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STRUCTURE AND PROPERTIES OF MAXIMAL OUTERPLANAR GRAPHS

By

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B.A., University of Louisville, 2003
M.A., University of Louisville, 2005

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STRUCTURE AND PROPERTIES OF MAXIMAL OUTERPLANAR GRAPHS

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ABSTRACT

STRUCTURE AND PROPERTIES OF MAXIMAL OUTERPLANAR GRAPHS

Benjamin Allgeier

August 11, 2009

Outerplanar graphs are planar graphs that have a plane embedding in which each vertex lies on the boundary of the exterior region. An outerplanar graph is maximal outerplanar if the graph obtained by adding an edge is not outerplanar. Maximal outerplanar graphs are also known as triangulations of polygons. The spine of a maximal outerplanar graph G is the dual graph of G without the vertex that corresponds to the exterior region.

In this thesis we study metric properties involving geodesic intervals, geodetic sets, Steiner sets, different concepts of boundary, and also relationships between the independence numbers and domination numbers of maximal outerplanar graphs and their spines. In Chapter 2 we find an extension of a result by Beyer, *et al.* [3] that deals with hamiltonian degree sequences in maximal outerplanar graphs. In Chapters 3 and 4 we give sharp bounds relating the independence number and domination number, respectively, of a maximal outerplanar graph to those of its spine. In Chapter 5 we discuss the boundary, contour, eccentricity, periphery, and extreme set of a graph. We give a characterization of the boundary of maximal outerplanar graphs that involves the degrees of vertices. We find properties that characterize the contour of a maximal outerplanar graph. The other main result of this chapter

gives characterizations of graphs induced by the contour and by the periphery of a maximal outerplanar graph. In Chapter 6 we show that the generalized intervals in a maximal outerplanar graph are convex. We use this result to characterize geodetic sets in maximal outerplanar graphs. We show that every Steiner set in a maximal outerplanar graph is a geodetic set and also show some differences between these types of sets. We present sharp bounds for geodetic numbers and Steiner numbers of maximal outerplanar graphs.

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CHAPTER 1
INTRODUCTION

1.1 Basic definitions and notation

In this section, the reader is provided basic graph theory definitions and notations used throughout this dissertation.

A **graph** G is a finite nonempty set of objects called **vertices** (the singular is **vertex**) together with a (possibly empty) set consisting of 2-element subsets of vertices of G called **edges**. The set of vertices of G, denoted by $V(G)$, is the **vertex set** of G, while the set of edges of G, denoted by $E(G)$, is the **edge set** of G. The cardinality of $V(G)$ is the **order** of G, while the cardinality of $E(G)$ is the **size** of G. A graph is **trivial** if the order of the graph is 1; otherwise, the graph is **nontrivial**.

Let u and v be distinct vertices of a graph G. If $\{u, v\}$ is an edge of G, then u and v are **adjacent vertices**. If e is an edge of G such that $e = \{u, v\}$, then e **joins** u and v, while u and e are **incident**, as are v and e. For convenience, we will denote the edge e by uv or vu rather than by $\{u, v\}$. Every vertex of G that is adjacent to v is a **neighbor** of v. The **neighborhood** $N_G(v)$ of v is the set of all neighbors of v, while the **closed neighborhood** $N_G[v]$ is $N_G(v) \cup \{v\}$. If the graph under consideration is clear, then we write more simply $N(v)$ or $N[v]$. If X is a set of vertices of G, then the **closed neighborhood** $N_G[X]$ of X, or simply $N[X]$, is the union of $N[v]$ over all $v \in X$. The **degree** $\deg_G(v)$ of v, or simply $\deg(v)$, is the number of edges incident to v; note that $\deg(v) = |N(v)|$.

If $\deg(v) = 1$, then v is an **end-vertex** or a **leaf**. If e is an edge incident to an end-vertex, then e is a **pendant edge**; otherwise, the edge e is a **nonpendant edge**. The **maximum degree** of G is the maximum degree among the vertices of G.

It is customary to define or describe a graph G by means of a diagram in which each vertex of G is represented by a point (which we draw as a small circle) and each edge is represented by a line segment or curve joining the points corresponding to the vertices incident to the edge. We then refer to this diagram as the graph G itself. In some instances, we will name a vertex by giving it a label. Two graphs may have the same structure, differing only in the way their vertices are labeled or in the way they are drawn. The next definition describes this more precisely. A graph G_1 is **isomorphic** to a graph G_2 if there exists a one-to-one mapping Φ, called an **isomorphism**, from $V(G_1)$ onto $V(G_2)$ such that Φ preserves adjacency and nonadjacency; that is, $uv \in E(G_1)$ if and only if $\Phi(u)\Phi(v) \in E(G_2)$ for every $u, v \in V(G_1)$. Figure 1.1 displays two isomorphic graphs.

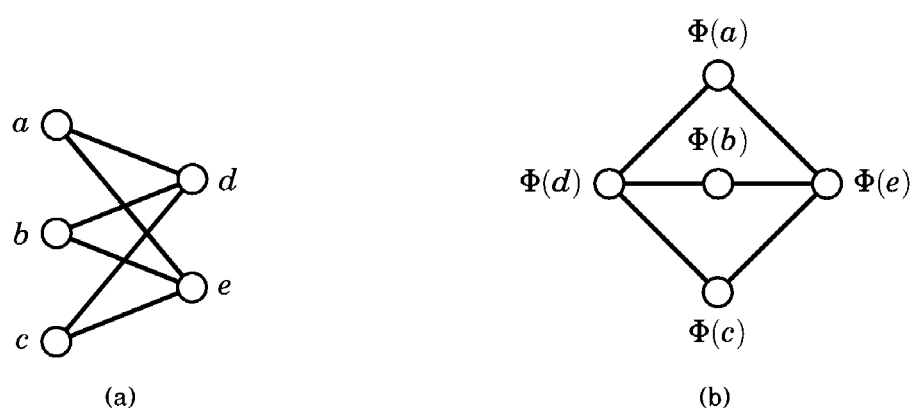


Figure 1.1 – Two isomorphic graphs.

Let G be a graph. A graph H is a **subgraph** of G, denoted by $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$; if H is a subgraph of G and $V(H) = V(G)$, then

H is a **spanning subgraph** of G. If U is a nonempty subset of the vertex set of G, then the subgraph $\langle U \rangle$ of G **induced** by U is the graph having vertex set U and whose edge set consists of those edges of G incident with two elements of U. A subgraph H of G is **induced** if $H = \langle U \rangle$ for some subset U of $V(G)$. Similarly, if X is a nonempty subset of the edge set of G, then the subgraph $\langle X \rangle$ **induced** by X is the graph whose vertex set consists of those vertices of G incident with at least one element of X and whose edge set is X. If $v \in V(G)$ and $|V(G)| \geq 2$, then $G - v$ denotes the subgraph of G induced by $V(G) - \{v\}$; if $e \in E(G)$, then $G - e$ denotes the subgraph of G induced by $E(G) - \{e\}$. The deletion of a set of vertices or a set of edges is defined analogously.

Let u and v be (not necessarily distinct) vertices of a graph G and let k be a nonnegative integer. A u–v **walk** W of G is a finite, alternating sequence

$$W : u = u_0, e_1, u_1, e_2, \ldots, u_{k-1}, e_k, u_k = v$$

of vertices and edges, beginning with the vertex u and ending with the vertex v, such that $e_i = u_{i-1}u_i$ for each i satisfying $1 \leq i \leq k$. The number k (the number of occurrences of edges) is the **length** of W. For convenience, we will only present the vertices of a walk since the edges present are evident. A u–v **path** P is a u–v walk in which no vertex is repeated; the vertices u and v are the **ends** of P and each vertex of P other than u or v is an **internal vertex** of P. A vertex u forms the **trivial** u–u **path**. A **cycle** C is a walk $v_1, v_2, \ldots, v_n, v_1$ ($n \geq 3$) whose n vertices v_i are distinct. The subgraph of G induced by the edges of a path or cycle is referred to as a **path** or **cycle** of G. A subgraph of a path P that is also a path is a **subpath** of P. A graph of order n that is a path is denoted by P_n. A cycle of length n is an n-**cycle**; a 3-cycle is commonly called a **triangle**. The graph G is **hamiltonian** if it has a cycle C containing all the vertices of G, while C is a **hamiltonian cycle** of G; if e is an edge of G that does not lie on C, then e is a **chord** of C.

The vertex u is **connected** to the vertex v in G if there exists a u–v path in G. A graph is **connected** if every two of its vertices are connected. A graph that is not connected is **disconnected**. A **component** of G is a connected subgraph of G that is not a subgraph of any other connected subgraph of G. A set U of vertices of G **separates** u and v if the removal of the elements of U from G produces a disconnected graph in which u and v lie in different components. The **connectivity** $\kappa(G)$ of G is the minimum number of vertices whose removal results in a disconnected or trivial graph. The graph G is k-**connected**, $k \geq 1$, if $\kappa(G) \geq k$. For an example to illustrate these concepts, we have for the graph G in Figure 1.2:

(1) $\{z\}$ separates a and b,

(2) $\{v, w\}$ separates u and y,

(3) $\kappa(G) = 1$, and

(4) $\kappa(G - b) = 2$.

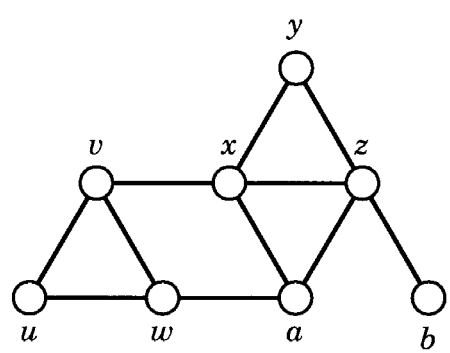


Figure 1.2 – An example of a graph.

Suppose that the graph G is connected. The **distance** $d_G(u, v)$, or simply $d(u, v)$, between the vertices u and v is the minimum of the lengths of the u–v paths of G. A u–v path of length $d(u, v)$ is a u–v **geodesic**. The **geodesic**

interval $I(u, v)$ is the set of vertices on some u–v geodesic. The **eccentricity** $\mathrm{ecc}(v)$ of the vertex v is the distance between v and a vertex farthest from v. The **radius** $\mathrm{rad}\, G$ of G is the minimum eccentricity among the vertices of G, while the **diameter** $\mathrm{diam}\, G$ of G is the maximum eccentricity. Returning to the graph G in Figure 1.2, we have:

(1) $d(v, a) = 2$,

(2) $I(v, a) = \{v, w, x, a\}$,

(3) the path z, x, v, u is a z–u geodesic,

(4) $\mathrm{ecc}(z) = 3$,

(5) $\mathrm{rad}\, G = 2$ and $\mathrm{diam}\, G = 4$..

A graph is a **tree** if it is connected and contains no cycles. It is well known that if T is a tree of order n, then T has $n - 1$ edges. Additionally, every pair of vertices in a tree T is connected by a unique path of T. A graph is **complete** if every two of its vertices are adjacent. A complete graph of order n is denoted by K_n. Note that if G is a triangle, then G is isomorphic to K_3. A graph G is **bipartite** if it is possible to partition $V(G)$ into two subsets V_1 and V_2 (called **partite sets**) such that every edge of G joins an element of V_1 and an element of V_2. A **complete bipartite graph** is a bipartite graph with partite sets V_1 and V_2 having the added property that if $u \in V_1$ and $v \in V_2$, then $uv \in E(G)$; if $|V_1| = r$ and $|V_2| = s$, then this graph is denoted by $K_{r,s}$. (The order in which the numbers r and s are written is not important.) The graph $K_{1,s}$ is a **star**.

We need to make the following comment. For two sets A and B, we use the convention that $A - B$ is the set of elements that are in A but not in B, even if $B \not\subseteq A$.

1.2 Maximal outerplanar graphs

A graph G is **planar** if G can be drawn in the plane with the property that no two edges intersect except at a vertex to which they are both incident; such a drawing is an **embedding** of G. A given embedding of a planar graph is a **plane graph**. The graph in Figure 1.1a is not a plane graph, although it is isomorphic to the planar graph $K_{2,3}$. On the other hand, the drawing of $K_{2,3}$ given in Figure 1.1b is a plane graph. Given a plane graph G, a **region** of G is a maximal portion of the plane for which any two points may be connected by a curve A such that each point of A neither corresponds to a vertex of G nor lies on any curve corresponding to an edge of G. For a plane graph G, the **boundary** of a region R consists of all those points x corresponding to vertices and edges of G having the property that x can be connected to a point of R by a curve, all of whose points different from x belong to R. Every plane graph G contains one unbounded region called the **exterior region**. Every other region is called an **interior region**.

A graph G is **outerplanar** if there exists a plane graph G' isomorphic to G such that every vertex of G' lies on the boundary of the exterior region; in this case, the graph G' is an **outerplane graph**. An outerplanar graph is **maximal outerplanar** if adding any edge results in a graph that is not outerplanar. See Figure 1.3 for an example of a planar graph, an outerplanar graph, and a maximal outerplanar graph. Note that the graph in Figure 1.3c is isomorphic to K_4.

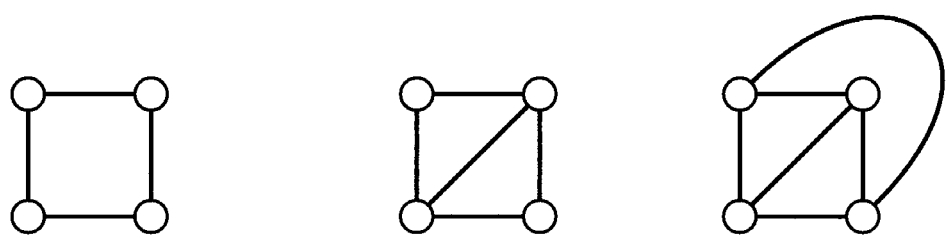
An **elementary subdivision** of a graph G is a graph obtained from G by removing some edge e, where $e = uv$, and adding a new vertex w and edges uw and vw; a **subdivision** of G is a graph obtained from G by succession of zero or more elementary subdivisions. A well known characterization of outerplanar graphs states that a graph is an outerplanar graph if and only if it contains no

subgraph that is a subdivision of either K_4 or $K_{2,3}$. In Theorem 1A, we state a characterization of maximal outerplanar graphs by Hopkins and Staton [12]. We will use condition 2 of this theorem throughout this thesis. A graph G is **2-degenerate** if every subgraph of G has a vertex of degree 2 or less.

Theorem 1A. *Let G be a connected graph with n vertices, $n \geq 2$. Then G is maximal outerplanar if and only if both conditions hold:*

(1) *G is 2-degenerate.*

(2) *For each vertex v of G, $N(v)$ induces a path in G.*



(a) An outerplanar graph that is not a maximal outerplanar graph.

(b) A maximal outerplanar graph.

(c) A planar graph that is not an outerplanar graph.

Figure 1.3 – Examples of planar graphs.

For convenience, we denote by **MOP** a maximal outerplanar graph with order at least 3. Observe that the reflection of an outerplane graph is also an outerplane graph. Because of an important set of vertices we will define later, namely $\mathrm{Arc}(u, v)$, we will assume that a MOP is a fixed outerplane graph. Since we will often need to refer to sets of vertices in a MOP G of a specified degree, we define $D_i(G)$, or simply D_i, to be the set of vertices of G of degree i for each integer i.

It is well known that every MOP is 2-connected. Moreover, each MOP has a unique hamiltonian cycle, which is the boundary of the exterior region, while the boundary of every interior region is a triangle. Because of this, a MOP is commonly called a triangulation of a polygon. The number of triangulations of a regular polygon with $n + 2$ sides is the nth Catalan number, which is a famous number in combinatorics. In [13] Guanzhang provides closed formulas involving the Catalan numbers for the number of both labeled and unlabeled MOPs of a fixed order.

Let G be a MOP and let C be the unique hamiltonian cycle of G. Every edge in G that is a chord of C is a **chord** of G, and each edge that lies on C is an **outer edge** of G. Observe that no two distinct vertices of G have more than two common neighbors since $K_{2,3}$ is not a subgraph of any outerplanar graph. Because each interior region of G is a triangle, it follows that the two vertices incident to a chord of G have exactly two common neighbors, while two vertices incident to an outer edge of G have exactly one common neighbor.

The number of edges and regions in a MOP can be determined from the order of the MOP. Euler's formula for graphs states that if G is a connected plane graph with n vertices, m edges, and r regions, then $n - m + r = 2$. Now let G be a MOP of order n. Using the fact that the boundary of the exterior region of G is a hamiltonian cycle and that the boundary of every interior region of G is a triangle, it follows that G has $2n - 3$ edges and $n - 1$ regions. Thus G has $n - 3$ chords and $n - 2$ interior regions (or triangles). The chords and interior regions of a MOP relate to an associated tree of the MOP, which we define next.

The **spine** S of a MOP G is the graph whose vertex set corresponds to the interior regions of G, where two vertices of S are adjacent if the boundaries of their corresponding regions have an edge in common. We remark that isomorphic MOPs have isomorphic spines. Because every interior region of a MOP is

a triangle, every vertex in a spine has degree at most 3. In fact, it is well known that the family of spines of MOPs consists of the family of trees of maximum degree at most 3. Since there are $n-2$ interior regions of G, the order of S is $n-2$. The $n-3$ edges of S correspond to the $n-3$ chords of G (the spine can be drawn in such a way that the edges of the spine cross the corresponding chords of G). If G is not isomorphic to K_3, then there is also a correspondence between the leaves of S and the vertices of degree 2 of G. When we want to include the spine of a MOP in a figure, we draw the spine with greyed vertices and edges as in Figure 1.4.

There are two classes of MOPs that have come up often in our study of MOPs. A graph F is a **fan** if F has a vertex v that is adjacent to every other vertex of F with the additional property that the graph induced by $N(v)$ is a path. Note that $\mathrm{ecc}(v) = 1$. If F is a fan of order n, $n \geq 3$, then F is a MOP with $n-2$ triangles; we denote the fan of order n by F_{n-2}. The fan F_5 is displayed in Figure 1.4 along with its spine. A vertex u of a connected graph G is a **central vertex** if $\mathrm{ecc}(u) = \mathrm{rad}\, G$. Observe that for $k \geq 3$, the fan F_k has a unique central vertex.

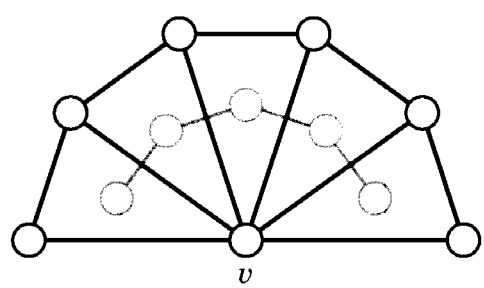
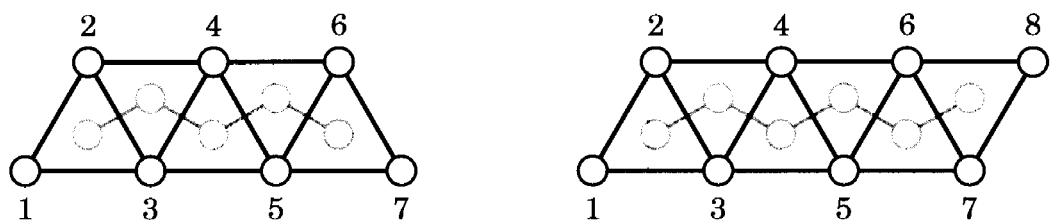


Figure 1.4 – The fan F_5 with its spine.

The **square** G^2 of a graph G is the graph whose vertex set is $V(G)$, where two vertices u and v are adjacent in G^2 if their distance in G is either 1 or 2. For $n \geq 3$, the graph P_n^2 is a **zig-zag**. The choice of the name zig-zag is

motivated by the nature of "turns" in the spine of P_n^2. Let G be a zig-zag. If the order of G is even, then G is an **even zig-zag**; otherwise, G is an **odd zig-zag**. Figure 1.5 displays the zig-zags of order 7 and 8 along with their spines. Both of these zig-zags are labeled so that one can easily verify that they are the square of P_7 and P_8, respectively. Notice that the spines of fans and zig-zags are paths. Informally, the turns of the edges in the spine of a fan are all in the same direction, whereas consecutive turns of the spine in a zig-zag alternate. In this sense, these family of MOPs represent two extremes when the spine of a MOP is a path.



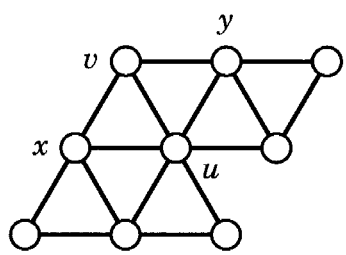
(a) An odd zig-zag. (b) An even zig-zag.

Figure 1.5 – Examples of zig-zags.

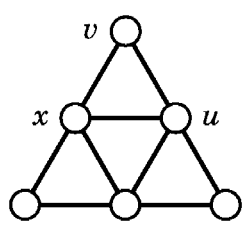
Let u and v be (not necessarily distinct) vertices of a MOP G and let C be the hamiltonian cycle of G. Recall that G is a fixed outerplane graph. We define $\mathrm{Arc}(u, v)$ to be the set of internal vertices of the u–v path in C that starts at u and traverses around C clockwise. To denote the sets of vertices consisting of $\mathrm{Arc}(u, v)$ along with u, v, or both u and v, we use closed parenthesis in the natural way. Note that $\mathrm{Arc}(u, u) = \emptyset$ and that if u and v are consecutive vertices on C, then either $\mathrm{Arc}(u, v) = \emptyset$ or $\mathrm{Arc}(v, u) = \emptyset$. Now suppose uv is a chord of G. Since G contains no subgraph that is a subdivision of K_4, we can observe that no edge of G joins a vertex in $\mathrm{Arc}(u, v)$ and a vertex in $\mathrm{Arc}(v, u)$. Therefore, the set $\{u, v\}$ separates every vertex in $\mathrm{Arc}(u, v)$ and every vertex in $\mathrm{Arc}(v, u)$; hence $\kappa(G) = 2$. We will use this property of chords in MOPs throughout this

thesis.

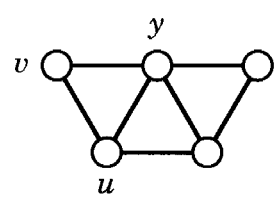
Another observation, which is useful in proofs by induction in MOPs, is that the subgraphs $\langle \mathrm{Arc}[u, v] \rangle$ and $\langle \mathrm{Arc}[v, u] \rangle$ are also MOPs. See Figure 1.6 for an illustration of this observation. Consequently, it is easy to prove by induction that both the sets $\mathrm{Arc}(u, v)$ and $\mathrm{Arc}(v, u)$ have a vertex of degree 2 in G and that unless G is isomorphic to K_3, no two vertices of degree 2 in G are adjacent. A special case of the observation above is when u and v have a common neighbor w of degree 2. If, say, $w \in \mathrm{Arc}(u, v)$, then the graph $\langle \mathrm{Arc}[v, u] \rangle$ is the graph $G - w$. So removing a vertex of degree 2 from a MOP (assuming the order of the MOP is at least 4) results in another MOP.



(a) A MOP with chord uv.



(b) The MOP $\langle \mathrm{Arc}[u, v] \rangle$.



