2_{H^2(\mathbb{R}^n)} ds
$$

Gronwall's Inequality implies $\|u^\epsilon(\cdot,t)-v^\epsilon(\cdot,t)\|^2_{H^2(\mathbb{R}^n)} = 0$ and $u^\epsilon = v^\epsilon$, that is the solution is unique. □

THEOREM 4.2. *Suppose $\mathcal{C}^\epsilon(\boldsymbol{x}) = 1$ on $\boldsymbol{x} \in \Omega_\epsilon$. Let $\hat{G}(\boldsymbol{\zeta},t) = e^{|\boldsymbol{\zeta}|^4(cp+cT+\frac{3}{2})}|\hat{g}(\boldsymbol{\zeta})|$ such that $G \in L^2(\mathbb{R}^n)$. Suppose*

$$
\mathcal{B}_p = \int_0^T \int_{\mathbb{R}^n} e^{2|\boldsymbol{\zeta}|^4(cT+cp+\frac{3}{2})} |\hat{F}_u(\boldsymbol{\zeta},s)|^2 d\boldsymbol{\zeta} ds < \infty
$$

If $u^\epsilon(\boldsymbol{x},t)$ and $u(\boldsymbol{x},t)$ are solution representations of the approximation and original problems respectively, then for $p > 0$

$$
\|u^\epsilon(\cdot,t) - u(\cdot,t)\|^2_{H^2(\mathbb{R}^n)} \leq \Phi(t) e^{-2|\boldsymbol{\zeta}_\epsilon|^4(ct+cp-\frac{3k^2T^2}{2})}
$$

where

$$
\Phi(t) = 2\|G(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + 2(T-t)\mathcal{B}_p
$$

and $\boldsymbol{\zeta}_\epsilon \in \Omega_\epsilon$ such that $e^{2c|\boldsymbol{\zeta}|^4(T-t)}$ has the maximum value at $\boldsymbol{\zeta} = \boldsymbol{\zeta}_\epsilon$.

Proof. Let $u^\epsilon(\boldsymbol{x},t)$ and $u(\boldsymbol{x},t)$ be solution representations of the approximation and original problems respectively. Then

$$\begin{aligned}
\|u^\epsilon(\cdot,t)-u(\cdot,t)\|^2_{H^2(\mathbb{R}^n)} &= \int_{\mathbb{R}^n}(1+|\boldsymbol{\zeta}|^2+|\boldsymbol{\zeta}|^4)|\hat{u}^\epsilon(\boldsymbol{\zeta},t)-\hat{u}(\boldsymbol{\zeta},t)|^2 d\boldsymbol{\zeta} \\
&= \int_{\Omega_\epsilon}(1+|\boldsymbol{\zeta}|^2+|\boldsymbol{\zeta}|^4)|\hat{u}^\epsilon(\boldsymbol{\zeta},t)-\hat{u}(\boldsymbol{\zeta},t)|^2 d\boldsymbol{\zeta} \\
&+ \int_{\Omega_\epsilon^c}(1+|\boldsymbol{\zeta}|^2+|\boldsymbol{\zeta}|^4)|\hat{u}^\epsilon(\boldsymbol{\zeta},t)-\hat{u}(\boldsymbol{\zeta},t)|^2 d\boldsymbol{\zeta} \\
&= \int_{\Omega_\epsilon} h(\boldsymbol{\zeta})|\hat{u}^\epsilon(\boldsymbol{\zeta},t)-\hat{u}(\boldsymbol{\zeta},t)|^2 d\boldsymbol{\zeta} + \int_{\Omega_\epsilon^c} h(\boldsymbol{\zeta})|\hat{u}^\epsilon(\boldsymbol{\zeta},t)-\hat{u}(\boldsymbol{\zeta},t)|^2 d\boldsymbol{\zeta}
\end{aligned}$$