(c) The MOP $\langle \mathrm{Arc}[v, u] \rangle$.

Figure 1.6 – Examples of $\langle \mathrm{Arc}[u, v] \rangle$ and $\langle \mathrm{Arc}[v, u] \rangle$.

The graph in Figure 1.6b will be discussed often in this thesis. We will refer to this graph as the 3-**sun**.

1.3 Motivation of work

Other than the families of outerplanar and planar graphs, the family of

MOPs is contained in some highly studied families of graphs such as chordal graphs and 2-trees. A graph G is **chordal** if every cycle of G of length greater than 3 has a chord. For a fixed k, $k \geq 1$, the family of k-**trees** is defined recursively as follows. A complete graph on k vertices is a k-tree. In addition, every other k-tree is obtained from a smaller k-tree G by first identifying a complete subgraph H on k vertices, and then adding a new vertex to G along with the edges that make this vertex adjacent to each vertex of H. Since each MOP is a 2-tree, the family of MOPs can be recursively defined as well. However, there is a slight difference in the construction of MOPs. Note that a complete graph on two vertices is simply an edge. We can construct a MOP from another MOP G by adding a new vertex and making it adjacent to both vertices of any outer edge e of G. If the edge e is not an outer edge of G, then the resulting graph contains $K_{2,3}$ as a subgraph, which implies it is not outerplanar.

One of the most important type of a graph is a tree, or a 1-tree. Trees have many applications to different fields including searching, sorting, and minimal connector problems. However, because of their simplicity, many graph theory problems that are difficult for general graphs have simple solutions for trees. In particular, problems involving distances in graphs and *boundaries* are easy for trees. A significant part of the thesis is devoted to studying those properties for MOPs, a subfamily of 2-trees. Another direction of the thesis is to find relationships between values of some graph parameters, namely independence number and domination number, in MOPs and in their spines, which are trees.

Although a MOP determines a unique spine, a given tree of maximum degree at most 3 may be the spine of several nonisomorphic MOPs. For a fixed tree T of maximum degree at most 3, Bange, *et al.* [2] gives bounds on the number of MOPs whose spine is isomorphic to T based on the number of vertices of T that have degree 1 or 2. In another paper, Bange, *et al.* [1] showed that

for a labeled MOP G, the number of spanning trees of G depends only on the spine of G. In Chapter 3 we will see that no such relationship holds between the independence number of a MOP and the independence number of its spine. However, we give sharp bounds relating these two numbers for a fixed MOP and its spine. Similarly, in Chapter 4 we establish sharp bounds relating the domination number of a MOP and the domination number of its spine.

In the last two chapters we shift our attention to several topics involving the notion of distance in graphs. The concepts we study there can be seen as analogies and generalizations of ideas from continuous mathematics. The fact that connected graphs can be seen as metric spaces by considering their shortest paths has lead to the study of the behavior of these structures as convexity spaces. Although there are different notions of convexity in a graph, we will say that a set W of vertices in a connected graph is **convex** if it contains every vertex on a u–v geodesic for every $u, v \in W$. From this point of view, a vertex v in a convex set W is **extreme** if $W - \{v\}$ is convex. Each nontrivial tree has at least two leaves, and the set of extreme vertices in a tree consists of the set of leaves. For MOPs, the set of extreme vertices consists of the vertices of degree 2. However, the vertices of higher degree play an important role in MOPs. Their role contributes to making the problems we have looked at for MOPs interesting, while these problems for trees can generally be solved by only considering their set of leaves. In Chapter 5 we study several types of boundary sets in MOPs, which include the set of extreme vertices. In Chapter 6 we show that generalized geodesic intervals are convex sets in MOPs. Using this result, we characterize which sets of vertices in a MOP are *geodetic sets* and also explore other concepts, *Steiner trees* and *Steiner sets*, which can be seen as extensions of geodesic concepts.

Before we discuss these topics further, the first main result in this thesis

concerns degree sequences in MOPs. This result will be used in the proof of one of the main theorems in Chapter 6.

CHAPTER 2
HAMILTONIAN DEGREE SEQUENCES

The degrees of vertices play an important role in the last two chapters of this thesis. In particular, for the proof of one of the main theorems in Chapter 6, we will need to determine the local structure of a MOP given the degrees of vertices on a path that lies on the hamiltonian cycle. We do this by applying the main result of this chapter, Theorem 2.1.

First, we start a result that can be seen as a special case of Theorem 2.1. Suppose we order the vertices of a MOP G such that $v_1, v_2, \ldots, v_n, v_1$ is the hamiltonian cycle. Let $D = (d_1, d_2, \ldots, d_n)$ be the corresponding sequence of degrees of these vertices in G, that is $\deg(v_i) = d_i$ for each $i \in \{1, 2, \ldots, n\}$. The sequence D is a **hamiltonian degree sequence** of G. Of course, any vertex of G can be labeled v_1 and there are two ways (clockwise and counterclockwise) to list a hamiltonian cycle of G starting with a chosen vertex for v_1. Since each MOP has a unique hamiltonian cycle, it follows that the set of hamiltonian degree sequences of two isomorphic MOPs coincide. Furthermore, Beyer, *et al.* [3] proved the converse of this statement.

Theorem 2A. *If $D = (d_1, d_2, \ldots, d_n)$ is a hamiltonian degree sequence of some MOP G, then G is unique up to isomorphism.*

It follows from Theorem 2A that if two MOPs have a common hamiltonian degree sequence, then the two MOPs are isomorphic. Now suppose $P : v_1, v_2, \ldots, v_k$ is a path that lies on the hamiltonian cycle of a MOP G and

that $D = (d_1, d_2, \ldots, d_k)$ is the degree sequence of these vertices in G, that is $\deg(v_i) = d_i$ for each $i \in \{1, 2, \ldots, k\}$. The sequence D is a **hamiltonian degree subsequence** of G. We will show in Theorem 2.1 that D determines the graph induced by $N[V(P)]$ up to isomorphism and that this subgraph is a MOP. Observe that if G has order k, then D is a hamiltonian degree sequence of G and the graph induced by $N[V(P)]$ is G. So Theorem 2A follows from the case of Theorem 2.1 where k is the order of G.

Recall that for a vertex v in a MOP G, the graph induced by $N(v)$ is a path. Note that $N(v)$ induces a path if and only if $N[v]$ induces a fan. So it follows that if $deg(v) = r$, then the graph induced by $N[v]$ is the fan of order $r + 1$. One of the main tools we use in the proof of Theorem 2.1 is condition 2 of Theorem 1A, which is essentially the case $k = 1$ of Theorem 2.1. We will need the following lemmas.

Lemma 2.1. *Let G be a MOP and let W be a nonempty set of vertices of G. If $G - W$ is 2-connected, then $W \cap D_2 \neq \emptyset$.*

Proof. We will prove the contrapositive. Let $G' = G - W$. Suppose that $W \cap D_2 = \emptyset$ and let $v \in W$. Hence $\deg(v) > 2$, which implies that there exists $u \in V(G)$ such that uv is a chord of G. Thus, there are vertices $x, y \in D_2$ such that $x \in \mathrm{Arc}(u, v)$ and $y \in \mathrm{Arc}(v, u)$. Furthermore, the set $\{u, v\}$ separates x and y in G. Since $W \cap D_2 = \emptyset$, the vertices x and y are in G'. Now note that $v \notin V(G')$. If $u \in W$, then G' contains no x–y path, implying that G' is disconnected. If $u \notin W$, then $G' - u$ contains no x–y path, implying that $G' - u$ is disconnected. Therefore, G' is not 2-connected. □

Lemma 2.1 implies that if H is a 2-connected induced subgraph of a MOP G, then there is some vertex of G of degree 2 that is not in H. Recall that if w is a vertex of degree 2 of a MOP G whose order is at least 4, then $G - w$ is also

a MOP. The next result uses this fact and the lemma above to show that the subgraph H must be a MOP.

Lemma 2.2. *Let G be a MOP. If H is a 2-connected induced subgraph of G, then H is a MOP.*

Proof. The proof is by induction on the order of G. For the base case, suppose that G is isomorphic to K_3. Since H is 2-connected, we must have $H = G$, and so the result is clear. So we assume that the lemma is valid whenever G' is a MOP of smaller order than G. Because H is an induced subgraph of G, there exists a set of vertices W of G such that $H = G - W$. It suffices to assume that W is nonempty. Thus, by Lemma 2.1, there exists a vertex of G of degree 2 that is in W. Let $G' = G - w$ and let $W' = W - \{w\}$. Observe that G' is a MOP and that $H = G' - W'$. Therefore, H is a 2-connected induced subgraph of the MOP G'. By the induction hypothesis, the graph H is a MOP. □

We use Lemma 2.2 in the proof of the lemma below.

Lemma 2.3. *Let G be a MOP and let $P : v_1, v_2, \ldots, v_k$ be a path that lies on the hamiltonian cycle of G. If H is the graph induced by $N[V(P)]$, then H is a MOP.*

Proof. As a consequence of Lemma 2.2, it suffices to show that H is 2-connected. Note that every vertex in H is in P or is adjacent to a vertex in P. It follows that H is connected. Let v be a vertex of H and let $H' = H - v$. We show that H' is connected. If $v \notin V(P)$, then for the same reasoning as for H we have that H' is connected. So assume that $v = v_i$ for some i and let $W = N(v_i)$. Since G is a MOP, the graph induced by W in G is a path. Hence the graph induced by W in H' is a path. Thus, if $k = 1$, then H' is a path and so H' is connected. Now we assume that $k > 1$, and, without loss of generality, assume that $i < k$. Since $v_i v_{i+1}$ is an edge of some triangle in G, there exists a vertex in W that is adjacent

to v_{i+1}. Therefore the vertices $v_{i+1}, \ldots, v_k$ are connected to W in H'. Similarly, if $i > 1$, then we can show that the vertices $v_1, \ldots, v_{i-1}$ are connected to W in H'. It follows that H' is connected. □

We are now ready for the main result of this chapter. Recall that if G is a MOP of order n with m edges, then $m = 2n - 3$.

Theorem 2.1. *Let $D = (d_1, d_2, \ldots, d_k)$ be a hamiltonian degree subsequence of some MOP G and let $P : v_1, v_2, \ldots, v_k$ be a path that lies on the hamiltonian cycle of G with $\deg(v_i) = d_i$ for each $i \in \{1, 2, \ldots, k\}$. If H is the graph induced by $N[V(P)]$, then H is a MOP and H is unique up to isomorphism.*

Proof. Let H be the graph induced by $N[V(P)]$. Lemma 2.3 implies that H is a MOP. We show by induction on the order of P that H is unique up to isomorphism. If $k = 1$, then H is the fan F_{s-1} where $s = \deg(v_1)$. So we assume that $k > 1$. Let $P' = P - v_k$ and let H' be the graph induced by $N[V(P')]$. Note that P' is a path that lies on the hamiltonian cycle of G and that $v_k \in V(H')$. By the induction hypothesis, the graph H' is uniquely determined. Thus, if $N(v_k) \subseteq V(H')$, then $H = H'$ and we are done. So we assume that there are neighbors of v_k that are not in H'. Lemma 2.3 implies that the graphs H and H' are MOPs. Therefore, the graph $\langle N_H(v_k)\rangle$ is a path with $\langle N_{H'}(v_k)\rangle$ as a subpath. Let $\langle N_H(v_k)\rangle$ be the path $u_1, u_2, \ldots, u_r, \ldots, u_{r+m}$ where $deg_{H'}(v_k) = r$ and $deg_H(v_k) = r + m$. Since the edge $v_{k-1}v_k$ is an outer edge of G, the vertices v_{k-1} and v_k have exactly one common neighbor in both H and H'. Thus the vertex v_{k-1} is a leaf in both the paths $\langle N_H(v_k)\rangle$ and $\langle N_{H'}(v_k)\rangle$. Without loss of generality, we assume that $u_1 = v_{k-1}$. Hence $\langle N_{H'}(v_k)\rangle$ is the path: $u_1, u_2, \ldots, u_r$. Observe that the m edges of the path $u_r, \ldots, u_{r+m}$ and the m edges $v_k u_{r+1}, \ldots, v_k u_{r+m}$ are in $E(H) - E(H')$. Since H and H' are MOPs and $|V(H)| - |V(H')| = m$, we have that $|E(H)| - |E(H')| = 2m$. Therefore, we have determined all $2m$ edges

in $E(H) - E(H')$. Since H' is uniquely determined, it follows that H is unique up to isomorphism (see Figure 2.1). □

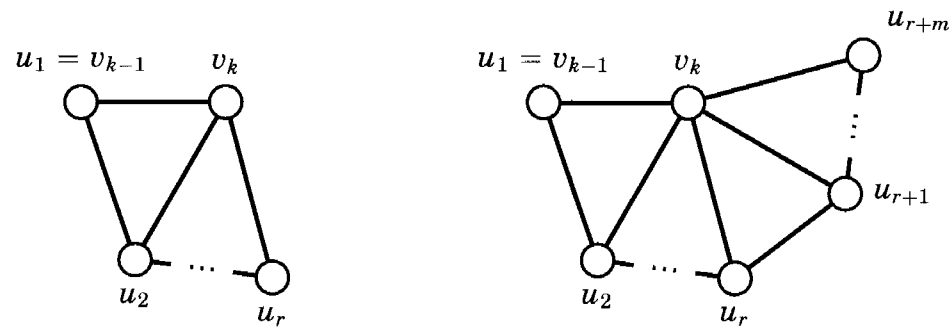


Figure 2.1 – The graphs induced by $N_{H'}[v_k]$ and $N[v_k]$ in the proof of Theorem 2.1.

Here we show our main application of Theorem 2.1. Before we do so, we make the following definition. Let G be a MOP. Let P be a u–v path that lies on the hamiltonian cycle of G for some distinct vertices u and v of G. If $\{u, v\} \subseteq D_3$ and $\mathrm{Arc}(u, v) \subseteq D_4$, then P is a **neutral segment** of G. Now, for an integer k, $k \geq 0$, let H be the odd zig-zag P^2_{2k+5}. Assume that $V(H) = \{v_1, v_2, \ldots, v_{2k+5}\}$ and that H is the square of the path $v_1, v_2, \ldots, v_{2k+5}$. Observe that the path $Q : v_2, v_4, \ldots, v_{2k+4}$ is a neutral segment of H with k vertices of degree 4 in H. Figure 2.2 illustrates H for the case $k = 1$. Since each vertex in the set $\{v_1, v_3, \ldots, v_{2k+5}\}$ is adjacent to a vertex in the set $\{v_2, v_4, \ldots, v_{2k+4}\}$, the graph H is the graph induced by $N[V(Q)]$. It follows from Theorem 2.1 that if P is a neutral segment of a MOP G with k vertices of degree 4 in G, then the graph induced by $N[V(P)]$ is isomorphic to H, which is the odd zig-zag of order $2k + 5$.

In this chapter we have considered degree sequences of a MOP with respect to its hamiltonian cycle. We remark that work on ordinary degree sequences has been done for MOPs as well. In [15], a recursive algorithm is given that determines if a sequence of numbers is the degree sequence of some MOP.

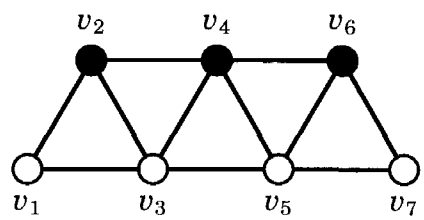


Figure 2.2 – The odd zig-zag of order 7. The graph induced by the colored vertices is a neutral segment with one vertex of degree 4.

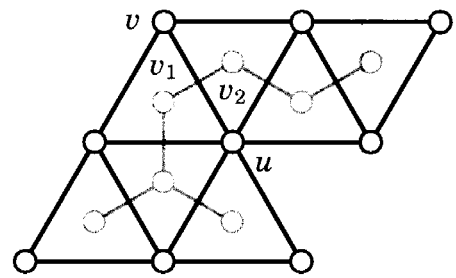
CHAPTER 3
INDEPENDENCE NUMBER

Let X be a set of vertices of a graph G. If no two vertices in X are adjacent, then X is **independent**. The maximum cardinality among all independent sets of vertices of G is the **independence number** of G and is denoted by $\alpha(G)$. If X is an independent set with size $\alpha(G)$, then X is a **α-set** of G. We remark that the problem of finding a maximum independent set in a chordal graph can be solved in polynomial time (see [10]), and thus this problem can be solved in polynomial time for trees and MOPs.

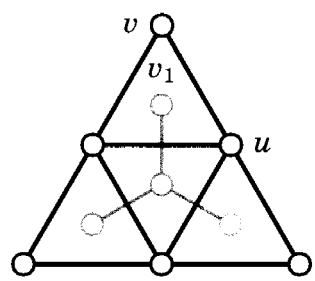
Let G be a MOP with spine S. In this chapter we investigate the relationship between $\alpha(G)$ and $\alpha(S)$. The main results of this chapter establish sharp bounds for $\alpha(G)$ in terms of $\alpha(S)$; namely, if S is nontrivial, then $\frac{\alpha(S)}{2} + 1 \leq \alpha(G) \leq \alpha(S) + 1$. Before we prove this, we need to define some notation that will be used in both this chapter and the next.

Recall that the vertices of S correspond to the triangles of G and that the edges of S correspond to the chords of G. Let $e = v_1v_2$ for some pair of adjacent vertices v_1 and v_2 of S. Let u and v be the ends of the chord of G corresponding to the edge e in S. Now consider the MOPs $\langle \text{Arc}[u, v] \rangle$ and $\langle \text{Arc}[v, u] \rangle$. These two MOPs have only the vertices u and v in common and every triangle of G is a triangle in exactly one of these MOPs. Furthermore, without loss of generality, the triangle of G corresponding to v_1 is $\langle \text{Arc}[u, v] \rangle$ and the triangle of G corresponding to v_2 is in $\langle \text{Arc}[v, u] \rangle$. For convenience, let G_1 be the MOP $\langle \text{Arc}[u, v] \rangle$ and let G_2 be the MOP $\langle \text{Arc}[v, u] \rangle$. For an example, Figure 3.1 displays the ex-

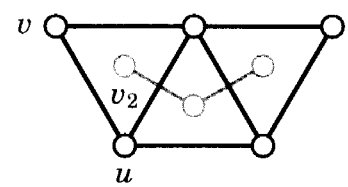
ample from Figure 1.6 with the spine and the vertices v_1 and v_2 labeled. We also let S_i be the spine of G_i for $i \in \{1, 2\}$. Note that $V(S) = V(S_1) \cup V(S_2)$ and that $v_i \in V(S_i)$. So S_i is the component of $S - e$ containing the vertex v_i. Whenever X is a set of vertices of S, we denote by X_i the set $X \cap V(S_i)$.



(a) A MOP G and its spine S with an edge v_1v_2.



(b) The MOP G_1.



(c) The MOP G_2.

Figure 3.1 – An example of G_1 and G_2 with spines S_1 and S_2.

Now lets look at some examples of MOPs and their spines and calculate their independence numbers. First let G be the fan F_6 which is shown in Figure 3.2. The colored vertices of G and the colored vertices of its spine S form α-sets of G and S, respectively. Thus $\alpha(G) = 4$ and $\alpha(S) = 3$. The fans prove to be a family of MOPs realizing one extreme. To be more clear, suppose G is the fan F_k, $k \geq 1$. We can easily compute both $\alpha(G)$ and $\alpha(S)$. Observe $\alpha(P_n) = \lceil \frac{n}{2} \rceil$. Therefore, since S is isomorphic to P_k, $\alpha(S) = \lceil \frac{k}{2} \rceil$. Now consider a central vertex v of the fan G. Note that $N(v)$ induces a path of order $k + 1$. It

follows that $\alpha(G) = \lceil \frac{k+1}{2} \rceil$. So when k is even, $\alpha(G) = \alpha(S) + 1$. Compared to $\alpha(S)$, we show next that this is the largest $\alpha(G)$ can be for any MOP G.

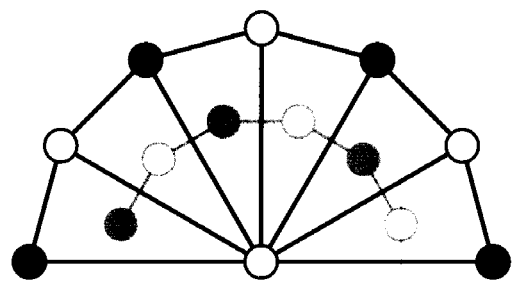


Figure 3.2 – The fan F_6 and its spine S.

Let G be a MOP of order n with spine S. Since G is hamiltonian, any independent set of vertices of G cannot contain two consecutive vertices on the hamiltonian cycle of G. Thus $\alpha(G) \leq \frac{n}{2}$. Note that S has order $n-2$. Because S is a tree and trees are bipartite, it follows that $\alpha(S) \geq \frac{n-2}{2} = \frac{n}{2} - 1 \geq \alpha(G) - 1$. Therefore, $\alpha(G) \leq \alpha(S) + 1$. Based on the calculations above for fans, we see that this bound is sharp.

The rest of this chapter focuses on showing a sharp lower bound for $\alpha(G)$ in terms of $\alpha(S)$ where G is any MOP whose spine is nontrivial. If the spine S of G is trivial, then G is isomorphic to K_3 and so $\alpha(G) = 1 = \alpha(S)$. Theorem 3.1 states that if S is nontrivial, then $\alpha(G) \geq \frac{\alpha(S)}{2} + 1$. To illustrate this bound is sharp, we now describe a family of MOPs achieving this bound.

For a positive even integer k, let H_k be the MOP with $2 + k + \frac{k}{2}$ vertices such that

$$V(H_k) = \{ v, v_0, \ldots, v_k, u_1, \ldots, u_{\frac{k}{2}} \}$$

and

$$E(H_k) = \{ vv_i : 0 \leq i \leq k \} \cup \{ v_0v_1, \ldots, v_{k-1}v_k \} \cup \{ u_iv_{2i-1}, u_iv_{2i} : 1 \leq i \leq \tfrac{k}{2} \}.$$

Observe that $H_k - \{u_1, \ldots, u_{\frac{k}{2}}\}$ is isomorphic to the fan F_k. Let S be the

spine of H_k with $V(S) = \{x_1, \ldots, x_k, y_1, \ldots, y_{\frac{k}{2}}\}$ such that x_i corresponds to the triangle induced by $\{v, v_{i-1}, v_i\}$ and y_i corresponds to the triangle containing u_i. Figure 3.3 shows H_6 with its spine. First we calculate $\alpha(H_k)$. Note that $D_2(H_k) = D_2 = \{v_0, u_1, \ldots, u_{\frac{k}{2}}\}$ and that the two neighbors of a vertex in D_2 are adjacent. Since no two vertices of degree 2 in H_k are adjacent, it follows that there exists a α-set of H_k containing D_2. Observe that D_2 is a maximal independent set. Thus D_2 is a α-set of G and so $\alpha(H_k) = \frac{k}{2} + 1$. Now we consider $\alpha(S)$. Because there exists a α-set of S containing all the leaves of S, it follows that $\{x_1, x_3, \ldots, x_{k-1}, y_1, \ldots, y_{\frac{k}{2}}\}$ is a α-set of S. Therefore, $\alpha(S) = \frac{k}{2} + \frac{k}{2} = k$, and so $\alpha(G) = \frac{\alpha(S)}{2} + 1$.

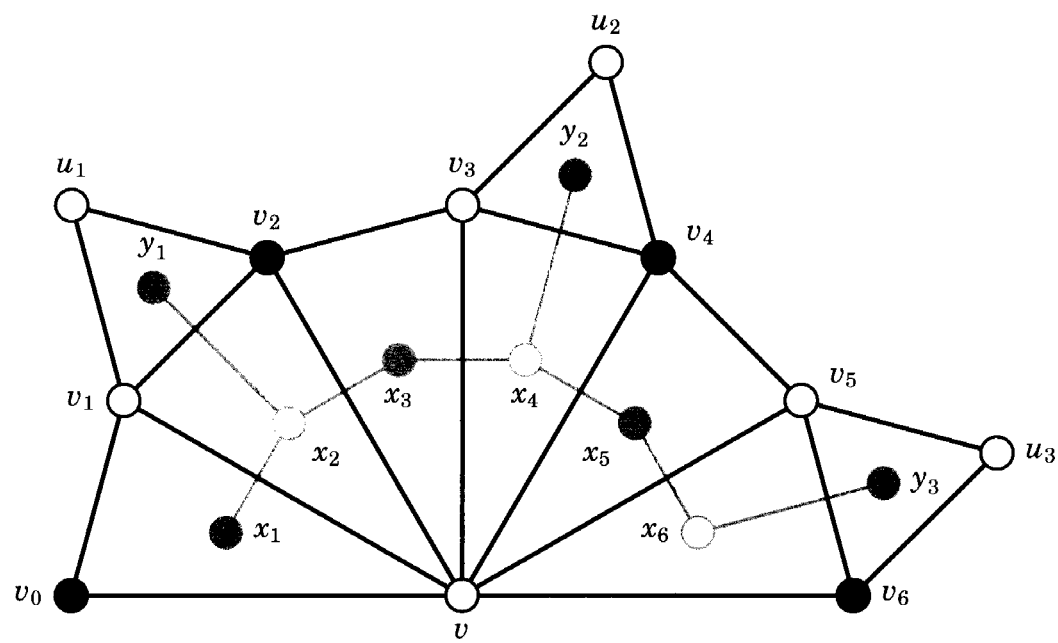


Figure 3.3 – The MOP H_6 and its spine. The sets of colored vertices of each graph are α-sets.

We will need some lemmas about independent sets in the spine of a MOP. When no MOP is mentioned, we will denote by **spine** a tree of maximum degree at most 3. If $e = v_1v_2$ is an edge of S, then we will also denote by S_i the

component of $S - e$ containing v_i.

Lemma 3.1. *Let S be a spine. If v_1v_2 is an edge of S, then*

$$\alpha(S) \leq \alpha(S_1) + \alpha(S_2) \leq \alpha(S) + 1.$$

Proof. Let X be a set of vertices of S and first suppose X is a α-set of S. Note that X_i is independent in S_i for $i \in \{1, 2\}$. Thus $\alpha(S) = |X| = |X_1| + |X_2| \leq \alpha(S_1) + \alpha(S_2)$. Now suppose the set X has the property that X_i is a α-set of S_i. Observe that $X - \{v_1\}$ is independent in S. It follows that $\alpha(S) \geq |X_1| + |X_2| - 1 = \alpha(S_1) + \alpha(S_2) - 1$. Hence $\alpha(S_1) + \alpha(S_2) \leq \alpha(S) + 1$, and the proof is complete. □

By Lemma 3.1, the sum $\alpha(S_1) + \alpha(S_2)$ is either $\alpha(S)$ or $\alpha(S) + 1$. For convenience, we say e is an edge of **type** 1 if $\alpha(S_1) + \alpha(S_2) = \alpha(S)$, and e is an edge of **type** 2 if $\alpha(S_1) + \alpha(S_2) = \alpha(S) + 1$.

Lemma 3.2. *Let $e = v_1v_2$ be an edge of type 1 of the spine S. If X is a α-set of S, then X_i is a α-set of S_i.*

Proof. Note that $|X_i|$ is independent in S_i which implies $\alpha(S_i) \geq |X_i|$. From the assumption that e is an edge of type 1, we conclude $|X_1| + |X_2| = |X| = \alpha(S) = \alpha(S_1) + \alpha(S_2)$. It follows that $|X_i| = \alpha(S_i)$ and thus X_i is a α-set of S_i. □

The parity of $\alpha(S_i)$ plays a role in the proof of Theorem 3.1. We say an edge e of a spine S is an **even edge** if at least one of the components of $S - e$ has an even independence number. The next lemma gives a sufficient condition for the existence of an even edge in a spine.

Lemma 3.3. *Let S be a spine such that $\alpha(S)$ is even. If there exists a vertex of degree 2 that is in some α-set of S, then S contains an even edge.*

Proof. Suppose first that S has an edge of type 2, say e, where $e = v_1v_2$. Because $\alpha(S)$ is even, it follows that $\alpha(S_1)+\alpha(S_2)$ is odd, which implies that one of $\alpha(S_1)$ and $\alpha(S_2)$ is even. Hence e is an even edge. Now suppose that every edge of S is of type 1. By our assumptions, there exists a path v_1, v_2, v_3 in S and a α-set X of S such that $\deg(v_2) = 2$ and $v_2 \in X$. Let $f = v_2v_3$. Let $S_2(f)$ denote the component of $S - f$ containing v_2 and let $X_2(f) = X \cap V(S_2(f))$. By Lemma 3.2, X_1 is a α-set of S_1 and $X_2(f)$ is a α-set of $S_2(f)$. It suffices to assume that e is not an even edge, which implies $|X_1|$ is odd. Observe that $X_2(f) = X_1 \cup \{v_2\}$. Therefore, $|X_2(f)|$ is even and so $\alpha(S_2(f))$ is even. Hence f is an even edge. □

The proof of Theorem 3.1 is by induction. When $\alpha(S)$ is even, we want to take advantage of the existence of an even edge. However, not all spines have an even edge. Observe that the spine S in Figure 3.4 has a unique α-set indicated by the colored vertices and has no even edge. The next lemma deals with this case directly. If G is the MOP in Figure 3.4, then note that D_2 is a α-set of G and also $\alpha(G) = 3 = \frac{4}{2} + 1 = \frac{\alpha(S)}{2} + 1$. Furthermore, observe that for any vertex w of G, there is an independent set of vertices of G with size three not containing w.

Lemma 3.4. *Let G be a MOP with spine S such that $\alpha(S)$ is even. Suppose that every α-set of S contains no vertex of degree 2. Then for any vertex w of G, there exists an independent set W of vertices of G such that $w \notin W$ and $|W| \geq \frac{\alpha(S)}{2} + 1$.*

Proof. Since $\alpha(S)$ is even, S is nontrivial. We define s_i to be the number of vertices of degree i in S for $i \in \{1, 2, 3\}$. Using the fact that every vertex of S has degree at most 3, a simple proof by induction shows $s_3 = s_1 - 2$. Note that $s_1 = |D_2|$. Observe no two vertices of degree 2 in G are adjacent since S is nontrivial. Therefore, D_2 is an independent set of vertices of G with size s_1. From our assumptions, we observe $\alpha(S) \leq s_1 + s_3$ where equality holds only

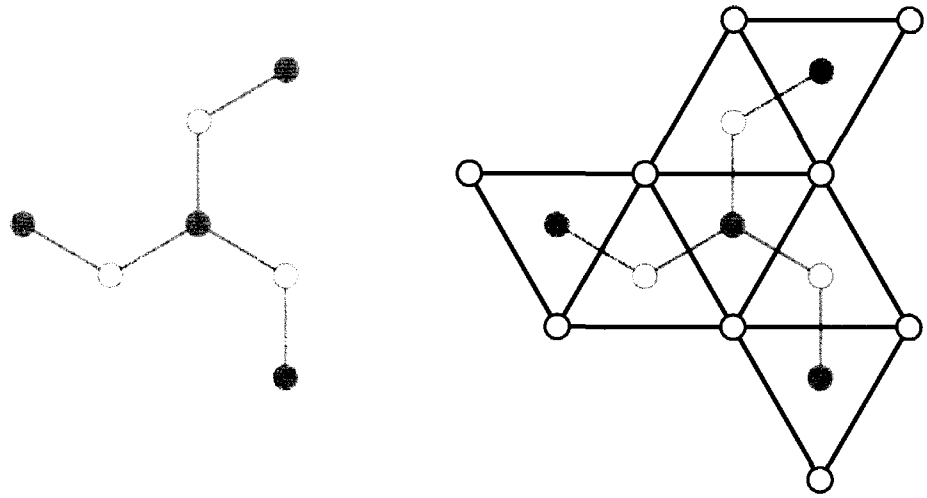


Figure 3.4 – A spine S with no even edges and a MOP whose spine is isomorphic to S.

if S has a unique α-set consisting of the vertices of S of degree 1 or 3. Hence $\frac{\alpha(S)}{2} + 1 \leq \frac{s_1+s_1-2}{2} + 1 = s_1 = |D_2|$. If $w \notin D_2$, then we let $W = D_2$. So we assume $w \in D_2$. Suppose first $\frac{\alpha(S)}{2} + 1 < s_1$. Since $\alpha(S)$ is even, it follows that $\frac{\alpha(S)}{2} + 1 \leq s_1 - 1$. In this case we let $W = D_2 - \{w\}$. Now suppose $\frac{\alpha(S)}{2} + 1 = s_1$. By the remark above, it follows that S has a unique α-set, say X, consisting of the vertices of S of degree 1 or 3. Let u be the leaf in S corresponding to the triangle in G containing w. Note that the neighbor of u in S has degree 2. It follows that there exists a neighbor of w in G, say v, such that $\deg(v) = 3$ and that w is the only vertex of degree 2 in G that is adjacent to v (see Figure 3.5). Then the set $W = (D_2 - \{w\}) \cup \{v\}$ is an independent set of vertices of S with size $\frac{\alpha(S)}{2} + 1$, and the proof is complete. □

We want to comment on the dashed curves in Figure 3.5. Recall that if uv is a chord of a MOP, then the graphs $\langle \mathrm{Arc}[u, v] \rangle$ and $\langle \mathrm{Arc}[v, u] \rangle$ are MOPs. When we want to indicate that an edge uv of a MOP may or may not be a chord, we will draw dashed curves whose ends are u and v.

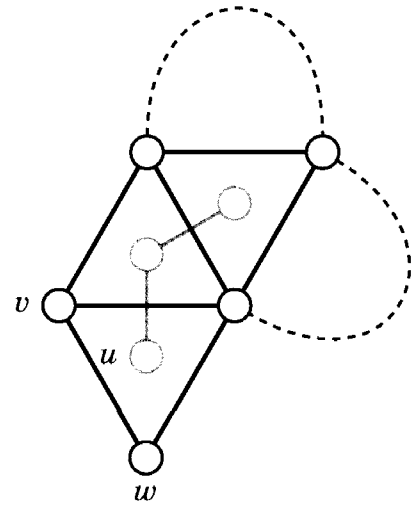


Figure 3.5 – The graph G in the proof of Lemma 3.4. The vertex u is a leaf in S and the neighbor of u has degree 2.

Lemma 3.5. *Let G be a MOP with spine S and let v_1v_2 be an edge of S. If U_i is an independent set of vertices of G_i for $i \in \{1, 2\}$ and $U = U_1 \cup U_2$, then U contains an independent set of vertices of G with size at least $|U_1| + |U_2| - 2$. Furthermore, if $V(G_1) \cap V(G_2) \nsubseteq U$, then U contains an independent set of vertices of G with size at least $|U_1| + |U_2| - 1$.*

Proof. Let $V(G_1) \cap V(G_2) = \{u, v\}$ (see Figure 3.6). Note uv is an edge in both G_1 and G_2. Hence $\{u, v\} \nsubseteq U_i$, which implies that $|U - \{u, v\}| \geq |U_1| + |U_2| - 2$. Since there is no edge in G that joins a vertex in $V(G_1) - \{u, v\}$ and a vertex in $V(G_2) - \{u, v\}$, it follows that $U - \{u, v\}$ is independent in G. So it suffices to assume $\{u, v\} \nsubseteq U$. If $U \cap \{u, v\} = \emptyset$, then observe that U is independent in G and that $|U| = |U_1| + |U_2|$. So we assume, without loss of generality, that $U \cap \{u, v\} = \{v\}$ and that $v \in U_1$. If $v \notin U_2$, then $U - \{v\}$ is independent in G and $|U - \{v\}| = |U_1| + |U_2| - 1$. If $v \in U_2$, then $v \in U_1 \cap U_2$ and thus U contains no neighbor of v in G. Therefore, U is independent in G and $|U| = |U_1| + |U_2| - 1$. $\square$

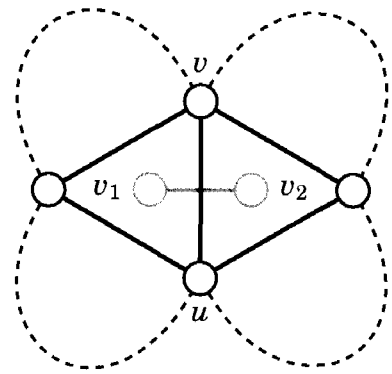


Figure 3.6 – The graph G in the proof of Lemma 3.5.

We are now ready for Theorem 3.1. Suppose that G is a MOP with nontrivial spine S and that $\alpha(S)$ is even. As we did in Lemma 3.4, we show that for every vertex w of G, there exists an independent set W of vertices of G such that $w \notin W$ and $|W| \geq \frac{\alpha(S)}{2} + 1$. Figure 3.7 displays the MOPs and their spines when S is isomorphic to a star. Out of these spines, we have that $\alpha(S)$ is even only if S is isomorphic to $K_{1,2}$. It is easy to verify the theorem is true for these three MOPs.

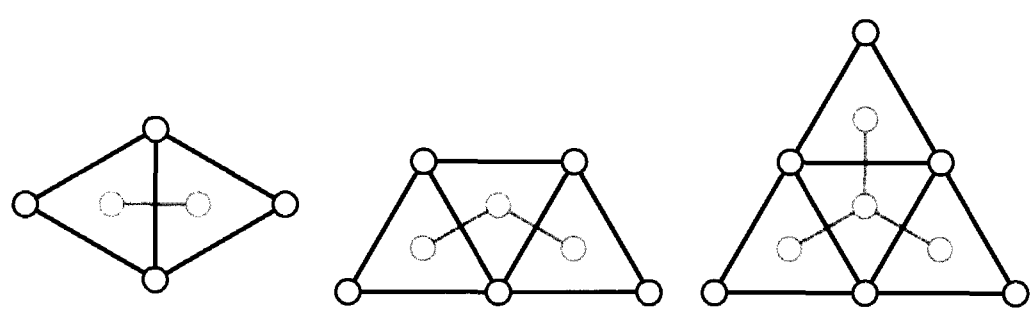


Figure 3.7 – The MOPs whose spines are nontrivial and isomorphic to stars.

Theorem 3.1. *Let G be a MOP with nontrivial spine S. Then $\alpha(G) \geq \frac{\alpha(S)}{2} + 1$.*

Moreover, if $\alpha(S)$ is even and w is any vertex of G, then there exists an independent set W of vertices of G such that $w \notin W$ and $|W| \geq \frac{\alpha(S)}{2} + 1$.

Proof. The proof is by induction on the order of S. From the analysis above, it suffices to assume S is not isomorphic to a star. Let $e = v_1v_2$ for some pair of adjacent vertices of S and let $V(G_1) \cap V(G_2) = \{u, v\}$.

Case 1: $\alpha(S)$ is odd.

Since S is not isomorphic to a star, we can select e to be a nonpendant edge of S. Then we apply the induction hypothesis to both G_1 and G_2. Suppose first e is an even edge of S and that $\alpha(S_1)$ is even. Let U_2 be a α-set of G_2. Without loss of generality, we assume $v \notin U_2$ since uv is an edge of G_2. By the induction hypothesis, there exists an independent set U_1 of vertices of G_1 such that $v \notin U_1$ and $|U_1| \geq \frac{\alpha(S_1)}{2} + 1$. Hence $\{u, v\} \not\subseteq U_1 \cup U_2$. By combining Lemma 3.5, the induction hypothesis, and Lemma 3.1, it follows that $\alpha(G) \geq |U_1| + |U_2| - 1 \geq \frac{\alpha(S_1)}{2} + 1 + \frac{\alpha(S_2)}{2} + 1 - 1 = \frac{\alpha(S_1)+\alpha(S_2)}{2} + 1 \geq \frac{\alpha(S)}{2} + 1$.

Now suppose that e is not an even edge. Therefore $\alpha(S_i)$ is odd for $i \in \{1, 2\}$. Let U_i be a α-set of G_i. As above, it follows that $\alpha(G) \geq |U_1| + |U_2| - 2 \geq \left\lceil \frac{\alpha(S_1)}{2} \right\rceil + 1 + \left\lceil \frac{\alpha(S_2)}{2} \right\rceil + 1 - 2 = \frac{\alpha(S_1)+1}{2} + \frac{\alpha(S_2)+1}{2} \geq \frac{\alpha(S)}{2} + 1$. Here we used that $\alpha(G_i)$ is an integer and that $\alpha(S_i)$ is odd.

Case 2: $\alpha(S)$ is even.

Let $w \in V(G)$. By Lemma 3.4, we can assume that there is a vertex of degree 2 in S that is in some α-set of S. Therefore, by Lemma 3.3, we can select e to be an even edge of S. We consider the two cases where e is an edge of type 1 or type 2. We note that e may or may not be a pendant edge of S.

Assume that e is an edge of type 1. Then it follows that $\alpha(S_1)$ and $\alpha(S_2)$ are both even, which implies that e is not a pendant edge of S. Suppose that $w \in \{u, v\}$. Using the induction hypothesis, we let U_i be an independent set of vertices of G_i such that $w \notin U_i$ and $|U_i| \geq \frac{\alpha(S_i)}{2} + 1$. Now suppose that

$w \notin \{u, v\}$. Without loss of generality, we assume that $w \in V(G_1) - V(G_2)$. Using the induction hypothesis, we let U_1 be an independent set of vertices of G_1 such that $w \notin U_1$ and $|U_1| \geq \frac{\alpha(S_1)}{2} + 1$. Since uv is an edge of G_1, we may assume $u \notin U_1$. Using the induction hypothesis, we let U_2 be an independent set of vertices of G_2 such that $u \notin U_2$ and $|U_2| \geq \frac{\alpha(S_2)}{2} + 1$. In both cases we have $\{u, v\} \nsubseteq U_1 \cup U_2$ and $w \notin U_1 \cup U_2$. By combining Lemma 3.1 and Lemma 3.5, $U_1 \cup U_2$ contains an independent set W of vertices of G such that $|W| \geq |U_1| + |U_2| - 1 \geq \frac{\alpha(S_1)}{2} + 1 + \frac{\alpha(S_2)}{2} + 1 - 1 = \frac{\alpha(S)}{2} + 1$.

Now we assume that e is an edge of type 2. Hence $\alpha(S) + 1 = \alpha(S_1) + \alpha(S_2)$. Thus, without loss of generality, we assume that $\alpha(S_1)$ is even and that $\alpha(S_2)$ is odd. Therefore, v_1 is not a leaf in S. If v_2 is a leaf in S, then $\alpha(S_2) = 1$ and so $\alpha(S_2) = \alpha(S)$. In this case, by the induction hypothesis, there exists an independent set W of vertices of G_2 such that $w \notin W$ (possibly $w \in V(G_1) - V(G_2)$) and $|W| \geq \frac{\alpha(S_2)}{2} + 1 = \frac{\alpha(S)}{2} + 1$. Since G_2 is an induced subgraph of G, W is also independent in G. So we assume that v_2 is not a leaf in S_2. Because $\alpha(S_1)$ is even, by the induction hypothesis, it follows that there exists an independent set U_i of vertices of G_i such that $|U_i| \geq \frac{\alpha(S_i)}{2} + 1$ and that $\{u, v\} \nsubseteq U_1 \cup U_2$. By combining Lemma 3.1 and Lemma 3.5, $U_1 \cup U_2$ contains an independent set W' of vertices of G such that $|W'| \geq |U_1| + |U_2| - 1 \geq \frac{\alpha(S_1)}{2} + 1 + \left\lceil \frac{\alpha(S_2)}{2} \right\rceil + 1 - 1 = \frac{\alpha(S_1)}{2} + \frac{\alpha(S_2)+1}{2} + 1 = \frac{\alpha(S)+2}{2} + 1$. Thus $W = W' - \{w\}$ is an independent set of vertices of G such that $|W| \geq \frac{\alpha(S)}{2} + 1$, and the proof is complete. $\square$

We close this chapter by pointing out that a similar condition when $\alpha(S)$ is odd does not hold. To see this, consider the MOP G in Figure 3.8. Note the spine S of G satisfies $\alpha(S) = 3$. Now observe that $3 = \left\lceil \frac{\alpha(S)}{2} \right\rceil + 1$ and that there is no independent set of three vertices of G not containing the vertex v.

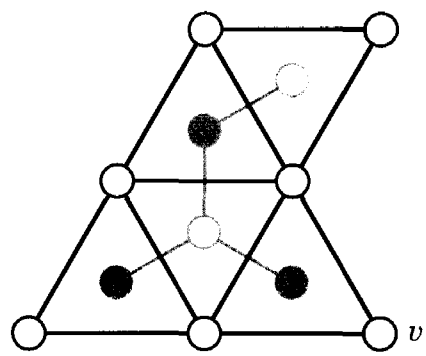


Figure 3.8 – Every independent set of vertices of G with size three contains the vertex v.

CHAPTER 4
DOMINATION NUMBER

A vertex v in a graph G **dominates** each vertex in its closed neighborhood $N[v]$. More generally, for a set X of vertices of G, X dominates every vertex in its closed neighborhood $N[X]$. The set X is a **dominating** set of G if $N[X] = V(G)$. The minimum cardinality among the dominating sets of G is the **domination number** of G and is denoted by $\gamma(G)$. If X is a dominating set of G with cardinality $\gamma(G)$, then X is a **γ-set** of G. We remark that there exists a linear time algorithm for finding a minimum dominating set in both trees and in MOPs (see [14]).

Let G be a MOP with spine S. In this chapter we investigate the relationship between $\gamma(G)$ and $\gamma(S)$ in a similar fashion to what we did for $\beta(G)$ and $\beta(S)$. The main result of this chapter shows that $\gamma(G) \leq \gamma(S) + 1$. Moreover, we characterize which MOPs achieve this bound. The main proof technique is to use induction on the order of the MOP G or the order of its spine S. We also use the notation defined in Chapter 3, that is G_i and S_i when v_1v_2 is an edge of S and X_i when X is a set of vertices of S.

First we consider the lower bound for $\gamma(G)$. If G is the fan F_k, then certainly $\gamma(G) = 1$. Since the spine S is isomorphic to P_k, we can observe that $\gamma(S) = \lceil \frac{k}{3} \rceil$. Therefore, for an arbitrary MOP G, 1 is the best lower bound for $\gamma(G)$ regardless of how large $\gamma(S)$ is. On the other hand, there is an upper bound for $\gamma(G)$ in terms of $\gamma(S)$. If G is a MOP of order at most 5, then G is a fan and so $\gamma(G) = 1$. However, now let G be the 3-sun shown in Figure 4.1. Note that G

is the only MOP whose spine is isomorphic to $K_{1,3}$. Observe that $\gamma(G) = 2$ and that $\gamma(K_{1,3}) = 1$. Thus $\gamma(G) = \gamma(S) + 1$. We will see that all MOPs satisfying this equation can be constructed by attaching copies of the 3-sun in a particular way, which we will describe more precisely later.