Let

$$\begin{aligned}
l_1 &= \int_{\Omega_\epsilon} h(\boldsymbol{\zeta})|\hat{u}^\epsilon(\boldsymbol{\zeta},t)-\hat{u}(\boldsymbol{\zeta},t)|^2 d\boldsymbol{\zeta} \\
l_2 &= \int_{\Omega_\epsilon^c} h(\boldsymbol{\zeta})|\hat{u}^\epsilon(\boldsymbol{\zeta},t)-\hat{u}(\boldsymbol{\zeta},t)|^2 d\boldsymbol{\zeta}
\end{aligned}$$

Consider

$$l_1 = \int_{\Omega_\epsilon} h(\boldsymbol{\zeta})|\hat{u}^\epsilon(\boldsymbol{\zeta},t)-\hat{u}(\boldsymbol{\zeta},t)|^2 d\boldsymbol{\zeta}$$

then

$$\begin{aligned}
l_1 &= \int_{\Omega_\epsilon} h(\boldsymbol{\zeta}) \Bigg| e^{c|\boldsymbol{\zeta}|^2(T-t)}\hat{g}(\boldsymbol{\zeta}) - \int_t^T \frac{e^{-c|\boldsymbol{\zeta}|^2 t}}{\epsilon^{\frac{s}{T}}+e^{-c|\boldsymbol{\zeta}|^2 s}}\hat{F}_{u^\epsilon}(\boldsymbol{\zeta},t)ds - e^{c|\boldsymbol{\zeta}|^2(T-t)}\hat{g}(\boldsymbol{\zeta}) \\
&+ \int_t^T e^{c|\boldsymbol{\zeta}|^2(s-t)}\hat{F}_u(\boldsymbol{\zeta},s)ds\Bigg|^2 d\boldsymbol{\zeta} \\
&= \int_{\Omega_\epsilon} h(\boldsymbol{\zeta})\left|\int_t^T e^{c|\boldsymbol{\zeta}|^2(s-t)}\hat{F}_u(\boldsymbol{\zeta},s)ds - \int_t^T \frac{e^{-c|\boldsymbol{\zeta}|^2 t}}{\epsilon^{\frac{s}{T}}+e^{-c|\boldsymbol{\zeta}|^2 s}}\hat{F}_{u^\epsilon}(\boldsymbol{\zeta},t)(\boldsymbol{\zeta})ds\right|^2 d\boldsymbol{\zeta} \\
&\leq \int_{\Omega_\epsilon} h(\boldsymbol{\zeta})\left|\int_t^T e^{c|\boldsymbol{\zeta}|^2(s-t)}\left(\hat{F}_u(\boldsymbol{\zeta},s)-\hat{F}_{u^\epsilon}(\boldsymbol{\zeta},t)\right)ds\right|^2 d\boldsymbol{\zeta} \\
&\leq (T-t)\int_{\Omega_\epsilon} h(\boldsymbol{\zeta})\int_t^T e^{2c|\boldsymbol{\zeta}|^2(s-t)}\left|\hat{F}_u(\boldsymbol{\zeta},s)-\hat{F}_{u^\epsilon}(\boldsymbol{\zeta},t)\right|^2 ds d\boldsymbol{\zeta}
\end{aligned}$$

We know that if $|\boldsymbol{\zeta}|\geq 1$, then $h(\boldsymbol{\zeta})\leq e^{3|\boldsymbol{\zeta}|^4}$, then

$$\leq (T-t)\int_{\Omega_\epsilon} e^{3|\boldsymbol{\zeta}|^4}\int_t^T e^{2c|\boldsymbol{\zeta}|^2(s-t)}\left|\hat{F}_u(\boldsymbol{\zeta},s)-\hat{F}_{u^\epsilon}(\boldsymbol{\zeta},t)\right|^2 ds d\boldsymbol{\zeta}$$

Let $p > 0$ and $\boldsymbol{\zeta}_\epsilon \in \Omega_\epsilon$ such that $e^{2c|\boldsymbol{\zeta}|^4(T-t)}$ has the maximum value at $\boldsymbol{\zeta} = \boldsymbol{\zeta}_\epsilon$. Also $h(\boldsymbol{\zeta})$ takes its maximum at $\boldsymbol{\zeta}_\epsilon$ on Ω_e. Therefore we rewrite the above inequality as follows:

$$\begin{aligned}
&\leq (T-t)e^{-2c|\boldsymbol{\zeta}_\epsilon|^4(t+p)}h(\boldsymbol{\zeta}_\epsilon)\int_{\Omega_\epsilon}\int_t^T e^{2c|\boldsymbol{\zeta}_\epsilon|^4(s+p)}|\hat{F}_u(\boldsymbol{\zeta},s)-\hat{F}_{u^\epsilon}(\boldsymbol{\zeta},t)|^2 ds d\boldsymbol{\zeta} \\
&\leq (T-t)e^{-2c|\boldsymbol{\zeta}_\epsilon|^4(t+p)}h(\boldsymbol{\zeta}_\epsilon)\int_t^T e^{2c|\boldsymbol{\zeta}_\epsilon|^4(s+p)}\int_{\mathbb{R}^n}|\hat{F}_u(\boldsymbol{\zeta},s)-\hat{F}_{u^\epsilon}(\boldsymbol{\zeta},t)|^2 d\boldsymbol{\zeta} ds \\
&= (T-t)e^{-2c|\boldsymbol{\zeta}_\epsilon|^4(t+p)}h(\boldsymbol{\zeta}_\epsilon)\int_t^T e^{2c|\boldsymbol{\zeta}_\epsilon|^4(s+p)}\|\hat{F}_u(\cdot,s)-\hat{F}_{u^\epsilon}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} ds \\
&\leq k^2(T-t)e^{-2c|\boldsymbol{\zeta}_\epsilon|^4(t+p)}h(\boldsymbol{\zeta}_\epsilon)\int_t^T e^{2c|\boldsymbol{\zeta}_\epsilon|^4(s+p)}\|u(\cdot,s)-u^\epsilon(\cdot,t)\|^2_{H^2(\mathbb{R}^n)} ds
\end{aligned}$$