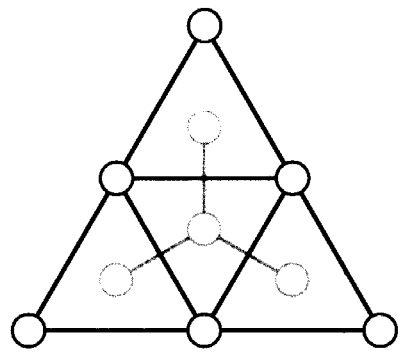


Figure 4.1 – The 3-sun.

We will use the following definition. Let G be a graph and let X be a set of vertices of G. For a vertex v in X, we define the X-**private neighborhood** of v as $\mathrm{pn}(v, X) = N[v] - N[X - \{v\}]$. Note that if $u \in \mathrm{pn}(v, X)$, then either $u = v$ and u is not adjacent to any other vertex in X, or $u \notin X$ and u is adjacent to v but not to any other vertex in X. For an example of this concept, let G be the graph in Figure 4.2, which is isomorphic to P_5. Let $X = \{b, d\}$ and observe that X is a dominating set of G since $N[b] \cup N[d] = \{a, b, c\} \cup \{c, d, e\} = V(G)$. In fact, X is a γ-set of G. Now observe that $\mathrm{pn}(b, X) = \{a, b\}$ and that $\mathrm{pn}(d, X) = \{d, e\}$.

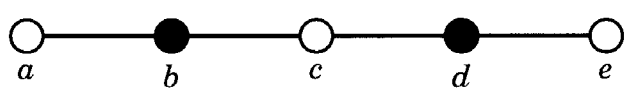


Figure 4.2 – A graph with domination number 2.

We will need some results on dominating sets in a spine. Let $e = v_1v_2$ be an edge of S. For convenience, we say the edge e is a **good cut** of the spine S if

there is a γ-set X of S with the property that $v_1 \notin \mathrm{pn}(v_2, X)$ and $v_2 \notin \mathrm{pn}(v_1, X)$. For such a set X, we say that X is a **good γ-set** with respect to e.

Lemma 4.1. *Let $e = v_1v_2$ be a good cut of the spine S. If X is a good γ-set with respect to e, then both the following two conditions hold.*

(1) $\gamma(S_1) + \gamma(S_2) = \gamma(S)$.

(2) *X_i is a γ-set of S_i for $i \in \{1, 2\}$.*

Proof.

(1) First observe that the union of a γ-set of S_1 and a γ-set of S_2 is a dominating set of S. Hence $\gamma(S) \leq \gamma(S_1) + \gamma(S_2)$. By the definition of a good γ-set, we have $v_1 \notin \mathrm{pn}(v_2, X)$ and $v_2 \notin \mathrm{pn}(v_1, X)$. It follows that there exists a vertex in X_i that dominates v_i for $i \in \{1, 2\}$). Thus X_i is a dominating set of S_i and so $\gamma(S_i) \leq |X_i|$. Because X is a γ-set of S, it follows that $\gamma(S) \leq \gamma(S_1) + \gamma(S_2) \leq |X_1| + |X_2| = |X| = \gamma(S)$. Therefore, $\gamma(S_1) + \gamma(S_2) = \gamma(S)$.

(2) From the proof above, we conclude that $\gamma(S_1) + \gamma(S_2) = |X_1| + |X_2|$. Since $\gamma(S_i) \leq |X_i|$, it follows that X_i is a γ-set of S_i. □

Next we show that if the spine S has a nonpendant edge, there exists a good cut of S. Note that every edge of S is a pendant edge if and only if S is isomorphic to a star. Therefore, if S has order at least 4 and is not isomorphic to $K_{1,3}$, then S has a nonpendant edge.

Lemma 4.2. *If the spine S is not isomorphic to a star, then there exists a good cut of S.*

Proof. Let $e = v_1v_2$ be an edge of S such that e is not a pendant edge. It suffices to assume that e is not a good cut of S. Let X be a γ-set of S. Without loss

of generality, we assume $v_1 \in \mathrm{pn}(v_1, X)$. Therefore, $v_1 \notin X$ and v_2 is the only neighbor of v_1 that is in X. Since e is not a pendant edge, the vertex v_1 is adjacent to some vertex v_3, where $v_3 \neq v_2$. Thus, v_1v_3 is an edge of S and neither v_1 nor v_3 is in X, which implies that X is a good γ-set with respect to the edge v_1v_3. Hence the edge v_1v_3 is a good cut of S. □

Recall that if G is the 3-sun with spine S, then $\gamma(G) = \gamma(S) + 1$. Now let G be the MOP in Figure 4.3 and let S be its spine. We can observe that the set $\{w_1, w_2, w_3, w_4\}$ of colored vertices of S is the unique γ-set of S. It can also be shown that $\gamma(G) = 5$, and so we have $\gamma(G) = \gamma(S) + 1$. Note that $\langle N[w_i] \rangle$ is isomorphic to $K_{1,3}$, $1 \leq i \leq 4$, and that $N[w_i] \cap N[w_j] = \emptyset$, $i \neq j$. The graph $K_{1,3}$ is commonly called a claw. This leads to the next definition.

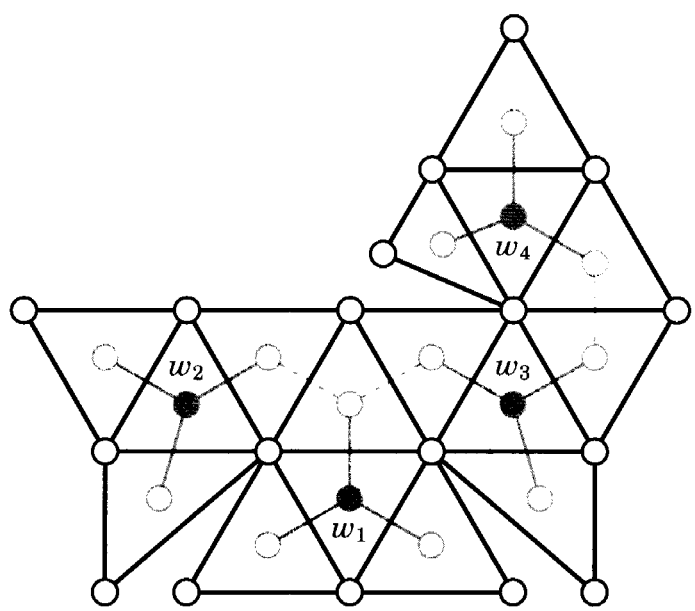


Figure 4.3 – A MOP G with spine S such that $\gamma(G) = 5$ and $\gamma(S) = 4$. The dashed edges of the spine represent the good cuts of S.

Let S be a spine. If there exists a set W of vertices of S with the property that $\{N[w]\}_{w \in W}$ partitions $V(S)$ and that $\langle N[w] \rangle$ is isomorphic to $K_{1,3}$ for all $w \in W$, then $V(S)$ has a **claw partition** and we say W **realizes** a claw partition

of $V(S)$. Equivalently, since S is a tree, W realizes a claw partition of $V(S)$ if $\{N[w]\}_{w\in W}$ partitions $V(S)$ and $\deg(w) = 3$ for all $w \in W$. If W realizes a claw partition of $V(S)$, then certainly W is a dominating set of S. In fact, W is the unique γ-set of S, which will be shown in Lemma 4.5. Also observe that if n is the order of S, then $n = 4|W|$.

Lemma 4.3. *Let S be a spine such that $V(S)$ has a claw partition. A set of vertices of S realizes a claw partition of $V(S)$ if and only if the set is a γ-set of S.*

Proof. Let n be the order of S. Let W be a set of vertices of S that realizes a claw partition of $V(S)$. Since every vertex of S has degree at most 3, each vertex of S can dominate at most four vertices. Therefore, $\gamma(S) \geq \lceil \frac{n}{4} \rceil$. Because W is also a dominating set of S and $n = 4|W|$, it follows that W is a γ-set of S.

Now let X be a γ-set of S. Note that $|X| = |W|$ and so $n = 4|X|$. It follows that

$$n = |V(S)| = \left|\bigcup_{v\in X} N[v]\right| \leq \sum_{v\in X} |N[v]| \leq 4|X| = n.$$

Observe that the inequalities above must be equalities. Therefore, we have that $\{N[v]\}_{v\in X}$ partitions $V(S)$ and that each vertex in X has degree 3. Thus X realizes a claw partition of $V(S)$. □

Let $e = v_1v_2$ be a good cut of the spine S. In order to use induction in the upcoming proofs, we will establish a condition for both $V(S_1)$ and $V(S_2)$ to have a claw partition.

Lemma 4.4. *Let $e = v_1v_2$ be a good cut of the spine S. Then $V(S_1)$ and $V(S_2)$ each have a claw partition if and only if $V(S)$ has a claw partition.*

Proof. Suppose first that $V(S)$ has a claw partition. Let X be a good γ-set with respect to e. Therefore, by Lemma 4.3, X realizes a claw partition of $V(S)$. We

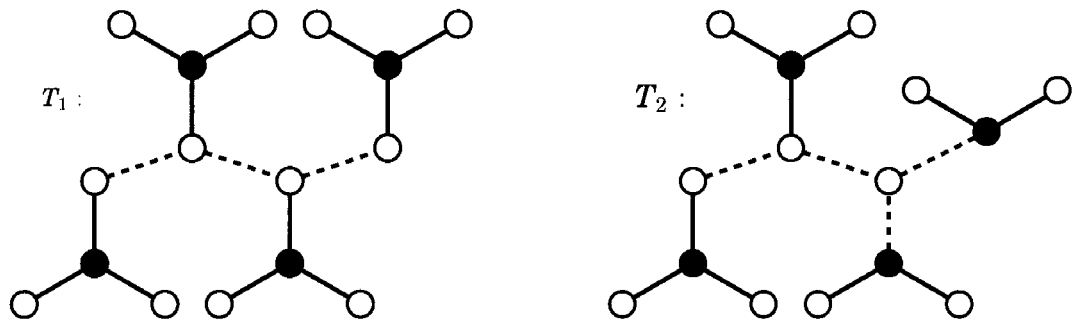
show that X_i realizes a claw partition of $V(S_i)$ for $i \in \{1, 2\}$. By Lemma 4.1 (2), X_i is a γ-set of S_i. It follows that $\{N[v]\}_{v \in X_i}$ is a partition of $V(S_i)$. It suffices to show that every vertex in X_i has degree 3 in S_i. Suppose this is not the case. Then, without loss of generality, there is a vertex v in X_1 whose degree is smaller than 3 in S_1. Because the vertex v has degree 3 in S, it must be the case that $v = v_1$. However, every vertex of S is dominated by exactly one vertex in X. Hence $v_2 \in \mathrm{pn}(v_1, X)$. This contradicts that X is a good γ-set with respect to e.

Now suppose $V(S_1)$ and $V(S_2)$ each have a claw partition. Let X be a set of vertices of S such that X_i is a γ-set of S_i for $i \in \{1, 2\}$. By Lemma 4.3, X_i realizes a claw partition of $V(S_i)$. Thus each vertex in X_i has degree 3 in S_i. Observe that v_i has at most two neighbors in S_i. Hence $v_i \notin X_i$. Since X is also a dominating set of S, it follows that $\{N[v]\}_{v \in X}$ partitions $V(S)$. Therefore, X realizes a claw partition of $V(S)$. □

To help illustrate these concepts, consider the two spines T_1 and T_2 in Figure 4.4. Observe that $V(T_1)$ has a claw partition and that $V(T_2)$ does not have a claw partition. Both spines have unique γ-sets indicated by the colored vertices and the dashed edges represent good cuts. Now observe that for a good cut v_1v_2 of T_1, both $V(S_1)$ and $V(S_2)$ have a claw partition. On the other hand, observe that for every good cut v_1v_2 of T_2, exactly one of $V(S_1)$ and $V(S_2)$ has a claw partition.

Lemma 4.5. *Let S be a spine and let W be a set of vertices of S that realizes a claw partition of $V(S)$. Then W is the unique γ-set of S.*

Proof. The proof is by induction on the order of S. The result is obvious when S has order 4, which implies that S is isomorphic to $K_{1,3}$. So we assume that the order of S is larger than 4. It follows that there exists a path v_1, v_2, v_3, v_4 in S such that v_4 is a leaf. Furthermore, since every vertex in W has degree 3 and



(a) The set of colored vertices realizes a claw partition of $V(T_1)$.

(b) The vertex set $V(T_2)$ has no claw partition.

Figure 4.4 – Two spines whose good cuts are marked with dashed edges.

W is a dominating set of S, it follows that $v_3 \in W$. Note that every vertex of S is dominated by exactly one vertex in W. Consequently, the vertices v_1 and v_2 are not in W. Let $e = v_1v_2$. Note that W is a γ-set of S by Lemma 4.3. Hence the edge e is a good cut of S and W is a good γ-set with respect to e. By Lemma 4.1 (2), the set $W_i = W \cap V(S_i)$ is a γ-set of S_i for $i \in \{1, 2\}$. Also $V(S_i)$ has a claw partition by Lemma 4.4. We conclude from the induction hypothesis that W_i is the unique γ-set of S_i. Since the good cut e does not depend on the choice of W, it follows that W is the unique γ-set of S. □

An immediate consequence of Lemma 4.5 is that if S is a spine such that $V(S)$ has a claw partition, then there is a unique set of vertices of S that realizes a claw partition of $V(S)$.

Lemma 4.6. *Let $e = v_1v_2$ be a good cut of the spine S. If at least one of $V(S_1)$ or $V(S_2)$ has a claw partition, then there exists a good cut f of S such that a component of $S - f$ is isomorphic to $K_{1,3}$.*

Proof. Let X be a good γ-set with respect to e and suppose that $V(S_1)$ has a claw partition. If S_1 is isomorphic to $K_{1,3}$, then we let $f = e$ and we are done. So we assume that S_1 is not isomorphic to $K_{1,3}$, which implies that S_1 has order larger

than 4. By Lemma 4.2, S_1 has a good cut $e' = v_3v_4$. By Lemma 4.1 (2), X_1 is a γ-set of S_1. Furthermore, by Lemma 4.5, X_1 is the unique γ-set of S_1 and also X_1 realizes the claw partition of $V(S_1)$. Hence X_1 is a good γ-set with respect to e' in S_1 and every vertex of S_1 is dominated by a unique vertex in X_1. It follows that $v_3, v_4 \notin X_1$, which implies $v_3, v_4 \notin X$. Thus e' is also a good cut of S. Now observe that one of the two components of $S_1 - e'$ is also a component of $S - e'$. We call this component S_3. Since $V(S_1)$ has a claw partition and e' is a good cut of S_1, we conclude $V(S_3)$ has a claw partition by Lemma 4.4. If S_3 is isomorphic to $K_{1,3}$, then we let $f = e'$ and we are done. Otherwise, we repeat this argument with e' replacing e and S_3 replacing S_1. Since the order of S_3 is smaller than the order of S_1, this process will eventually terminate and after finitely many steps to produce a good cut f of S whose removal leaves $K_{1,3}$ as a component. □

Recall by Lemma 4.1 (1) that if S is a spine and v_1v_2 is a good cut of S, then $\gamma(S) = \gamma(S_1) + \gamma(S_2)$. We use this result in the next four proofs. Now suppose S is a spine such that $V(S)$ has a claw partition and S is not isomorphic to $K_{1,3}$. Observe that S has a good cut v_1v_2 by Lemma 4.2. It follows from Lemma 4.4 that both $V(S_1)$ and $V(S_2)$ have a claw partition. Therefore, we can apply Lemma 4.6 to the spine S.

Lemma 4.7. *Let G be a MOP with spine S. If $V(S)$ does not have a claw partition, then $\gamma(G) \leq \gamma(S)$.*

Proof. The proof is by induction on the order of S. If S has order less than 4, then G is a fan and so $\gamma(G) = 1 = \gamma(S)$. So we assume that the order of S is at least 4. Because we are assuming $V(S)$ has no claw partition, the spine S is not isomorphic to $K_{1,3}$. Hence S has a nonpendant edge. By Lemma 4.2, there exists a good cut $e = v_1v_2$ of S. Suppose first that both $V(S_1)$ and $V(S_2)$ have no claw partition. Then $\gamma(G_i) \leq \gamma(S_i)$ for $i \in \{1, 2\}$ by the induction hypothesis.

Since G_1 and G_2 are subgraphs of G such that $V(G) = V(G_1) \cup V(G_2)$, it follows that $\gamma(G) \leq \gamma(G_1) + \gamma(G_2) \leq \gamma(S_1) + \gamma(S_2) = \gamma(S)$ as desired.

Now suppose, without loss of generality, that $V(S_1)$ has a claw partition. Then $V(S_2)$ does not have a claw partition by Lemma 4.4. Furthermore, by virtue of Lemma 4.6, we may select e such that S_1 is isomorphic to $K_{1,3}$. Note that G_1 is isomorphic to the 3-sun. Let v be the vertex in $V(G_1) \cap V(G_2)$ that has degree 2 in G_1. Since $G_1 - v$ and G_2 are subgraphs of G such that $V(G_1 - v) \cup V(G_2) = V(G)$, it follows that $\gamma(G) \leq \gamma(G_1 - v) + \gamma(G_2)$. Observe that $\gamma(G_1 - v) = 1 = \gamma(S_1)$. Therefore, by the induction hypothesis, $\gamma(G) \leq 1 + \gamma(S_2) = \gamma(S_1) + \gamma(S_2) = \gamma(S)$. $\square$

Let G be a MOP with spine S such that $W = \{w_1, \ldots, w_r\}$ realizes a claw partition of $V(S)$. Then the MOP $H_j = \langle N[w_j] \rangle$, $1 \leq j \leq r$, is isomorphic to the 3-sun. Note that each vertex in H_j has degree 2 or 4. We denote by $\mathcal{A}_2(G)$ the set of vertices of G that have degree 2 in H_j for some j. We define $\mathcal{A}_4(G)$ similarly. Note that $\mathcal{A}_2(G)$ and $\mathcal{A}_4(G)$ are well defined by Lemma 4.5. Now suppose $e = v_1v_2$ is a good cut of S. Lemma 4.4 implies that $V(S_i)$ has a claw partition for $i \in \{1, 2\}$. Combining Lemma 4.1 (2) and Lemma 4.5 proves that the set $W_i = W \cap V(S_i)$ realizes the claw partition of $V(S_i)$. Therefore, for $j \in \{2, 4\}$, $\mathcal{A}_j(G_i) \subseteq \mathcal{A}_j(G)$. Without loss of generality, suppose w_i is the vertex in W that dominates v_i. Note $v_i \notin W$ since every vertex of S is dominated by a unique vertex in W and e is a good cut of S. There are two possibilities for how H_1 and H_2 are joined. Let uv be the edge G_1 and G_2 have in common and suppose $u \in \mathcal{A}_2(H_1)$ and $v \in \mathcal{A}_4(H_2)$. Then either $u \in \mathcal{A}_2(H_2)$ and $v \in \mathcal{A}_4(H_2)$, or $u \in \mathcal{A}_4(H_2)$ and $v \in \mathcal{A}_2(H_2)$. Figure 4.5 displays both possibilities.

Observe for the MOP J_1 of Figure 4.5 that $\mathcal{A}_2(J_1) \cap \mathcal{A}_4(J_1) = \emptyset$. On the other hand, for the MOP J_2 of Figure 4.5, $\mathcal{A}_2(J_2) \cap \mathcal{A}_4(J_2) = \{u, v\}$. Hence $\mathcal{A}_2(J_2) \cap \mathcal{A}_4(J_2) \neq \emptyset$. Also observe $\gamma(J_1) = 3$ and $\gamma(J_2) = 2$, while the spine

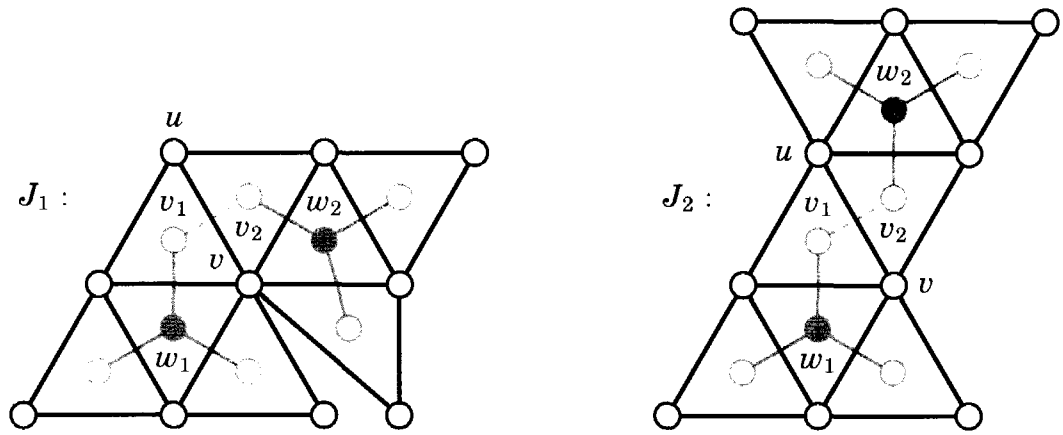


Figure 4.5 – The two ways H_1 and H_2 can be joined.

of these MOPs has domination number equal to 2. However, $\gamma(J_1 - x) = 2$ whenever $x \in \mathcal{A}_2(J_1)$. We generalize these observations in the next two lemmas and theorem.

Lemma 4.8. *Let G be a MOP with spine S. Suppose that $V(S)$ has a claw partition and that $\mathcal{A}_2(G) \cap \mathcal{A}_4(G) = \emptyset$. If $v \in \mathcal{A}_2(G)$, then $\gamma(G - v) \leq \gamma(S)$.*

Proof. The proof is by induction on the order of G. If G is isomorphic to the 3-sun, then $v \in D_2$ and so $\gamma(G - v) = 1 = \gamma(S)$. Now we assume that G is not isomorphic to the 3-sun. Let $e = v_1v_2$ be a good cut of S and note that both $V(S_1)$ and $V(S_2)$ have a claw partition by Lemma 4.4. Our assumptions imply $\mathcal{A}_2(G_i) \cap \mathcal{A}_4(G_i) = \emptyset$ for $i \in \{1, 2\}$. Therefore, we can apply the induction hypothesis to both G_1 and G_2. Let u be the vertex in $V(G_1) \cap V(G_2)$ that is in $\mathcal{A}_2(G)$ and observe $u \in \mathcal{A}_2(G_1) \cap \mathcal{A}_2(G_2)$ (see Figure 4.6). Without loss of generality, we assume $v \in V(G_2)$, which implies $v \in \mathcal{A}_2(G_2)$ and $v \notin V(G_1 - u)$. Let U_1 be a γ-set of $G_1 - u$ and let U_2 be a γ-set of $G_2 - v$. Then $|U_i| \leq \gamma(S_i)$ by the induction hypothesis. If $u \neq v$, then U_2 dominates u. Hence $U_1 \cup U_2$ is a dominating set of $G - v$. Furthermore, $|U_1 \cup U_2| \leq \gamma(S_1) + \gamma(S_2) = \gamma(S)$. Thus $\gamma(G - v) \leq \gamma(S)$ as desired. $\square$

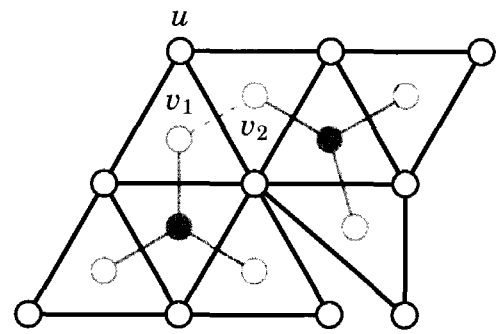


Figure 4.6 – A subgraph of G in the proof of Lemma 4.8.

Lemma 4.9. *Let G be a MOP with spine S and suppose that $V(S)$ has a claw partition. If $\mathcal{A}_2(G) \cap \mathcal{A}_4(G) \neq \emptyset$, then $\gamma(G) \leq \gamma(S)$.*

Proof. The proof is by induction on the order of G. If G is isomorphic to the 3-sun, then $\mathcal{A}_2(G) \cap \mathcal{A}_4(G) = \emptyset$ and so the statement is true vacuously. Now suppose that G is not isomorphic to the 3-sun. By virtue of Lemma 4.6, we may assume that $e = v_1v_2$ is a good cut of S such that S_1 is isomorphic to $K_{1,3}$ and that $V(S_2)$ has a claw partition. Hence G_1 is isomorphic to the 3-sun. Let $\{u, v\} = V(G_1) \cap V(G_2)$ such that $u \in \mathcal{A}_2(G_1)$ and $v \in \mathcal{A}_4(G_1)$. Observe that there exists $w \in V(G_1 - u)$ such that w dominates $V(G_1 - u)$. If $\gamma(G_2) \leq \gamma(S_2)$, then, for a γ-set U of G_2, $U \cup \{w\}$ is a dominating set of G and $|U \cup \{w\}| \leq \gamma(S_1) + \gamma(S_2) = \gamma(S)$. So we assume $\gamma(G_2) > \gamma(S_2)$, which implies that $\mathcal{A}_2(G_2) \cap \mathcal{A}_4(G_2) = \emptyset$ by the induction hypothesis. However, since $\mathcal{A}_2(G) \cap \mathcal{A}_4(G) \neq \emptyset$, it follows that $u \in \mathcal{A}_4(G_2)$ and $v \in \mathcal{A}_2(G_2)$ (see Figure 4.7). Hence $\gamma(G_2 - v) \leq \gamma(S_2)$ by Lemma 4.8. Thus, for a γ-set U of $G_2 - v$, $U \cup \{w\}$ is a dominating set of G and $|U| \leq \gamma(S_1) + \gamma(S_2) = \gamma(S)$. Therefore, $\gamma(G) \leq \gamma(S)$. □

Suppose G is a MOP with spine S. If it is not the case that both $V(S)$ has a claw partition and $\mathcal{A}_2(G) \cap \mathcal{A}_4(G) = \emptyset$, then combining Lemma 4.7 and Lemma 4.9 implies $\gamma(G) \leq \gamma(S)$. On the other hand, suppose $V(S)$ has a claw

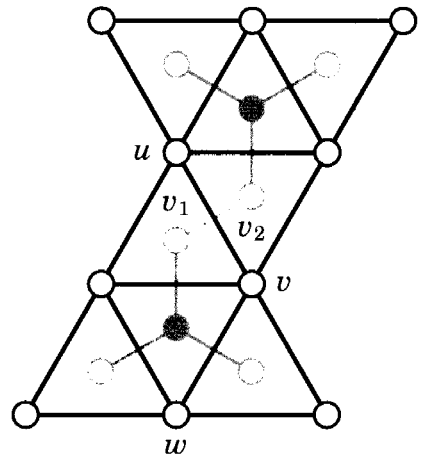


Figure 4.7 – A subgraph of the MOP G in the proof of Lemma 4.9

partition and $\mathcal{A}_2(G) \cap \mathcal{A}_4(G) = \emptyset$. Let $v \in \mathcal{A}_2(G)$ and let U be a γ-set of $G - v$. Observe that $U \cup \{v\}$ is a dominating set of G. Thus, by Lemma 4.8, $\gamma(G) \leq \gamma(S) + 1$.

Theorem 4.1. *Let G be a MOP with spine S. Then $\gamma(G) \leq \gamma(S) + 1$ with equality if and only if S has a claw partition and $\mathcal{A}_2(G) \cap \mathcal{A}_4(G) = \emptyset$.*

Proof. From the discussion above, it suffices to assume S has a claw partition and $\mathcal{A}_2(G) \cap \mathcal{A}_4(G) = \emptyset$, and then show $\gamma(G) \geq \gamma(S) + 1$. To do so, we prove the following stronger claim. If $U \subseteq V(G)$ and U dominates $\mathcal{A}_2(G)$, then $|U| \geq \gamma(S) + 1$. The proof of the claim is by induction on the order of G. If G is isomorphic to the 3-sun, then observe $\mathcal{A}_2(G) = D_2$ and that D_2 cannot be dominated by one vertex in G. Since $\gamma(S) = 1$, the result follows.

Now assume that G is not isomorphic to the 3-sun. By virtue of Lemma 4.6, we assume that $e = v_1v_2$ is a good cut of S such that S_1 is isomorphic to $K_{1,3}$ and $V(S_2)$ has a claw partition. Let $V(G_1)$ be labeled as in Figure 4.8 such that $V(G_1) \cap V(G_2) = \{u, v\}$. Now suppose $U' \subseteq V(G)$ such that U' dominates $\mathcal{A}_2(G)$. Because $x, z \in \mathcal{A}_2(G)$ and from considering the neighborhoods of the vertices in

$V(G_1)$, we conclude that there exists $U \subseteq V(G)$ such that $U \cap \{w, x, y, z\} = \{y\}$, U dominates $\mathcal{A}_2(G)$, and $|U| \leq |U'|$. Observe that $U - \{y\} \subseteq V(G_2)$ and that v is the only vertex of G_2 that y dominates. Since $v \in \mathcal{A}_4(G)$, it follows from our assumptions that $v \notin \mathcal{A}_2(G_2)$. Consequently, $U - \{y\}$ dominates $\mathcal{A}_2(G_2)$. Note $\mathcal{A}_2(G_2) \cap \mathcal{A}_4(G_2) = \emptyset$. Thus, by the induction hypothesis, $|U - \{y\}| \geq \gamma(S_2) + 1$. Therefore, $|U| \geq \gamma(S_1) + \gamma(S_2) + 1 = \gamma(S) + 1$. This proves the claim above, which completes the proof. □

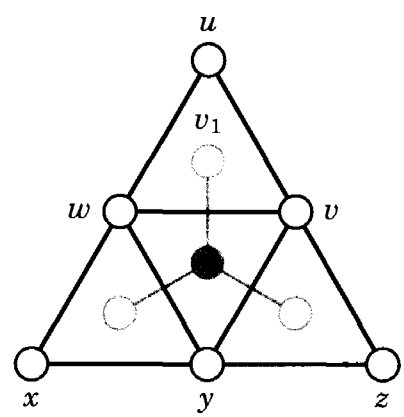


Figure 4.8 – The graph G_1 in the proof of Theorem 4.1

We remark that if S is a spine such that $V(S)$ has a claw partition, then there exists a MOP G whose spine is isomorphic to S and $\gamma(G) = \gamma(S) + 1$.

CHAPTER 5
BOUNDARY-TYPE SETS

5.1 Introduction

In this chapter we discuss the boundary of MOPs and several subsets of the boundary, which we refer to as boundary-type sets. In section 5.2 we give a characterization of the boundary of a MOP that involves degrees of vertices. We use this result to further study other boundary-type sets. The boundary-type set we focus on the most is called the *contour*, which has been found to be useful in rebuilding convex sets in graphs. We discuss this in more details and also find properties that characterize the contour of a MOP in section 5.3. One of the main results in section 5.4 is a characterization of graphs induced by the contour of a MOP, which also leads to a characterization of graphs induced by the *periphery*. For the rest of this section, we give the necessary definitions and examples of boundary-type sets in MOPs.

Let u and v be vertices in a connected graph G. Recall that a u–v geodesic is a shortest u–v path. The vertex v is said to be a **boundary vertex** of u if no neighbor of v is further away from u than v. Equivalently, the vertex v is a boundary vertex of u if no u–v geodesic can be extended at v to a longer geodesic. The vertex v is a **boundary vertex** of G if it is a boundary vertex of some vertex in G, and the **boundary** $\partial(G)$ of G is the set of all its boundary vertices;

$$\partial(G) = \{v \in V(G) \mid \exists u \in V(G), \forall w \in N(v) : d(u,w) \leq d(u,v)\}.$$

Recall that the eccentricity $\mathrm{ecc}(v)$ of v is the distance between v and a vertex farthest from v. The vertex u is an **eccentric vertex** of v if $d(u,v) = \mathrm{ecc}(v)$. We denote by $\mathrm{Ecc}(v)$ the set of eccentric vertices of v. The vertex v is an **eccentric vertex** of G if it is an eccentric vertex of some vertex in G, and the **eccentricity** $\mathrm{Ecc}(G)$ of G is the set of all of its eccentric vertices;

$$\mathrm{Ecc}(v) = \{u \in V(G) \mid d(u,v) = \mathrm{ecc}(v)\},$$

$$\mathrm{Ecc}(G) = \bigcup_{v \in V(G)} \mathrm{Ecc}(v).$$

We remark that it is easy to show that if G is a MOP, then there exists an eccentric vertex of v of degree 2. This fact implies that in order to compute the eccentricity of a vertex v in a MOP, it is sufficient to calculate the maximum of $d(x,v)$ over all vertices x of degree 2. We mention this to help the reader calculate eccentricities of vertices in the figures of this chapter.

Suppose that the vertex v is an eccentric vertex of u and let w be a neighbor of v. Observe that $d(u,w) \leq \mathrm{ecc}(u) = d(u,v)$. It follows directly from this observation that v is a boundary vertex of u, and so v is a boundary vertex of G. Therefore, we have that $\mathrm{Ecc}(G) \subseteq \partial(G)$. On the other hand, it is not the case that every boundary vertex of G is necessarily an eccentric vertex of G. For example, let G be the MOP in Figure 5.1, where each vertex is labeled with its eccentricity. We can observe that the distance from the vertex v to any other vertex is not the eccentricity of that vertex. Thus, v is not an eccentric vertex of G. Now observe that u is adjacent to v and to both of the other two neighbors of v. So $d(u,w) \leq 1 = d(u,v)$ for every neighbor w of v, which implies that v is a boundary vertex of u.

The vertex v is a **contour vertex** of G if the eccentricity of any neighbor of v is at most $\mathrm{ecc}(v)$. The **contour** $\mathrm{Ct}(G)$ of G is the set of all of its contour

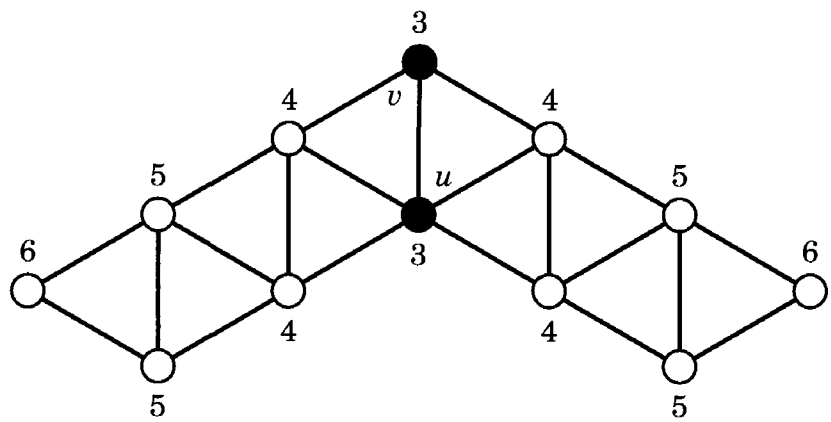


Figure 5.1 – The vertex v is a boundary vertex of u.

vertices;

$$\mathrm{Ct}(G) = \{v \in V(G) \mid \forall u \in N(v) : \mathrm{ecc}(u) \leq \mathrm{ecc}(v)\}.$$

If v is a contour vertex of G and u is an eccentric vertex of v, then it follows that v is a boundary vertex of u. Hence $\mathrm{Ct}(G) \subseteq \partial(G)$. Note that the vertex v in the MOP of Figure 5.1 has eccentricity 3 and that some of v's neighbors have eccentricity 4. So, as with eccentric vertices, it is possible for a vertex to be a boundary vertex of G but not a contour vertex of G.

Recall that the maximum eccentricity among the vertices of G is the diameter $\mathrm{diam}\, G$ of G. The vertex v is a **peripheral vertex** of G if $\mathrm{ecc}(v) = \mathrm{diam}\, G$, and the **periphery** $\mathrm{Per}(G)$ of G is the set of all of its peripheral vertices;

$$\mathrm{Per}(G) = \{v \in V(G) \mid \mathrm{ecc}(v) = \mathrm{diam}\, G\}.$$

Certainly, every peripheral vertex of G is a contour vertex of G. If v is a peripheral vertex of G and $u \in \mathrm{Ecc}(v)$, then u is a peripheral vertex of G and $v \in \mathrm{Ecc}(u)$. Hence every peripheral vertex of G is an eccentric vertex of G. Now let G be the MOP in Figure 5.2, where each vertex is labeled by its eccentricity. Observe that $\mathrm{diam}\, G = 6$. So the vertices with eccentricity 6 are peripheral

vertices of G and hence eccentric vertices of G. Next observe that $\mathrm{ecc}(v) = 4$ and that the eccentricity of every neighbor of v is at most 4. Thus v is a contour vertex of G but v is not a peripheral vertex of G. In fact, we can observe that v is not an eccentric vertex of G.

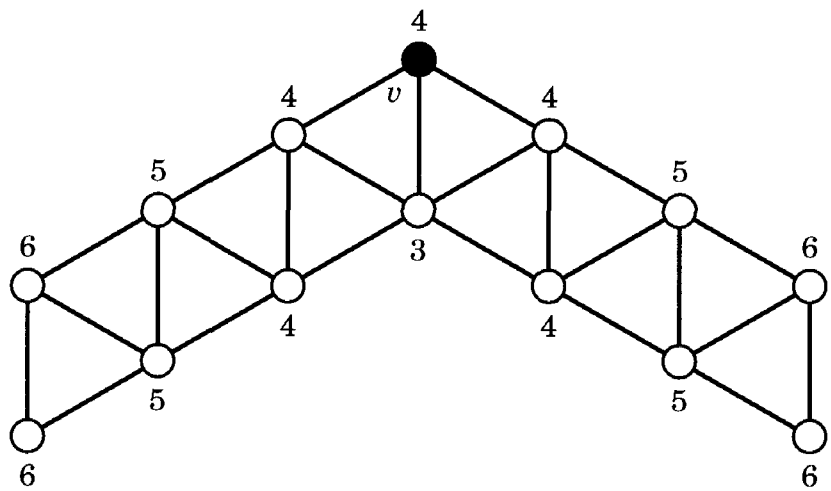


Figure 5.2 – The vertex v is a contour vertex but not an eccentric vertex of G.

Finally, the vertex v is an **extreme vertex** if the subgraph induced by its neighborhood is a complete graph. This definition coincides with the one given in Chapter 1 for an extreme vertex in the context of convexity. An extreme vertex is also commonly called a **simplicial vertex**. The **extreme set** $\mathrm{Ext}(G)$ of G is the set of all its extreme vertices;

$$\mathrm{Ext}(G) = \{v \in V(G) \mid \langle N[v] \rangle \text{ is a complete graph}\}.$$

Observe that every leaf in G is an extreme vertex. It is easy to show that if v is an extreme vertex, then v is a boundary vertex of every other vertex in G. Moreover, the vertex v is a contour vertex of G. Note that all the vertices of degree 2 of the MOPs in both figures above are extreme vertices and also peripheral vertices. However, the MOP in Figure 5.2 has two peripheral vertices of degree 3 that are not extreme vertices.

Each of the sets we have defined above is a subset of the boundary of G. The following proposition, which is Proposition 6 in [4], gives a summary of the relationships between these boundary-type sets.

Proposition 5A. *If G is a connected graph, then the following statements hold.*

(1) $\mathrm{Ext}(G) \subseteq \mathrm{Ct}(G)$.

(2) $\mathrm{Per}(G) \subseteq \mathrm{Ct}(G) \cap \mathrm{Ecc}(G)$.

(3) $\mathrm{Ecc}(G) \cup \mathrm{Ct}(G) \subseteq \partial(G)$.

The containments established in the above proposition are illustrated in Figure 5.3. We point out that if G is a tree, then the vertex v is a boundary vertex if and only if v is a leaf. So it follows from Proposition 5A that if G is a tree, then $\partial(G) = \mathrm{Ct}(G) = \mathrm{Ext}(G)$, where these sets consist of the leaves of G. Also the eccentric vertices and peripheral vertices of G form subsets of the leaves of G. For MOPs, there are no general containments between these boundary-type sets other than the containments stated in Proposition 5A. We will demonstrate this in section 5.5 by showing that for every region of the Venn diagram in Figure 5.3, there exists a vertex in a MOP satisfying the properties indicated by that region.

5.2 Characterization of the boundary

The main result in this section is a characterization of the boundary of a MOP. The degree of a vertex in a MOP plays a significant role in determining if the vertex is in the boundary. The next three lemmas concern vertices in a MOP of degree 2, 3, and 4, respectively. These results will be used throughout this chapter.

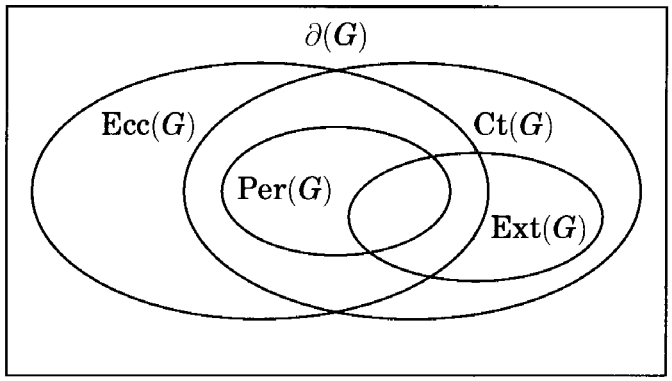
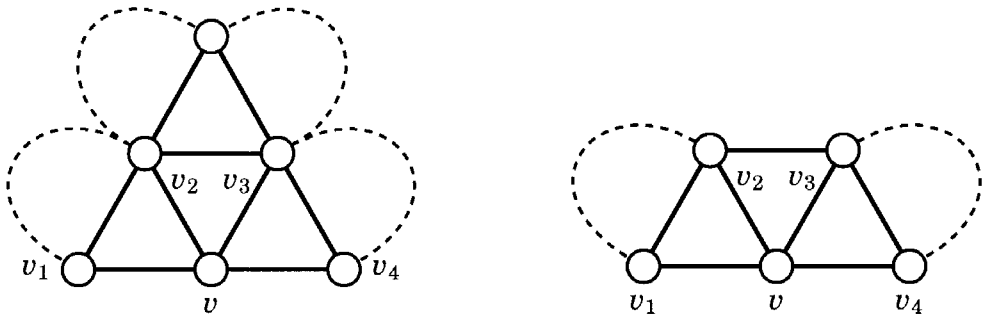


Figure 5.3 – Basic containments between boundary-type sets.

Lemma 5.1. *If G is a MOP, then* $\mathrm{Ext}(G) = D_2$.

Proof. If $v \in D_2$, then $N[v]$ induces a triangle in G. Thus both of the neighbors of v are adjacent and so $v \in \mathrm{Ext}(G)$. Now assume that $v \in \mathrm{Ext}(G)$ and note that $\deg(v) \geq 2$. Therefore, the subgraph of G induced by $N[v]$ is a complete graph. Since outerplanar graphs contain no copy of K_4, it follows that $\deg(v) \leq 2$. Hence $v \in D_2$, and the proof is complete. □

Proposition 5A implies that $\mathrm{Ext}(G) \subseteq \mathrm{Ct}(G) \subseteq \partial(G)$ for any connected graph G. Therefore, by Lemma 5.1, we have that $D_2 \subseteq \partial(G)$ for any MOP G. Now let G be a MOP. In order to describe $\partial(G)$, we will define a special subset of the vertices of degree 4. Let v be a vertex of degree 4 in G and let the path induced by $N(v)$ be v_1, v_2, v_3, v_4. We call v_2 and v_3 the **internal neighbors** of v. We denote by D_4' the set of vertices of degree 4 with the property that the edge joining the internal neighbors is a chord of G. Figure 5.4a illustrates a MOP with a vertex v in D_4' and Figure 5.4b illustrates a MOP with a vertex v in $D_4 - D_4'$. The edge v_2v_3 is a chord in Figure 5.4a, but the edge v_2v_3 is an outer edge in Figure 5.4b. Note that in both MOP embeddings we have that $N[v] \subseteq \mathrm{Arc}[v, v_4]$.



(a) A vertex v in D_4'.

(b) A vertex v in $D_4 - D_4'$.

Figure 5.4 – Vertices of degree 4 in a MOP.

Lemma 5.2. *If v is a vertex of degree 3 in a MOP G and the path induced by $N(v)$ is v_1, v_2, v_3, then the following three conditions hold.*

(1) *The vertex v is a boundary vertex of v_2.*

(2) *If $v \in \mathrm{Ct}(G)$ and $u \in \mathrm{Ecc}(v)$, then $v_2 \in I(u, v)$.*

(3) *If $v \in \mathrm{Ct}(G)$, then $v_2 \notin D_2 \cup D_3 \cup D_4'$.*

Proof.

(1) Note that v_2 is adjacent to both v_1 and v_3. Therefore, $d(v_2, w) \leq 1 = d(v_2, v)$ for every $w \in N(v)$. Hence v is a boundary vertex of v_2.

(2) The proof is by contradiction. Assume that $v \in \mathrm{Ct}(G)$ and let $u \in \mathrm{Ecc}(v)$. Without loss of generality, we assume that $N[v] \subseteq \mathrm{Arc}[v, v_3]$ and that $u \in \mathrm{Arc}(v_1, v_2)$ (see Figure 5.5). Note that the set $\{v_1, v_2\}$ separates u and v and also u and v_3. Now suppose that $v_2 \notin I(u, v)$. Thus $v_1 \in I(u, v)$ and so $d(u, v_1) < d(u, v_2)$. It follows that $v_1 \in I(u, v_3)$. Since v_1 is closer to v than to v_3, we have that $d(u, v) < d(u, v_3)$. Because $u \in \mathrm{Ecc}(v)$, this implies that $\mathrm{ecc}(v) < \mathrm{ecc}(v_3)$, which contradicts that $v \in \mathrm{Ct}(G)$.