Also, we have

$$\begin{aligned}
l_2 &= \int_{\Omega_\epsilon^c} h(\boldsymbol{\zeta})\left|e^{c|\boldsymbol{\zeta}|^2(T-t)}\hat{g}(\boldsymbol{\zeta}) - \int_t^T \frac{e^{-c|\boldsymbol{\zeta}|^2 t}}{\epsilon^{\frac{s}{T}}+e^{-c|\boldsymbol{\zeta}|^2 s}}\hat{F}_{u^\epsilon}(\boldsymbol{\zeta},t)ds - e^{c|\boldsymbol{\zeta}|^2(T-t)}\hat{g}(\boldsymbol{\zeta})\right. \\
&\quad \left. + \int_t^T e^{c|\boldsymbol{\zeta}|^2(s-t)}\hat{F}_u(\boldsymbol{\zeta},s)ds\right|^2 d\boldsymbol{\zeta} \\
&= \int_{\Omega_\epsilon^c} h(\boldsymbol{\zeta})\left|-e^{c|\boldsymbol{\zeta}|^2(T-t)}\hat{g}(\boldsymbol{\zeta}) + \int_t^T e^{c|\boldsymbol{\zeta}|^2(s-t)}\hat{F}_u(\boldsymbol{\zeta},s)ds\right|^2 d\boldsymbol{\zeta} \\
&\leq \int_{\Omega_\epsilon^c} h(\boldsymbol{\zeta})\left(2e^{2c|\boldsymbol{\zeta}|^2(T-t)}|\hat{g}(\boldsymbol{\zeta})|^2 + 2\left|\int_t^T e^{c|\boldsymbol{\zeta}|^2(s-t)}\hat{F}_u(\boldsymbol{\zeta},s)ds\right|^2\right)d\boldsymbol{\zeta} \\
&\leq \int_{\Omega_\epsilon^c} h(\boldsymbol{\zeta})\left(2e^{2c|\boldsymbol{\zeta}|^2(T-t)}|\hat{g}(\boldsymbol{\zeta})|^2 + 2(T-t)\int_t^T e^{2c|\boldsymbol{\zeta}|^2(s-t)}|\hat{F}_u(\boldsymbol{\zeta},s)|^2 ds\right)d\boldsymbol{\zeta} \\
&= \int_{\Omega_\epsilon^c} 2h(\boldsymbol{\zeta})e^{2c|\boldsymbol{\zeta}|^2(T-t)}|\hat{g}(\boldsymbol{\zeta})|^2 d\boldsymbol{\zeta} + 2(T-t)\int_{\Omega_\epsilon^c} h(\boldsymbol{\zeta})\int_t^T e^{2c|\boldsymbol{\zeta}|^2(s-t)}|\hat{F}_u(\boldsymbol{\zeta},s)|^2 ds d\boldsymbol{\zeta} \\
&\leq \int_{\Omega_\epsilon^c} 2e^{3|\boldsymbol{\zeta}|^4}e^{2c|\boldsymbol{\zeta}|^4(T-t)}|\hat{g}(\boldsymbol{\zeta})|^2 d\boldsymbol{\zeta} + 2(T-t)\int_{\Omega_\epsilon^c} e^{3|\boldsymbol{\zeta}|^4}\int_t^T e^{2c|\boldsymbol{\zeta}|^4(s-t)}|\hat{F}_u(\boldsymbol{\zeta},s)|^2 ds d\boldsymbol{\zeta} \\
&\leq 2e^{-2c|\boldsymbol{\zeta}_\epsilon|^2(t+p)}\int_{\Omega_\epsilon^c} e^{2|\boldsymbol{\zeta}|^4(cT+cp+\frac{3}{2})}|\hat{g}(\boldsymbol{\zeta})|^2 d\boldsymbol{\zeta} \\
&\quad + 2(T-t)e^{-2c|\boldsymbol{\zeta}_\epsilon|^2(t+p)}\int_t^T\int_{\Omega_\epsilon^c} e^{2|\boldsymbol{\zeta}|^4(cT+cp+\frac{3}{2})}|\hat{F}_u(\boldsymbol{\zeta},s)|^2 d\boldsymbol{\zeta} ds \\
&\leq 2e^{-2c|\boldsymbol{\zeta}_\epsilon|^2(t+p)}\int_{\mathbb{R}^n} e^{2|\boldsymbol{\zeta}|^4(cT+cp+\frac{3}{2})}|\hat{g}(\boldsymbol{\zeta})|^2 d\boldsymbol{\zeta} \\
&\quad + 2(T-t)e^{-2c|\boldsymbol{\zeta}_\epsilon|^2(t+p)}\int_t^T\int_{\mathbb{R}^n} e^{2|\boldsymbol{\zeta}|^4(cT+cp+\frac{3}{2})}|\hat{F}_u(\boldsymbol{\zeta},s)|^2 d\boldsymbol{\zeta} ds \\
&= 2e^{-2c|\boldsymbol{\zeta}_\epsilon|^2(t+p)}\|G(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + 2(T-t)e^{-2c|\boldsymbol{\zeta}_\epsilon|^2(t+p)}\mathcal{B}_p
\end{aligned}$$