(3) Assume that $v \in \mathrm{Ct}(G)$. Observe that $v_2 \notin D_2$ since $N(v_2) \supseteq \{v, v_1, v_3\}$. Suppose first that $v_2 \in D_3$. Since $\deg(v) = 3$, it follows that $G = \langle N[v] \rangle$, which implies that $\mathrm{ecc}(v) = 1$ and that $\mathrm{ecc}(v_1) = 2$. This contradicts that $v \in \mathrm{Ct}(G)$. Now suppose $v_2 \in D_4'$. Observe that v is an internal neighbor of v_2. However, the edges vv_1 and vv_3 are not chords of G. This contradicts that $v_2 \in D_4'$. Therefore, $v_2 \notin D_2 \cup D_3 \cup D_4'$. □

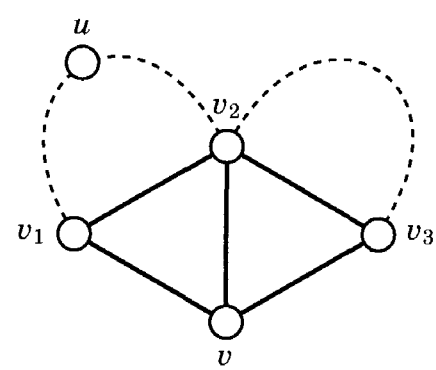


Figure 5.5 – A MOP with a vertex v of degree 3 and a vertex u in $Arc(v_1, v_2)$.

Let v and v_2 be vertices of a MOP as in the statement of Lemma 5.2 (3). We will show in Theorem 5.1 that $\partial(G) = D_2 \cup D_3 \cup D_4'$ for any MOP G. So combined with this theorem, Lemma 5.2 (3) implies that $v_2 \notin \partial(G)$, and therefore $v_2 \notin \mathrm{Ct}(G)$.

Lemma 5.3. *Let G be a MOP. Let $v \in D_4'$ such that the path induced by $N(v)$ is v_1, v_2, v_3, v_4 and that $N[v] \subseteq \mathrm{Arc}[v, v_4]$. Let w be the common neighbor of v_2 and v_3 other than v. Then the following three conditions hold.*

(1) *The vertex v is a boundary vertex of w.*

(2) *If $v \in \mathrm{Ct}(G)$ and $u \in \mathrm{Ecc}(v)$, then $u \in \mathrm{Arc}(v_2, v_3)$ and $w \in I(u, v)$.*

(3) *If* $\{v, v_i\} \subseteq \mathrm{Ct}(G)$ *for some* $i \in \{2, 3\}$ *and* $\deg(v_i) = 4$, *then the component of* $\langle \mathrm{Ct}(G) \rangle$ *that contains* v *has a triangle.*

Proof.

(1) Observe that $d(w, v_i) \leq 2$ for each i, $1 \leq i \leq 4$, and that $d(v, w) = 2$. Therefore, v is a boundary vertex of w.

(2) Assume that $v \in \mathrm{Ct}(G)$ and let $u \in \mathrm{Ecc}(v)$. Observe that $u \notin N(v)$ since $\mathrm{ecc}(v) \geq 2$. Suppose that $u \notin \mathrm{Arc}(v_2, v_3)$. Without loss of generality, we assume that $u \in \mathrm{Arc}(v_1, v_2)$, which implies that the set $\{v_1, v_2\}$ separates u and v and also u and v_4. Since both v_1 and v_2 are closer to v than to v_4, we have that that $d(u, v) < d(u, v_4)$. Because $u \in \mathrm{Ecc}(v)$, this implies that $\mathrm{ecc}(v) < \mathrm{ecc}(v_4)$, which contradicts that $v \in \mathrm{Ct}(G)$. So we conclude that $u \in \mathrm{Arc}(v_2, v_3)$. Next we show that $w \in I(u, v)$. Without loss of generality, we assume that $u \in \mathrm{Arc}(v_2, w)$. Note that the set $\{v_2, w\}$ separates u and v and also u and v_4. Now suppose that $w \notin I(u, v)$, which implies that $v_2 \in I(u, v)$. If $v_2 \in I(u, w)$, then $d(u, v) < d(u, v_4)$ since v_2 is closer to v than to v_4. If $w \in I(u, v_4)$, then it follows that $d(u, v) < d(u, w) + 2 = d(u, v_4)$ since w is distance 2 from both v and v_4. Therefore, it follows that $\mathrm{ecc}(v) < \mathrm{ecc}(v_4)$, which contradicts that $v \in \mathrm{Ct}(G)$.

(3) Without loss of generality, suppose that $\{v, v_2\} \subseteq \mathrm{Ct}(G)$ and that $\deg(v_2) = 4$. Observe that the edges vv_1 and v_1v_2 are outer edges of G and that the edge vv_2 is a chord of G. This implies that $v_1 \in D_2$. Hence $v_1 \in \mathrm{Ct}(G)$ by Lemma 5.1. Therefore, the triangle induced by $\{v, v_1, v_2\}$ is a subgraph of $\langle \mathrm{Ct}(G) \rangle$, and the proof is complete. □

We now present a characterization of the boundary of a MOP.

Theorem 5.1. *If* G *is a MOP, then* $\partial(G) = D_2 \cup D_3 \cup D_4'$.

Proof. Let $v \in V(G)$. If $v \in D_2$, then Lemma 5.1 implies that $v \in \partial(G)$. If $v \in D_3$, then Lemma 5.2 (1) implies that $v \in \partial(G)$. If $v \in D_4'$, then Lemma 5.3 (1) implies that $v \in \partial(G)$. Thus, it suffices to show that if $v \in D_4 - D_4'$ or if $\deg(v) \geq 5$, then $v \notin \partial(G)$. To do so, we show that for every vertex u, $u \neq v$, that there is a neighbor of v whose distance from u is larger than its distance from v. Suppose first $v \in D_4 - D_4'$ and let $N(v)$ be labeled as in Figure 5.4b. Note that $V(G) - \{v\} = \text{Arc}[v_1, v_2] \cup \text{Arc}[v_3, v_4]$. Without loss of generality, suppose that $u \in \text{Arc}[v_1, v_2]$. As in the proof of Lemma 5.3 (2), we have that $d(u, v) < d(u, v_4)$. Hence v is not a boundary vertex of G. Now suppose that $\deg(v) \geq 5$. Let the path induced by $N(v)$ be $v_1, v_2, \ldots, v_k$ and assume that $N[v] \subseteq \text{Arc}[v, v_k]$ (see Figure 5.6). Note that $V(G) - \{v\} = \text{Arc}[v_1, v_2] \cup \text{Arc}[v_2, v_3] \cdots \cup \text{Arc}[v_{k-1}, v_k]$. Suppose $u \in \text{Arc}[v_1, v_2] \cup \text{Arc}[v_2, v_3]$. Then for some $i \in \{1, 2\}$ we have that the set $\{v_i, v_{i+1}\}$ separates u and v and also u and v_k. Since $k \geq 5$, we have that both v_i and v_{i+1} are closer to v than to v_k. Hence $d(u, v) < d(u, v_k)$. If $u \notin \text{Arc}[v_1, v_2] \cup \text{Arc}[v_2, v_3]$, then a similar argument shows that $d(u, v) < d(u, v_1)$. Therefore, v is not a boundary vertex of G as desired. □

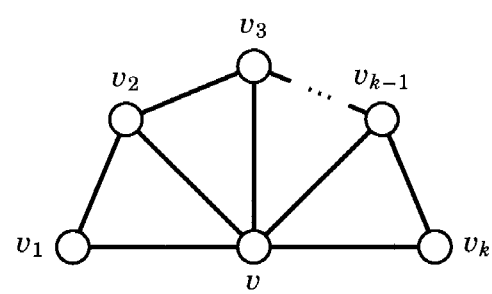


Figure 5.6 – The graph $\langle N[v] \rangle$ for the case $\deg(v) = k \geq 5$.

5.3 Characterization of the contour

The notion of contour of a graph can be generalized to define the *contour* of a set of vertices. A **convex geometry** is a convexity space with the additional property that every convex set is the convex hull of its extreme points. This property is referred to as the **Minkowski-Krein-Milman** property. Recall that we say a set W of vertices in a graph is convex if it contains every vertex on a u–v geodesic for every $u, v \in W$. We discuss convex sets of a MOP in detail in section 6.1. Farber and Jamison [9] proved that a graph has the Minkowski-Krein-Milman property if and only if G is chordal and has no induced 3-fan (these graphs are called **Ptolemaic graphs**). However, Cáceres, *et al.* [6] proved that for every graph, every convex set is the convex hull of its contour.

In this section we focus on finding a characterization of the contour of a MOP. Let v be a vertex in a connected graph G and consider the two properties below.

P1: Each neighbor u of v which is on a geodesic between v and some eccentric vertex of v satisfies $N[v] \subseteq N[u]$.

P2: For each eccentric vertex u of v, there exists $w \in V(G)$ such that v is a boundary vertex of w and $w \in I(u, v)$.

A graph G is **distance hereditary** if for every connected induced subgraph H of G and every two vertices u, v in H, $d_H(u, v) = d_G(u, v)$. Proposition 3 in [6] shows that for distance hereditary graphs without induced 4-cycles, the set of contour vertices is precisely the set of vertices that satisfy property P1. Cáceres, *et al.* [6] pose the problem of finding a characterization of contour vertices for various classes of graphs similar to the one above for distance hereditary graphs. They remark that property P1 is a sufficient condition for the vertex v to be in $\mathrm{Ct}(G)$ for any graph G. They also point out that this charac-

terization of contour vertices does not extend to the family of all chordal graphs since, for example, it does not extend to the fan F_3. To see this, observe that F_3 is isomorphic to the subgraph induced by $N[v]$ in the MOP of Figure 5.4b. Using the labels in this figure, we see that v_2 and v_3 are contour vertices and that each lie on a geodesic between the other vertex and an eccentric vertex of the other. However, their closed neighborhoods are not equal. Therefore, property P1 is not a necessary condition for a vertex to be a contour vertex of a MOP. Like property P1, the next result shows that property P2 is a sufficient condition for a vertex to be a contour vertex of any connected graph. On the other hand, Theorem 5.2 shows that property P2 is a necessary condition for a vertex to be a contour vertex of a MOP.

Proposition 5.1. *If v is a vertex of a connected graph G such that v satisfies property P2, then v is a contour vertex of G.*

Proof. Assume, to the contrary, that there exists a neighbor x of v such that $\mathrm{ecc}(x) > \mathrm{ecc}(v)$. Let $u \in \mathrm{Ecc}(x)$ and note that $d(u,x) \leq d(u,v) + 1$. We claim that $u \in \mathrm{Ecc}(v)$; for otherwise, $\mathrm{ecc}(x) = d(u,x) \leq d(u,v) + 1 \leq \mathrm{ecc}(v)$, which is a contradiction. Hence there exists a vertex w of G such that v is a boundary vertex of w and $w \in I(u,v)$. Since u is an eccentric vertex of x and v, it follows that $d(u,x) = d(u,v) + 1$. Thus, there exists a u–x geodesic of the form $u, \ldots, w, \ldots, v, x$. However, this implies that $d(w,x) > d(w,v)$, which contradicts that v is a boundary vertex of w. □

Recall that every extreme vertex of a connected graph is a boundary vertex of every other vertex in the graph. It follows that every extreme vertex of a connected graph satisfies property P2.

Theorem 5.2. *If G is a MOP, then* $\mathrm{Ct}(G)$ *is the set of vertices satisfying prop-*

erty P2.

Proof. Let $v \in V(G)$. Proposition 5.1 shows that property P2 is a sufficient condition for v to be a contour vertex of G. Now assume that $v \in \mathrm{Ct}(G)$. Note that $v \in D_2 \cup D_3 \cup D_4'$ by Theorem 5.1. If $v \in D_2$, then $v \in \mathrm{Ext}(G)$ by Lemma 5.1, and so v satisfies property P2 by the comment preceding the theorem. If $v \in D_3$, then the first two conditions of Lemma 5.2 imply that v satisfies property P2. If $v \in D_4'$, then the first two conditions of Lemma 5.3 imply that v satisfies property P2. Thus, property P2 is a necessary condition for v to be a contour vertex of G. □

Let G be a MOP and consider property P2. We remark that if $v \in D_3 \cup D_4$, then the combination of Lemma 5.2 and Lemma 5.3 implies that v satisfies the following condition, which is stronger than property P2: There exists $w \in V(G)$ such that for each eccentric vertex u of v, the vertex v is a boundary vertex of w and $w \in I(u, v)$. However, if $v \in D_2$, this is not necessarily the case. For example, consider the 3-sun in Figure 5.7. Note that both u_1 and u_2 are eccentric vertices of v. However, the unique v–u_1 geodesic and the unique v–u_2 geodesic have only the vertex v in common. Hence the candidates for w depend on the choice of u.

5.4 Graphs induced by the periphery, contour, and eccentricity

For a connected graph G, recall that $\mathrm{Per}(G)$ is the set of vertices of maximum eccentricity, while the **center** of a graph is the subgraph induced by the vertices of minimum eccentricity. Proskurowski [19] showed that the center of a maximal outerplanar graph can be one of only seven graphs. In this section we examine which graphs can be induced subgraphs of the periphery, contour, or eccentricity of a MOP. The main results of this section gives a characteriza-

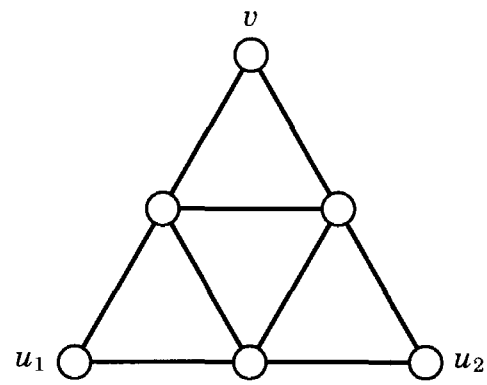


Figure 5.7 – There is no vertex w such that v is a boundary vertex of w and w is in $I(v, u_1)$ and $I(v, u_2)$.

tion of which graphs can be isomorphic to $\langle \mathrm{Ct}(G) \rangle$ or to $\langle \mathrm{Per}(G) \rangle$ when G is a MOP.

Let G be a connected graph and consider a component C of $\langle \mathrm{Ct}(G) \rangle$. Observe that all the vertices of C have the same eccentricity. Thus, either each vertex of C is a peripheral vertex or each vertex of C is not a peripheral vertex. Since $\mathrm{Per}(G) \subseteq \mathrm{Ct}(G)$, it follows that every component of $\langle \mathrm{Per}(G) \rangle$ is a component of $\langle \mathrm{Ct}(G) \rangle$. So by finding necessary conditions for the components of $\langle \mathrm{Ct}(G) \rangle$, we also find necessary conditions for the components of $\langle \mathrm{Per}(G) \rangle$.

Next, suppose that every vertex of G has the same eccentricity. Note that every vertex of G is a peripheral vertex. Hence $V(G) = \mathrm{Per}(G)$, which is true if and only if $V(G) = \mathrm{Per}(G) = \mathrm{Ct}(G) = \mathrm{Ecc}(G) = \partial(G)$. There are two MOPs that have the property that every vertex has the same eccentricity; namely, K_3, and the 3-sun which is displayed in Figure 5.7. However, if G is a MOP that is isomorphic to neither K_3 nor the 3-sun, then there can be no triangle in $\langle \mathrm{Per}(G) \rangle$ (see Lemma 5.5). Therefore, K_3 and the 3-sun are the only MOPs that satisfy the equations above. On the other hand, there are other MOPs whose subgraph induced by their contour vertices contain a triangle, which we discuss next.

For the rest of this section we denote by Λ the MOP in Figure 5.8. In Λ,

observe that every vertex in the set $\{v, v_1, v_2, x_1, x_2\}$ has eccentricity 3 and that $\mathrm{ecc}(w) = 2$. Hence $\{v, v_1, v_2\} \subseteq \mathrm{Ct}(\Lambda)$. Now observe that $\mathrm{ecc}(z_1) = \mathrm{ecc}(z_2) = 4$. It follows from these observations that $x_1, x_2, w \notin \mathrm{Ct}(\Lambda)$, and thus the triangle induced by the set $\{v, v_1, v_2\}$ is a component of $\langle \mathrm{Ct}(\Lambda) \rangle$. Furthermore, $\langle \mathrm{Ct}(\Lambda) \rangle$ has only one triangle and $\langle \mathrm{Per}(\Lambda) \rangle$ contains no triangles. The next three lemmas generalize the observations we have made about the MOP Λ.

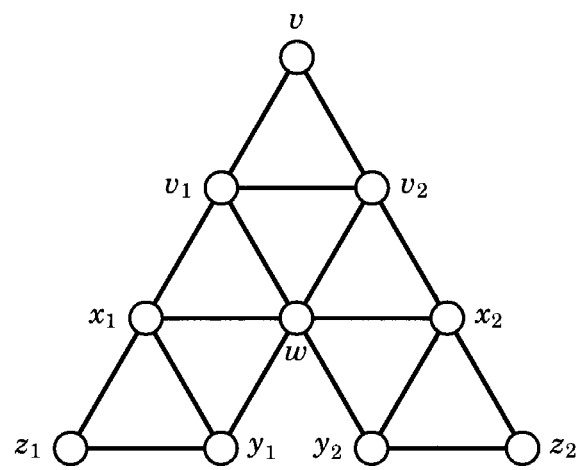


Figure 5.8 – The MOP Λ.

Lemma 5.4. *Let G be a MOP that is isomorphic to neither K_3 nor the 3-sun. If $v \in D_2$ such that $N[v] \subseteq \mathrm{Ct}(G)$, then v is in a subgraph of G that is isomorphic to Λ, where the neighbors of v have degree 4 in G.*

Proof. Let $N(v) = \{v_1, v_2\}$ such that $v \in \mathrm{Arc}(v_1, v_2)$ as in the MOP Λ. Since we are assuming G is not isomorphic to K_3, the edge v_1v_2 is a chord of G. Let w be the common neighbor of v_1 and v_2 other than v. Since $v_1, v_2 \in \mathrm{Ct}(G)$, Lemma 5.2 (3) implies that $v_1, v_2 \notin D_3$. Hence $v_1, v_2 \in D_4'$ by Theorem 5.1. Thus, for $i \in \{1, 2\}$, there exists a vertex x_i that is a common neighbor of v_i and w, where $x_1 \in \mathrm{Arc}(w, v_1)$ and $x_2 \in \mathrm{Arc}(v_2, w)$. Since $\deg(v_i) = 4$ and $\deg(v) = 2$, it follows that the edges of the path x_1, v_1, v, v_2, x_2 are all outer edges of G, and so

this path lies on the hamiltonian cycle of G. Because G is not isomorphic to the 3-sun, we can assume, without loss of generality, that x_1 and w have a common neighbor y_1, where $y_1 \neq v_1$. Note that $d(v_1, y_1) = 2$. Thus, by Lemma 5.3 (2), we have $\mathrm{Ecc}(v_1) \subseteq \mathrm{Arc}(v_2, w)$ and that $\mathrm{ecc}(v_1) \geq 3$. It follows that x_2 and w have a common neighbor y_2, where $y_2 \neq v_2$. It remains to show that the edges x_1y_1 and x_2y_2 are not outer edges of G. Suppose, without loss of generality, that x_2y_2 is an outer edge of G. Let $u \in \mathrm{Ecc}(v_1)$. Since the edge v_2x_2 is an outer edge of G, we have that $u \in \mathrm{Arc}(y_2, w)$ by Lemma 5.3 (2). However, because both y_2 and w are closer to v_1 than they are to v, it follows that $d(u, v_1) < d(u, v)$. Therefore, $\mathrm{ecc}(v_1) < \mathrm{ecc}(v)$, which contradicts that $v_1 \in \mathrm{Ct}(G)$. □

Let G be a MOP that is isomorphic to neither K_3 nor the 3-sun. As in the MOP Λ, the next lemma shows that every component of $\langle \mathrm{Ct}(G) \rangle$ which contains a triangle is isomorphic to K_3 and no such component of $\langle \mathrm{Per}(G) \rangle$ exists.

Lemma 5.5. *Let G be a MOP that is isomorphic to neither K_3 nor the 3-sun. If C is a component of $\langle \mathrm{Ct}(G) \rangle$ that has a triangle, then there exists a vertex of C in D_2 and C is isomorphic to K_3. Furthermore,* $\mathrm{Per}(G) \cap V(C) = \emptyset$.

Proof. Let x, y, z be vertices of the component C that induce a triangle in G. We show that at least one of these vertices has degree 2 in G. Suppose this is not the case. Without loss of generality, we may assume the edges xy and xz are chords of G since G is not isomorphic to K_3. Observe that $\deg(x) \geq 4$, which implies $x \in D_4'$ by Theorem 5.1. Thus, the edge yz is also a chord of G, and so $y, z \in D_4'$ by Theorem 5.1 (see Figure 5.9). It follows that G is isomorphic to the 3-sun, which is a contradiction.

So we assume that C has a vertex v of degree 2 in G such that $N[v] \subseteq \mathrm{Ct}(G)$. By Lemma 5.4, we assume that v is in a subgraph isomorphic to the MOP Λ and that the neighbors of v have degree 4. We refer to the vertices of

this subgraph as they are labeled in Λ. Since $\deg(w) > 4$, we have $w \notin \mathrm{Ct}(G)$ by Theorem 5.1. Next we show that $x_1, x_2 \notin \mathrm{Ct}(G)$. Without loss of generality, suppose $x_1 \in \mathrm{Ct}(G)$. Note that $\mathrm{ecc}(v) = \mathrm{ecc}(v_1)$. Theorem 5.1 implies $x_1 \in D_4'$. Now let $u \in \mathrm{Ecc}(x_1)$. By Lemma 5.3 (2), we have $u \in \mathrm{Arc}(w, y_1)$. Since both w and y_1 are closer to x_1 than they are to v, it follows that $d(u, x_1) < d(u, v)$. This implies that $\mathrm{ecc}(x_1) < \mathrm{ecc}(v) = \mathrm{ecc}(v_1)$, which contradicts our assumption that $x_1 \in \mathrm{Ct}(G)$. So we conclude that $x_1, x_2 \notin \mathrm{Ct}(G)$. Hence C is isomorphic to K_3. Finally, we show that $\mathrm{Per}(G) \cap V(C) = \emptyset$. To do so, it suffices to show $\mathrm{ecc}(v_1) < \mathrm{ecc}(z_1)$ since all the vertices in a component of $\langle \mathrm{Ct}(G) \rangle$ have the same eccentricity. So let $u \in \mathrm{Ecc}(v_1)$. Lemma 5.3 (2) implies that $u \in \mathrm{Arc}(v_2, w)$. Since both v_2 and w are closer to v_1 than they are to z_1, we have $d(u, v_1) < d(u, z_1)$. This implies $\mathrm{ecc}(v_1) < \mathrm{ecc}(z_1)$ as desired. □

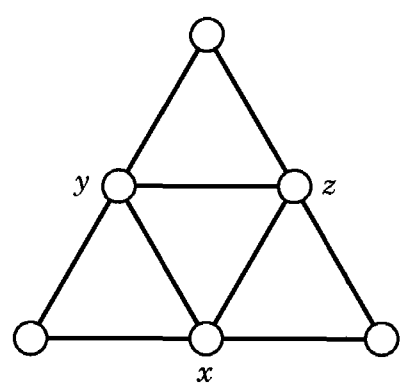


Figure 5.9 – The vertices x, y, z are each in D_4'.

Let G be a MOP. As in the MOP Λ, the next lemma shows that at most one component of $\langle \mathrm{Ct}(G) \rangle$ is isomorphic to K_3.

Lemma 5.6. *If G is a MOP, then at most one component of $\langle \mathrm{Ct}(G) \rangle$ is isomorphic to K_3.*

Proof. The proof is by contradiction. Suppose C_u and C_v are distinct components

in $\langle \mathrm{Ct}(G) \rangle$ and that both of these components are isomorphic to K_3. Hence G is isomorphic to neither K_3 nor the 3-sun. By Lemma 5.5, there exists a vertex of degree 2 in C_u and in C_v. By Lemma 5.4, we may assume that $V(C_u) = \{u, u_1, u_2\}$ and that $V(C_v) = \{v, v_1, v_2\}$, where $u, v \in D_2$ and $u_1, u_2, v_1, v_2 \in D_4$. Let w be the common neighbor of v_1 and v_2 other than v and assume that $v \in \mathrm{Arc}(v_1, v_2)$. Lemma 5.3 (2) implies that $\mathrm{Ecc}(v_1) \subseteq \mathrm{Arc}(v_2, w)$ and $\mathrm{Ecc}(v_2) \subseteq \mathrm{Arc}(w, v_1)$, and hence $\mathrm{Ecc}(v_1) \cap \mathrm{Ecc}(v_2) = \emptyset$. By interchanging the roles of C_u and C_v, we also have $\mathrm{Ecc}(u_1) \cap \mathrm{Ecc}(u_2) = \emptyset$. Since the vertices in a component of $\langle \mathrm{Ct}(G) \rangle$ have the same eccentricity, we may assume that the vertices of C_v have eccentricity at least as large as the vertices of C_u. Furthermore, without loss of generality, we may assume that $V(C_u) \subseteq \mathrm{Arc}(w, v_1)$ (see Figure 5.10). Let $x \in \mathrm{Ecc}(v_1)$ and $i \in \{1, 2\}$. So we have that $x \in \mathrm{Arc}(v_2, w)$. Note that the set $\{v_2, w\}$ separates x and u_i and also x and v_1. Since v_1 is adjacent to v_2 and w, it follows that $d(u_i, x) \geq d(v_1, x) = \mathrm{ecc}(v_1)$. Because $\mathrm{Ecc}(u_1) \cap \mathrm{Ecc}(u_2) = \emptyset$, we may assume that $x \notin \mathrm{Ecc}(u_1)$. However, this implies that $\mathrm{ecc}(u_1) > d(u_1, x) \geq \mathrm{ecc}(v_1)$, which is a contradiction. □

Let G be a MOP that is isomorphic to neither K_3 nor the 3-sun. We are now ready to give a necessary condition for a graph to be isomorphic to either $\langle \mathrm{Ct}(G) \rangle$ or $\langle \mathrm{Per}(G) \rangle$. A **linear forest** is a graph in which each component is a path. Theorem 5.3 shows that $\langle \mathrm{Per}(G) \rangle$ is a linear forest and that $\langle \mathrm{Ct}(G) \rangle$ is either a linear forest or the union of a linear forest and a triangle. In fact, we prove that a triangle-free component of $\langle \mathrm{Ct}(G) \rangle$ is a path by showing that it is a subgraph of the hamiltonian cycle.

Theorem 5.3. *Let G be a MOP that is not isomorphic to the 3-sun. Then both the following conditions hold.*

(1) *Every component of $\langle \mathrm{Ct}(G) \rangle$ is isomorphic to either K_3 or a path. Further-*

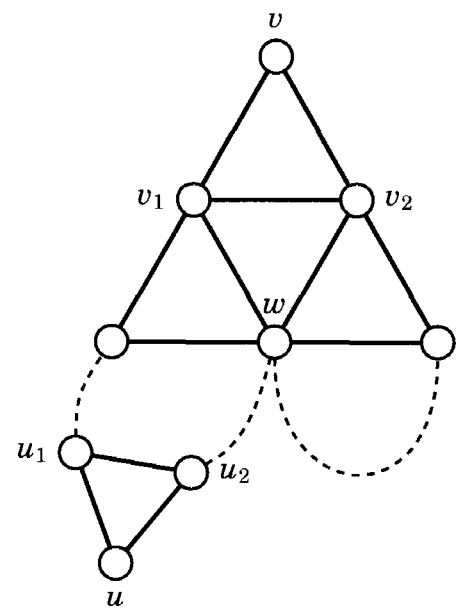


Figure 5.10 – The MOP G in the proof of Lemma 5.6

more, at most one component of $\langle \mathrm{Ct}(G) \rangle$ *is isomorphic to* K_3.

(2) *If* G *is not isomorphic to* K_3*, then every component of* $\langle \mathrm{Per}(G) \rangle$ *is isomorphic to a path.*

Proof. Lemma 5.6 shows that $\langle \mathrm{Ct}(G) \rangle$ has at most one component isomorphic to K_3. Recall that every component of $\langle \mathrm{Per}(G) \rangle$ is a component of $\langle \mathrm{Ct}(G) \rangle$. Thus, if G is not isomorphic to K_3, then it follows from Lemma 5.5 that $\langle \mathrm{Per}(G) \rangle$ does not contain a component isomorphic to K_3. Now let C be a component of $\langle \mathrm{Ct}(G) \rangle$ and suppose C is not isomorphic to K_3. To complete the proof, it remains to show that C is a path. Note that C is triangle-free by Lemma 5.5, which implies that $V(C) \neq V(G)$. Consequently, it suffices to prove that C is a subgraph of the hamiltonian cycle of G. Let $e = uv$ be an edge of C. By Theorem 5.1, we have $\{u, v\} \subseteq D_2 \cup D_3 \cup D_4'$. Observe that if $\{u, v\} \cap D_2 \neq \emptyset$, then e is an outer edge of G. If $\{u, v\} \cap D_3 \neq \emptyset$, then it follows from Lemma 5.2 (3) that e is an outer

edge of G. Thus, we consider the case that $u, v \in D_4'$. If e is not an outer edge of G, then it follows from Lemma 5.3 (3) that C is not triangle-free, which is a contradiction. We conclude that every edge of C is an outer edge of G, and hence C is a subgraph of the hamiltonian cycle of G. □

Theorem 5.3 shows there are limitations on $\langle \mathrm{Ct}(G) \rangle$ and $\langle \mathrm{Per}(G) \rangle$ when G is a MOP. If G is a connected graph with at least three vertices, then there are some basic restrictions for $\langle \mathrm{Per}(G) \rangle$ and $\langle \mathrm{Ct}(G) \rangle$, which we discuss now. Suppose $\langle \mathrm{Per}(G) \rangle$ is a path. Clearly this path must have at least two vertices. However, it cannot contain exactly two vertices since this implies that G is isomorphic to K_2. We can also observe that $\langle \mathrm{Per}(G) \rangle$ cannot be a path on three vertices. Since every component of $\langle \mathrm{Per}(G) \rangle$ is a component of $\langle \mathrm{Ct}(G) \rangle$, it follows that $\langle \mathrm{Ct}(G) \rangle$ cannot be a path on less than four vertices either. Now suppose that G is a MOP and that $\langle \mathrm{Ct}(G) \rangle$ contains a triangle as a component. Since $\mathrm{Ext}(G) \subseteq \mathrm{Ct}(G)$, it follows from Lemma 5.4 that $\langle \mathrm{Ct}(G) \rangle$ has at least two other components, which must be paths by Theorem 5.3. The next two theorems show that with respect to Theorem 5.3, these are the only restrictions for $\langle \mathrm{Per}(G) \rangle$ and $\langle \mathrm{Ct}(G) \rangle$, respectively.

Theorem 5.4. *If H is a graph from the following list:*

(1) *the complete graph K_3,*

(2) *the 3-sun,*

(3) *the paths P_n, $n \geq 4$*

(4) *the disjoint union of paths $P_{i_1} \cup P_{i_2} \cdots \cup P_{i_k}$, where $k \geq 2$ and $i_j \geq 1$,*

then there exists a MOP G such that $\langle \mathrm{Per}(G) \rangle$ is isomorphic to H.

Theorem 5.5. *If H is a graph in one of the cases of Theorem 5.4 or if H is the disjoint union $K_3 \cup P_{i_1} \cup P_{i_2} \cdots \cup P_{i_k}$, where $k \geq 2$ and $i_j \geq 1$, then there exists a MOP G such that $\langle \mathrm{Ct}(G) \rangle$ is isomorphic to H.*

Cases 1 and 2 have been already established for both theorems. Now consider the fan F_k, $k \geq 3$. It is easy to see that every vertex except the central vertex has eccentricity 2 and that the central vertex has eccentricity 1. So we have that $\langle \mathrm{Per}(G) \rangle$ and $\langle \mathrm{Ct}(G) \rangle$ are isomorphic to P_{k+1}. This establishes case 3 for both theorems. We have decided to omit the proofs of case 4 for both theorems and the extra case in Theorem 5.5 since they are technical. Instead, we provide examples of MOPs to demonstrate the main idea of the proofs. The constructions that follow all have the property that the set of contour vertices is equal to the set of peripheral vertices.

Consider case 4 and suppose H is the disjoint union $P_2 \cup P_5$. Consider the MOP G in Figure 5.11. Each vertex of G is labeled with its eccentricity and so $\mathrm{Per}(G)$ and $\mathrm{Ct}(G)$ are represented by the set of colored vertices. Therefore, $\langle \mathrm{Ct}(G) \rangle$ and $\langle \mathrm{Per}(G) \rangle$ are isomorphic to H. Note that the graph formed from G by removing the colored vertices is an even zig-zag. Even zig-zags form the core of the construction of G for case 4. Observe that vertices of the even zig-zag have the property that if two are on a line with a 60 degree angle with a horizontal line, then both vertices have the same eccentricity. Also, each set of black vertices along with their common neighbor form a fan. If H has only two components, then we can alter the size of these fans to get the desired components for $\langle \mathrm{Per}(G) \rangle$ and $\langle \mathrm{Ct}(G) \rangle$.

If H has more than two components, then we have to add to this construction. For example, suppose H is the disjoint union $P_2 \cup P_5 \cup P_3 \cup P_4$ and consider the MOP G in Figure 5.12. The set of black vertices form both $\mathrm{Per}(G)$ and $\mathrm{Ct}(G)$, and so H is isomorphic to $\langle \mathrm{Per}(G) \rangle$ and to $\langle \mathrm{Ct}(G) \rangle$. We can always

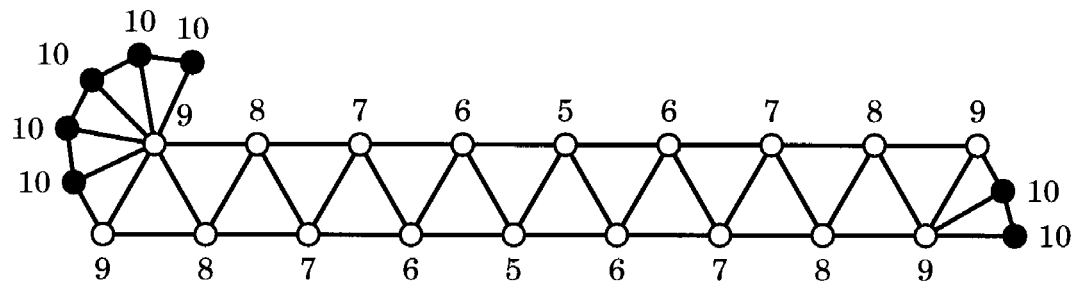


Figure 5.11 – A MOP G with $\langle \mathrm{Ct}(G) \rangle$ and $\langle \mathrm{Per}(G) \rangle$ isomorphic to $P_2 \cup P_5$.

build on the core by just using the side that is to the right of the central vertices of G (in this case, the vertices with eccentricity 5). Again, note that the noncolored vertices on a line with a 60 degree slope have the property that they have the same eccentricities. If H has a large number of components, we can modify this construction so that the core is a larger even zig-zag.

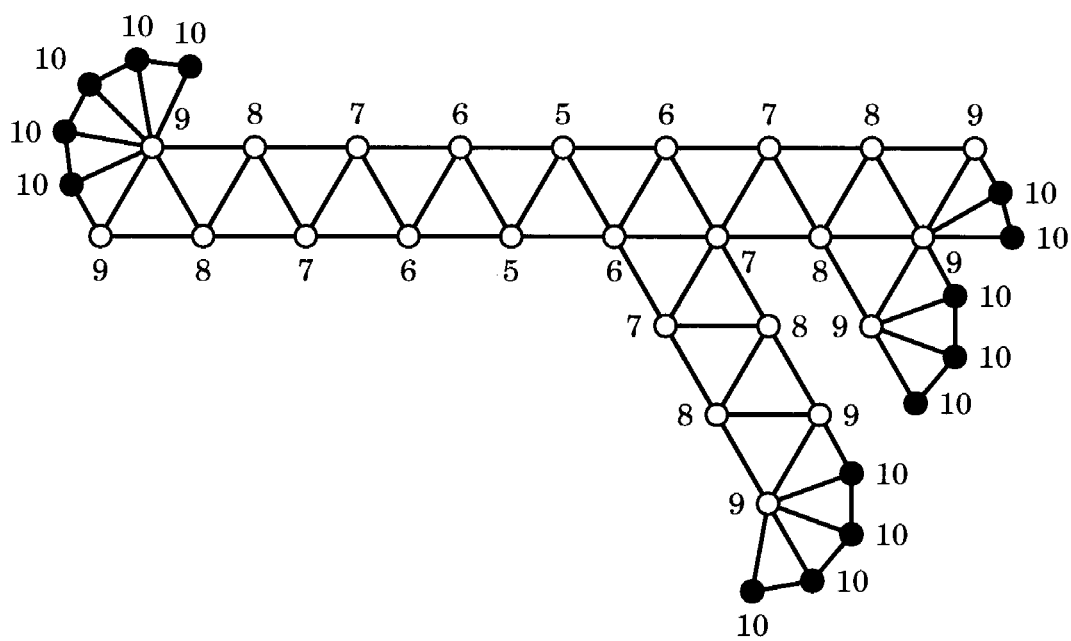


Figure 5.12 – A MOP G with $\langle \mathrm{Ct}(G) \rangle$ and $\langle \mathrm{Per}(G) \rangle$ isomorphic to $P_2 \cup P_5 \cup P_3 \cup P_4$.

Now consider the extra case in Theorem 5.5. For example, suppose H is

the disjoint union $K_3 \cup P_2 \cup P_5 \cup P_3 \cup P_4$ and let G be the MOP in Figure 5.13. The MOP G is a slight modification of the MOP in Figure 5.12. The black vertices of G form $\mathrm{Ct}(G)$ and so $\langle \mathrm{Ct}(G) \rangle$ is isomorphic to H. The core of G for this case is an odd zig-zag with an edge missing and a copy of the 3-sun attached as in Figure 5.13. As in the previous case, if H has a large number of components, then we can increase the size of the core and can use only the right side of the core to build onto.

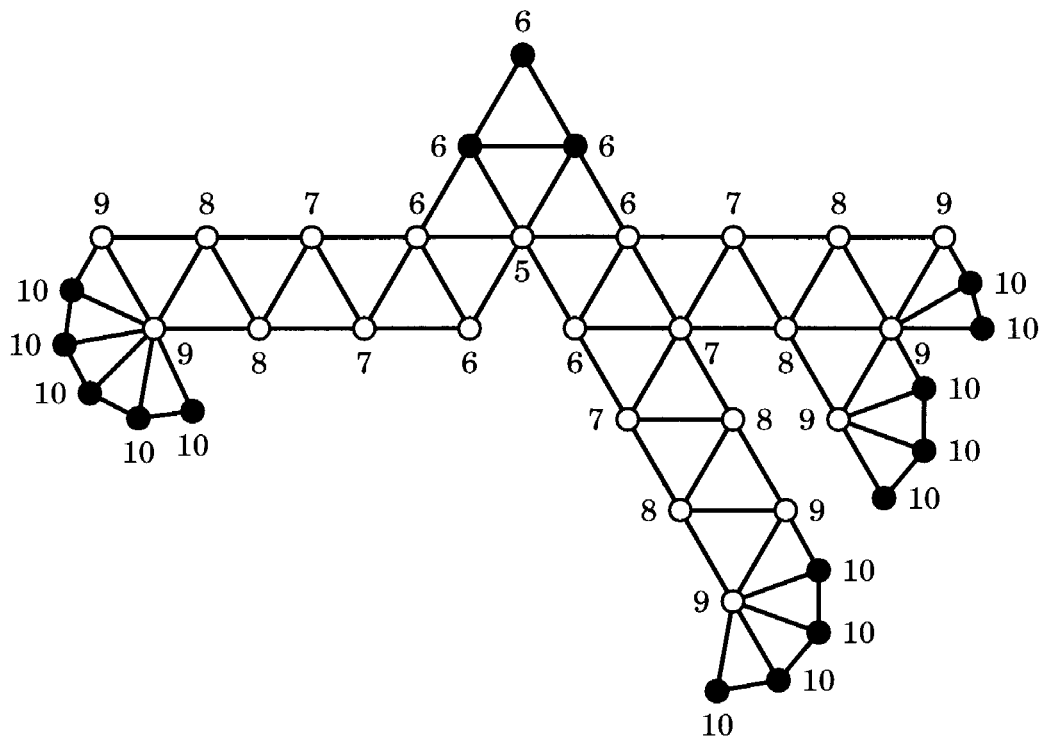


Figure 5.13 – A MOP G with $\langle \mathrm{Ct}(G) \rangle$ isomorphic to $K_3 \cup P_2 \cup P_5 \cup P_3 \cup P_4$.

Let G be a MOP. We have used the fact that every component of $\langle \mathrm{Per}(G) \rangle$ is a component of $\langle \mathrm{Ct}(G) \rangle$ to find necessary conditions on the components of $\langle \mathrm{Per}(G) \rangle$. Recall we also have that $\mathrm{Per}(G) \subseteq \mathrm{Ecc}(G)$. However, it is not the case that every component of $\langle \mathrm{Per}(G) \rangle$ is a component of $\langle \mathrm{Ecc}(G) \rangle$. For example, let G be the MOP in Figure 5.14. Each vertex of G is labeled with its eccentricity

and the colored vertex of G is the only vertex not in $\mathrm{Ecc}(G)$. The vertices of eccentricity 3 form $\mathrm{Per}(G)$. Also note that $\langle \mathrm{Ecc}(G) \rangle$ is connected and is isomorphic to neither a path nor K_3. So Theorem 5.3 cannot be extended to the components of $\langle \mathrm{Ecc}(G) \rangle$. Now observe that for every eccentric vertex v of G, there exists a vertex in D_2 that is distance at most 2 from v. We show in the last result of this section that this is true for every MOP. This property is not true for contour vertices of a MOP. For example, let G be the MOP in Figure 5.1 and consider the vertex v of G. The vertex v is in $\mathrm{Ct}(G)$ but the distance from v to the closest vertex in D_2 is 3. We remark that there exist contour vertices of MOPs for which this distance is arbitrarily large.

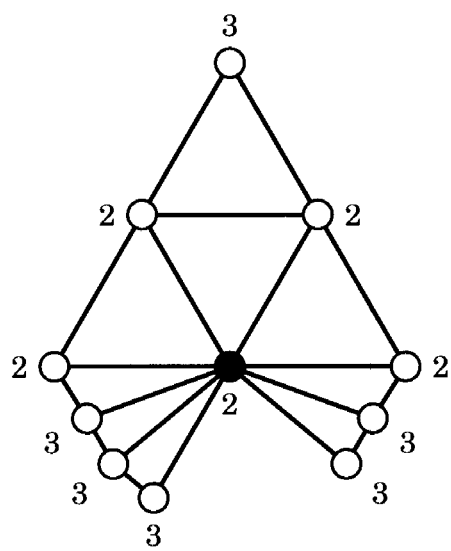


Figure 5.14 – The colored vertex is the only vertex that is not an eccentric vertex.

Proposition 5.2. *If G is a MOP and $v \in \mathrm{Ecc}(u)$ for some vertex u of G, then there exists $w \in \mathrm{Ecc}(u)$ such that $\deg(w) = 2$ and that $d(v, w) \leq 2$.*

Proof. Let $v \in \mathrm{Ecc}(G)$. By Theorem 5.1, we have that $v \in D_2 \cup D_3 \cup D_4'$. If $v \in D_2$, then the result is obvious. Now let u be a vertex in G such that $v \in \mathrm{Ecc}(u)$. Suppose first that $v \in D_3$. Let v_1, v_2, v_3 be the path induced by $N(v)$ and assume

that $N[v] \subseteq \text{Arc}[v, v_3]$. Without loss of generality, suppose $u \in \text{Arc}[v_1, v_2]$. Let $x \in Arc(v_2, v_3]$. Note that $\{v, v_2\}$ separates u and x. This implies that $d(u, x) \geq d(u, v)$. However, since $v \in \text{Ecc}(u)$, it follows that $d(u, x) = d(u, v)$. Thus, $x \in \text{Ecc}(u)$ and $v_2 \in I(u, x)$. Therefore, x is adjacent to v_2. Because there exists a vertex of degree 2 in $Arc(v_2, v_3]$, it follows that there exists $w \in \text{Ecc}(u)$ such that $\deg(w) = 2$ and that $d(v, w) \leq 2$. Suppose now that $v \in D_4'$. We assume that v_1, v_2, v_3, v_4 is the path induced by $N(v)$ and that $N[v] \subseteq \text{Arc}[v, v_4]$. If $u \in \text{Arc}[v_1, v_2]$, then, as in the proof of Lemma 5.3 (2), we have that $d(u, v) < d(u, v_4)$. This contradicts that $v \in \text{Ecc}(u)$. Similarly, we have that $u \notin \text{Arc}[v_3, v_4]$, and thus $u \in \text{Arc}(v_2, v_3)$. A similar argument to the case $v \in D_3$ shows that if $x \in \text{Arc}[v_1, v_2)$, then x is adjacent to v_2 and $x \in \text{Ecc}(u)$. Because there exists a vertex of degree 2 in $\text{Arc}[v_1, v_2)$, it follows that there exists a vertex w with the desired properties. □

5.5 Sharpness of containments of different boundary-type sets

Consider the Venn diagram in Figure 5.15. Note that the regions are labeled $1 - 8$. As mentioned in the section 5.1, each region of this Venn diagram has the property that there exists a vertex in a MOP such that this vertex belongs to this region.

For region 1, it follows from Proposition 5.2 that every MOP G has a vertex in $\text{Ext}(G) \cap \text{Per}(G)$.

For region 2, let G be the MOP in Figure 5.1. We established that the vertex v of G satisfies $v \in \partial(G)$ but $v \notin \text{Ecc}(G) \cup \text{Ct}(G)$.

For region 3, let G be the MOP in Figure 5.2. We established that the vertex v of G is a contour vertex but not an eccentric vertex of G. Since $\deg(v) \neq 2$, we also have that $v \notin \text{Ext}(G)$. Thus $v \in \text{Ct}(G) - (\text{Ecc}(G) \cup \text{Ext}(G))$.