Since $\|u^\epsilon(\cdot,t)-u(\cdot,t)\|^2_{H^2(\mathbb{R}^n)} = l_1 + l_2$, therefore,

$$\leq k^2(T-t)e^{-2c|\boldsymbol{\zeta}_\epsilon|^4(t+p)}h(\boldsymbol{\zeta}_\epsilon)\int_t^T e^{2c|\boldsymbol{\zeta}_\epsilon|^4(s+p)}\|u(\cdot,s)-u^\epsilon(\cdot,t)\|^2_{H^2(\mathbb{R}^n)}ds$$
$$+ 2e^{-2c|\boldsymbol{\zeta}_\epsilon|^2(t+p)}\|G(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + 2(T-t)e^{-2c|\boldsymbol{\zeta}_\epsilon|^2(t+p)}\mathcal{B}_p$$

and

$$e^{2c|\boldsymbol{\zeta}_\epsilon|^4(t+p)}\|u^\epsilon(\cdot,t)-u(\cdot,t)\|^2_{H^2(\mathbb{R}^n)} \leq \Phi_\epsilon + k^2(T-t)h(\boldsymbol{\zeta}_\epsilon) \times \int_t^T e^{2c|\boldsymbol{\zeta}_\epsilon|^4(s+p)}\|u(\cdot,s)-u^\epsilon(\cdot,t)\|^2_{H^2(\mathbb{R}^n)}ds$$

where

$$\Phi(t) = 2\|G(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + 2(T-t)\mathcal{B}_p$$

Gronwall's Inequality implies

$$e^{2c|\boldsymbol{\zeta}_\epsilon|^4(t+p)}\|u^\epsilon(\cdot,t)-u(\cdot,t)\|^2_{H^2(\mathbb{R}^n)} \leq \Phi(t)e^{k^2(T-t)^2h(\boldsymbol{\zeta}_\epsilon)}$$

since all $|\boldsymbol{\zeta}| > 1$ we have $h(\boldsymbol{\zeta}) \leq 3|\boldsymbol{\zeta}|^4$, then

$$\begin{aligned}\|u^\epsilon(\cdot,t)-u(\cdot,t)\|^2_{H^2(\mathbb{R}^n)} &\leq \Phi(t)e^{k^2T^2h(\boldsymbol{\zeta}_\epsilon)}e^{-2c|\boldsymbol{\zeta}_\epsilon|^4(t+p)}\\ &= \Phi(t)e^{3k^2T^2|\boldsymbol{\zeta}_\epsilon|^4}e^{-2c|\boldsymbol{\zeta}_\epsilon|^4(t+p)}\\ &\leq \Phi(t)e^{-2|\boldsymbol{\zeta}_\epsilon|^4(ct+cp-\frac{3k^2T^2}{2})}\end{aligned}$$

□

Example 4.3. *Suppose $n = 1$ case, choose $\epsilon > 0$ such that $|\ln(\frac{1}{\epsilon})| > 1$ and consider $\Omega_\epsilon = [-\left(\ln(\frac{1}{\epsilon})\right)^{\frac{1}{4}}, \left(\ln(\frac{1}{\epsilon})\right)^{\frac{1}{4}}]$. It is clear that $\Omega_\epsilon \to \mathbb{R}$ whenever $\epsilon \to 0$. Also, we have $\boldsymbol{\zeta}_\epsilon = \left(\ln(\frac{1}{\epsilon})\right)^{\frac{1}{4}}$ and choose $p > \frac{3k^2T^2}{2c}$. then $cp - \frac{3k^2T^2}{2} > 0$, therefore*

$$\|u^\epsilon(\cdot,t)-u(\cdot,t)\|^2_{H^2(\mathbb{R}^n)} \leq \Phi(t)e^{-2(ct+cp-\frac{3k^2T^2}{2})\ln(\frac{1}{\epsilon})}$$

and

$$\|u^\epsilon(\cdot,t)-u(\cdot,t)\|^2_{H^2(\mathbb{R}^n)} \leq \Phi(t)\epsilon^{2(ct+cp-\frac{3k^2T^2}{2})}$$

EXAMPLE 4.4. *Choose* $n = 1$ *and* $0 < \epsilon < 1$*, consider* $\Omega_\epsilon = [-e^{\frac{1}{4\epsilon}}, e^{\frac{1}{4\epsilon}}]$*, then* $\boldsymbol{\zeta}_\epsilon = e^{\frac{1}{4\epsilon}}$ *and*

$$\|u^\epsilon(\cdot,t) - u(\cdot,t)\|^2_{H^2(\mathbb{R}^n)} \leq \Phi(t)e^{-2(ct+cp-\frac{3k^2T^2}{2})e^{\frac{1}{\epsilon}}}$$

EXAMPLE 4.5. *Choose* $n = 1$*, and choose* $\epsilon > 0$ *such that* $|\ln\ln(\frac{1}{\epsilon})| > 1$ *consider* $\Omega_\epsilon = [-\left(\ln(\ln\frac{1}{\epsilon})\right)^{\frac{1}{4}}, \left(\ln(\ln\frac{1}{\epsilon})\right)^{\frac{1}{4}}]$*, then* $\boldsymbol{\zeta}_\epsilon = \left(\ln(\ln\frac{1}{\epsilon})\right)^{\frac{1}{4}}$

$$\begin{aligned}\|u^\epsilon(\cdot,t) - u(\cdot,t)\|^2_{H^2(\mathbb{R}^n)} &\leq \Phi(t)e^{-2\left(ct+cp-\frac{3k^2T^2}{2}\right)\left(\ln(\ln\frac{1}{\epsilon})\right)} \\ &= \Phi(t)e^{\ln\left(\ln\frac{1}{\epsilon}\right)^{-2\left(ct+cp-\frac{3k^2T^2}{2}\right)}} \\ &= \Phi(t)\left(\ln\frac{1}{\epsilon}\right)^{-2\left(ct+cp-\frac{3k^2T^2}{2}\right)} \\ &= \frac{\Phi(t)}{\left(\ln\frac{1}{\epsilon}\right)^{2\left(ct+cp-\frac{3k^2T^2}{2}\right)}}\end{aligned}$$

The above examples show that the approximation solution converges to the original solution whenever $\epsilon \to 0$.

CHAPTER 5
CONCLUSION AND FUTURE DIRECTIONS

5.1 Conclusion

We consider a quasi- boundary value method and a modified quasi-boundary value method to regularize time and space dependent thermal conductivity heat equation with a terminal condition.

We use a modified quasi-boundary value method to regularize time dependent thermal conductivity heat equation with a terminal condition. In many earlier works on the nonlinear problems at any fixed time $t > 0$, an explicit error estimate at $t = 0$ is still difficult. Our calculation also implies we cannot prove main results when $t = 0$. But all results are valid for $0 < t \leq T$.

In contrast, we prove that the explicit error estimates are valid for all $t \in [0, T]$ when we use a quasi-boundary value method to regularize the space dependent thermal conductivity heat equation with a terminal condition. But we do not have a direct integral form to see the solution representation of the original problem. So it is necessary to use a transformation method to convert this problem to either the time or constant dependent thermal conductivity heat equation.

5.2 Future Research

Backward Heat equation is one of the ill-posed problems in partial differential equations. There are many ill-posed problems in PDE such as Cauchy problem for elliptic equation, parabolic equation with a terminal condition etc., requiring regularization. There are many applications in Physics, Engineering, Neuroscience etc, especially, the parabolic equation with a terminal condition,

For my future research, I would like to continue this research area and would like to consider the following problems.

- For given $T > 0$ and $\Omega \subset \mathbb{R}^n$, for $n \geq 1$ is an open bounded domain with a smooth boundary Γ. Set $Q = \Omega \times (0, T)$ and $\Sigma = \Gamma \times (0, T)$; Σ is called the lateral boundary of the cylinder Q. Now we consider the question of finding the function $u(\boldsymbol{x}, t) \in \Omega \times [0, T]$, satisfying the problem (3.5) and (4.4) with $u \equiv 0$ on Σ.

- Regularization of parabolic equations with a locally Lipschitz continuous source function.

In my future research, I plan to explore regularization mechanisms for these problems.

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