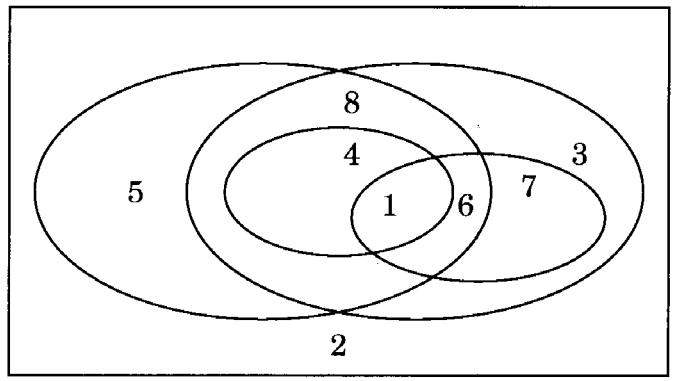


Figure 5.15 – The basic relationships of boundary-type sets along with labeled regions.

For region 4, let G be the MOP in Figure 5.11. Each of the colored vertices that are not of degree 2 are in $\mathrm{Per}(G) - \mathrm{Ext}(G)$.

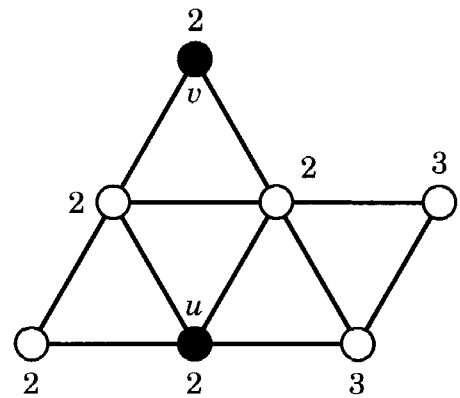
For region 5, let G be the MOP in Figure 5.14. The noncolored vertices with eccentricity 2 are in $\mathrm{Ecc}(G)$. Since each of these vertices has a neighbor with eccentricity 3, these vertices are in $\mathrm{Ecc}(G) - \mathrm{Ct}(G)$.

For the remaining three regions, we will refer the the MOPs in Figure 5.16. Each vertex of these MOPs is labeled with its eccentricity.

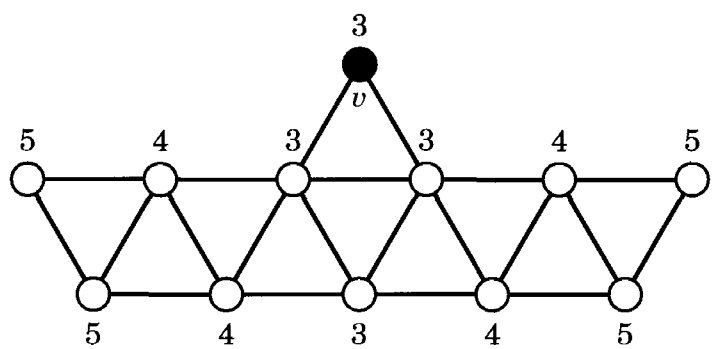
For region 6, let G be the MOP in Figure 5.16a. Note that $\mathrm{diam}\, G = 3$ and so v is not a peripheral vertex. Since $\deg(v) = 2$ and v is an eccentric vertex of u, we have that $v \in (\mathrm{Ext}(G) \cap \mathrm{Ecc}(G)) - \mathrm{Per}(G)$.

For region 7, let G be the MOP in Figure 5.16b. Note that $\deg(v) = 2$. However, the distance from v to any other vertex is less than the eccentricity of that vertex. So we have that $v \in \mathrm{Ext}(G) - \mathrm{Ecc}(G)$.

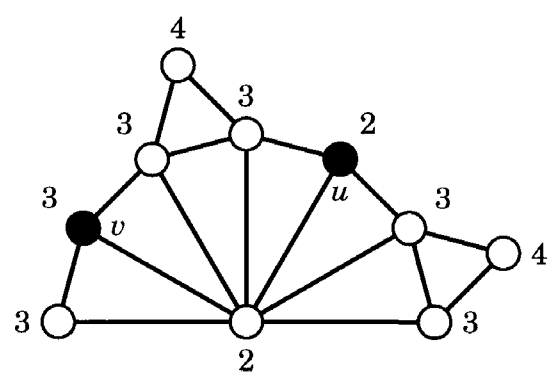
For region 8, let G be the MOP in Figure5.16c. Note that v is an eccentric vertex of u and that $v \in \mathrm{Ct}(G)$. However, $v \notin \mathrm{Per}(G)$, and so it follows that $v \in (\mathrm{Ecc}(G) \cap \mathrm{Ct}(G)) - \mathrm{Per}(G)$.



(a) The vertex v belongs to region 6.



(b) The vertex v belongs to region 7.



(c) The vertex v belongs to region 8.

Figure 5.16 – MOPs with a vertex v that belong to regions 6, 7, and 8, respectively.

CHAPTER 6
INTERVALS, GEODETIC SETS, AND STEINER SETS

Several notions of convexities for the vertex set of a graph have been studied. The term **geodesically convex** is sometimes used for what we have been referring to as simply *convex*. For vertices u and v of a graph G, recall that the geodesic interval $I(u, v)$ is the set of vertices on some u–v geodesic. Using this definition, we say that a set W of vertices of G is convex if and only if $I(u, v) \subseteq W$ for any pair $u, v \in W$. If W is a nonempty set of vertices of G, then the **geodetic closure** $I(W)$ of W is the union of all geodesic intervals $I(u, v)$ over all pairs $u, v \in W$; if $W = \emptyset$, then $I(W) = \emptyset$. The set W is a **geodetic set** if $I(W) = V(G)$. In section 6.1 we show that for any set W of vertices of a MOP, the set $I(W)$ is convex. This result helps to characterize geodetic sets of a MOP, which is done in section 6.2.

Now suppose W is a nonempty set of vertices of G. A subgraph of G with the minimum number of edges that contains all of W is necessarily a tree and is called a **Steiner W-tree**. The **Steiner interval** $S(W)$ of W consists of all vertices that lie on some Steiner W-tree. If $S(W) = V(G)$, then W is called a **Steiner set** for G. Observe that if $|W| = 2$, then $S(W) = I(W)$. Therefore, Steiner sets can be thought of as extensions of geodesic concepts and give us another way of studying the structure of graphs by means of distance. These topics relate to the topics discussed in the previous chapter. For example, Cáceres, *et al.* [4] proved that the boundary of every graph G is a geodetic set and that none of the other boundary-type sets we defined have this property. However, if G

is a chordal graph, then one of the main results in [5] proves that Ct(G) is a geodetic set of G, which implies that Ct(G) is a geodetic set for any MOP G. In section 6.2 we show that for a MOP G, every Steiner set is also a geodetic set. In section 6.3, we discuss some differences in geodetic sets and Steiner sets in MOPs.

6.1 Convexity in MOPs

Let W be a set of vertices of a graph G. We define $I^k(W)$ recursively as follows: $I^1(W) = I(W)$ and $I^k(W) = I(I^{k-1}(W))$ for $k > 1$. Since the vertex u forms the only u–u geodesic, we have $I(u, u) = \{u\}$. Hence $W \subseteq I(W)$. Observe this implies that W is convex if and only if $I(W) = W$. The **geodesic iteration number** $gin(W)$ of W is the smallest positive integer n such that $I^n(W) = I^{n+1}(W)$. Note that $I(W)$ is convex if and only if $gin(W) = 1$. The **geodesic iteration number** $gin(G)$ of G is defined as max $\{gin(W) : W \subseteq V(G)\}$. The graph G in Figure 6.1 shows that not all geodesic intervals are convex in an arbitrary graph. To see this, observe that $x, z \in I(u, v)$ and $y \in I(x, z)$. However, $y \notin I(u, v)$, which altogether shows that $I(u, v)$ is not convex. This example and the next definition come from Mulder [16].

A graph is **interval monotone** if every geodesic interval is convex. Note that the n-cube Q_n and trees are examples of interval monotone graphs. Proposition 1.1.8 of [16] shows that the bipartite graph $K_{2,3}$ plays a role in some known interval monotone graphs, which we give next.

Proposition 6A. *If a graph G does not contain a subgraph that is a subdivision of $K_{2,3}$, then G is interval monotone.*

Recall that outerplanar graphs satisfy the assumptions of Proposition 6A. Hence all MOPs are interval monotone. Now observe for a graph G that if

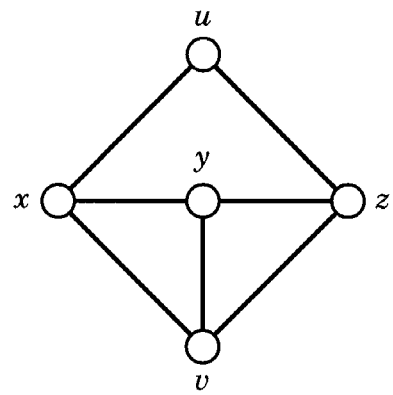


Figure 6.1 – The geodesic interval $I(u, v)$ is not convex.

$gin(G) = 1$, then G must be interval monotone. However, the converse is not true. Parvathy and Vijayakumar [17] give the following counterexample in the cube Q_3 displayed in Figure 6.2. As we mentioned above, the cube Q_3 is interval monotone. Let $W = \{a_2, b_1, d_1\}$. Then observe that $I(W) = V(G) - \{c_2\}$ and $I^2(W) = V(G)$. It follows that $gin(Q_3) = 2$. Moreover, Parvathy and Vijayakumar state that there are interval monotone graphs G with $W \subseteq V(G)$ and $|W| = 3$ such that $gin(W)$ is arbitrarily large. On the other hand, the main result in [17] shows a sufficient condition for an interval monotone graph G to satisfy $gin(G) = 1$. However, to apply this result, the graph G is required to be **geodetic**, which means that G is connected and that every pair of vertices is joined by a unique geodesic. Since MOPs with at least four vertices have $K_4 - e$ as an induced subgraph, we see that such MOPs are not geodetic. The main result of this section implies that if G is a MOP, then $gin(G) = 1$.

Let G be a MOP and let u and v be vertices of G. For convenience, we define $L(u, v) = \text{Arc}(u, v) - I(u, v)$. We intuitively think of $L(u, v)$ as the vertices in $\text{Arc}(u, v)$ that are on the left side of $I(u, v)$. See Figure 6.3 for an example of $L(u, v)$ and $L(v, u)$. Since $\text{Arc}(u, u) = \emptyset$, we have $L(u, u) = \emptyset$. These sets of vertices play an important role in the next three lemmas and the first theorem.

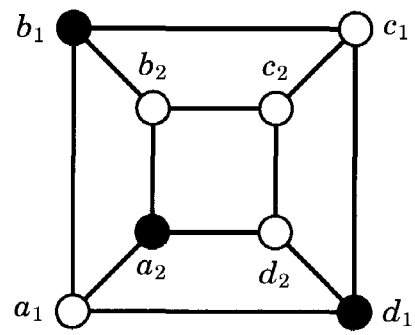


Figure 6.2 – The cube Q_3 is interval monotone but $gin(Q_3) \neq 1$.

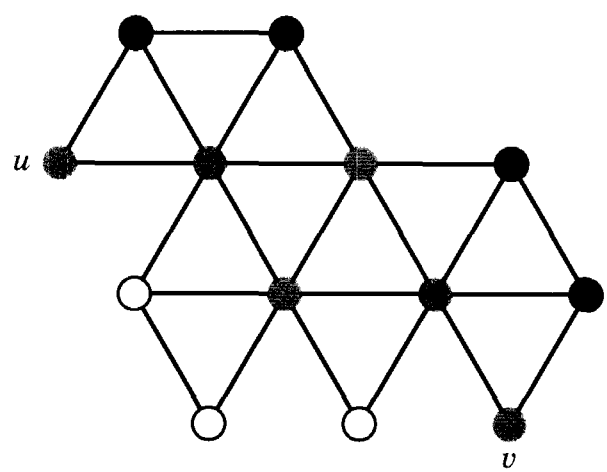


Figure 6.3 – A MOP with $L(u, v)$ represented by the set of black vertices, $I(u, v)$ represented by the set of grey vertices, and $L(v, u)$ represented by the set of white vertices.

Lemma 6.1. *Let G be a MOP and let $u, v, x, y \in V(G)$. If $x \in L(u, v)$ and $y \notin L(u, v)$, then all x–y paths contain a vertex from $I(u, v)$.*

Proof. If $y \in I(u, v)$, then the result is obviously true. So it suffices to assume that $y \notin I(u, v)$. Observe that every vertex in $\mathrm{Arc}[u, v]$ is in $L(u, v)$ or $I(u, v)$, but not both. In particular $u, v \in I(u, v)$ and by our assumptions about y, we have $y \notin \mathrm{Arc}[u, v]$. Since $x \in L(u, v)$, it follows that there exist vertices s and t in $I(u, v)$ with the property that $x \in \mathrm{Arc}(s, t) \subseteq L(u, v)$. Let P be a s–t geodesic. By Proposition 6A, G is interval monotone and so $V(P) \subseteq I(s, t) \subseteq I(u, v)$, which implies $V(P) \cap \mathrm{Arc}(s, t) = \emptyset$. Now observe that $\mathrm{Arc}[t, s] = \mathrm{Arc}[t, y] \cup \mathrm{Arc}[y, s]$. Since P is a s–t path and $y \notin V(P)$, it follows there exists an edge $s't'$ of P such that $t' \in \mathrm{Arc}[t, y)$ and $s' \in \mathrm{Arc}(y, s]$. Note that the edge $s't'$ is a chord of G. Thus, $y \in \mathrm{Arc}(t', s')$ and $x \in \mathrm{Arc}(s', t')$, which implies that $\{s', t'\}$ separates the vertices x and y. Therefore, all x–y paths contain a vertex of $I(u, v)$. □

Figure 6.4 helps to illustrate the proof of Lemma 6.1. Note that it is possible to have $s = u, t = v, s = s'$, or $t = t'$. On our way to show that $I(W)$ is convex when G is a MOP, we first show that the vertex subsets of the form $V(G) - L(u, v)$ are convex.

Lemma 6.2. *Let G be a MOP and let $u, v \in V(G)$. Then $V(G) - L(u, v)$ is convex.*

Proof. Assume, to the contrary, that $V(G) - L(u, v)$ is not convex. Then there exist vertices $x, y \notin L(u, v)$ such that $I(x, y) \cap L(u, v) \neq \emptyset$. Let P be a x–y geodesic containing the vertex w, where $w \in L(u, v)$. It follows by Lemma 6.1 that the x–w subpath of P contains a vertex, say s, from $I(u, v)$. Similarly, the w–y subpath of P contains a vertex, say t, from $I(u, v)$. Now note that the s–t subpath of P is a geodesic containing the vertex w. Therefore, $w \in I(s, t) \subseteq I(u, v)$ since G is interval monotone. This contradicts that $w \in L(u, v)$. □

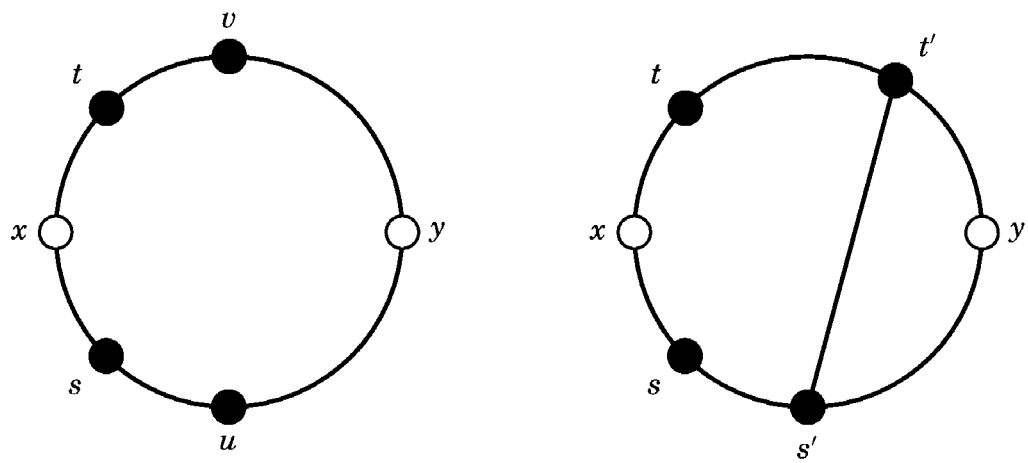


Figure 6.4 – Illustration for Lemma 6.1.

For a MOP G, let $W = \{w_0, w_1, \ldots, w_{k-1}\}$ be a set of vertices of G, $k \geq 2$. In the next lemma and theorem, we list the vertices of W in the order of traversing the hamiltonian cycle clockwise starting at some w_0 in W. This property is equivalent to having $\mathrm{Arc}(w_i, w_{i+1}) \cap W = \emptyset$ for each i, $0 \leq i \leq k-1$, where subscripts are taken modulo k. Suppose $w \in I(W)$ and that $w \in \mathrm{Arc}(w_i, w_{i+1})$ for some i. The next lemma shows that in fact $w \in I(w_i, w_{i+1})$.

Lemma 6.3. *Let G be a MOP and let $W \subseteq V(G)$ be as above. If $w \in I(W)$, then $w \in V(G) - L(w_i, w_{i+1})$ for all i.*

Proof. Let $u, v \in W$ such that $w \in I(u, v)$. Note $L(w_i, w_{i+1}) \subseteq \mathrm{Arc}(w_i, w_{i+1})$ for all i. Therefore, it suffices to assume that $w \in \mathrm{Arc}(w_i, w_{i+1})$ for some i and then show that $w \notin L(w_i, w_{i+1})$. By our assumptions about W, we have $u, v \notin \mathrm{Arc}(w_i, w_{i+1})$. Hence $u, v \in V(G) - L(w_i, w_{i+1})$. By Lemma 6.2, it follows that $w \in I(u, v) \subseteq V(G) - L(w_i, w_{i+1})$, which gives the desired result. □

Theorem 6.1. *If G is a MOP with $W \subseteq V(G)$ as in Lemma 6.3, then*

$$\bigcup_{i=0}^{k-1} I(w_i, w_{i+1}) = I(W) = I(I(W)).$$

Proof. By the definition of geodetic closure it follows that

$$\bigcup_{i=0}^{k-1} I(w_i, w_{i+1}) \subseteq I(W) \subseteq I(I(W)).$$

Therefore, it suffices to show

$$I(I(W)) \subseteq \bigcup_{i=0}^{k-1} I(w_i, w_{i+1}).$$

Let $w \in I(I(W))$. Then there exist vertices $u, v \in I(W)$ such that $w \in I(u, v)$. If $w \in W$, then $w = w_i$ for some i. So it suffices to assume that $w \notin W$, which implies that $w \in \mathrm{Arc}(w_i, w_{i+1})$ for some i. Hence $u, v \in V(G) - L(w_i, w_{i+1})$ by Lemma 6.3. It now follows from Lemma 6.2 that $w \in I(u, v) \subseteq V(G) - L(w_i, w_{i+1})$. Therefore, $w \in I(w_i, w_{i+1})$, which proves the necessary containment. □

The first part of Theorem 6.1 implies that in order to find $I(W)$, it is enough to compute k geodesic intervals, as oppose to $\binom{k}{2}$. The second equality in Theorem 6.1 shows that $I(W)$ is convex. This implies that for any MOP G, $gin(G) = 1$. We note that the proofs in this section did not require the interior regions of G to be triangles. We are only using the fact that MOPs are hamiltonian and that vertices of a chord form a separating set. So these results apply to all hamiltonian outerplanar graphs.

6.2 Geodetic Sets and Steiner Sets

In the next two sections, we discuss geodetic sets and Steiner sets in MOPs. Pelayo and Ignacio [18] have shown that there are graphs containing

Steiner sets which are not geodetic sets. Hernando, *et al.* [11] pose the problem of finding conditions for every Steiner set to be also a geodetic set. Many classes of graphs, especially chordal graphs, are considered in this paper with respect to this problem. A graph is **distance hereditary** if every chordless path is also a geodesic. A graph is an **interval graph** if the vertex set corresponds to a set of closed intervals on the real line and edges correspond to nonempty intersection of the intervals. It is shown in [11] for both of these families of graphs that every Steiner set is also a geodetic set. This result cannot be extended to all chordal graphs, so subclasses of chordal graphs are considered. The chordal distance hereditary graphs (also known as *Ptolemaic graphs*) and interval graphs both belong to the class of strongly chordal graphs, where **strongly chordal graphs** are those chordal graphs in which every cycle on six or more vertices contains a chord joining two vertices whose distance on the cycle is odd. Since the 3-sun is a forbidden induced subgraph of strongly chordal graphs, the family of MOPs is not contained in the family of strongly chordal graphs. In this section, we show that for MOPs, every Steiner set is a geodetic set. This is done finding a necessary condition for Steiner sets in MOPs that is a sufficient (and necessary) condition for geodetic sets in MOPs.

Recall that a vertex v is an extreme vertex if the graph induced by $N[v]$ is a complete graph. The following lemma is due to Hernando, *et al.* [11].

Lemma 6A. *Let G be a connected graph. If $W \subseteq V(G)$ and W is a geodetic set or a Steiner set, then* $\mathrm{Ext}(G) \subseteq W$.

Since leaves are extreme vertices, Lemma 6A implies that every geodetic set and every Steiner set in a tree must contain the set of leaves. In fact, for every tree the set of leaves is the unique minimum geodetic set and the unique minimum Steiner set. Lemma 5.1 states that if G is a MOP, then $\mathrm{Ext}(G) = D_2$.

Therefore, Lemma 6A implies that all vertices of degree 2 in a MOP are in every geodetic set and every Steiner set. Unlike trees, however, the extreme vertices do not form a geodetic set and a Steiner set in every MOP.

The next lemma will help in establishing other necessary conditions for Steiner sets. Let G be a connected graph with Steiner set W. Suppose that for some $A \subseteq V(G)$, we wish to show $A \cap W \neq \emptyset$. We use the following strategy to deal with this. Assume, to the contrary, that $W \cap A = \emptyset$. We take a Steiner W-tree T containing some subset of A. Then we form a new graph T' from T by removing some vertices from A and then adding some new edges (possibly new vertices if the new edges are incident to vertices not in T). Since we only removed some vertices from A, we have $W \subseteq V(T')$. If T' is connected and T' has less edges than T, then this contradicts that T is a Steiner W-tree. The next lemma gives a sufficient condition for T' to be connected.

Since we want to add edges in the process of forming T', we describe the notation we will use for this procedure. Let G be a graph with $H \subseteq G$ and $Y \subseteq E(G)$. Then we write $H + Y$ to indicate the graph induced by $E(H) \cup Y$. If $Y = \{e\}$ for some edge e of G, then we write $H + e$ instead. For convenience, we also use the following notation throughout the rest of the Chapter. If T' is formed from a tree T as described above, we let

$$\mathcal{C} = (V(T') - V(T)) \cup \{v \in V(T') \cap V(T) : N_T(v) \neq N_{T'}(v)\}.$$

The set $\mathcal{C}$ is essentially the set of vertices of T' whose neighborhoods have changed (with respect to T). For an example, let G be the MOP in both subfigures of Figure 6.5. Let T be the graph induced by the red edges in the MOP of Figure 6.5a and let T' be the graph induced by the red edges in the MOP of Figure 6.5b. Note that T' can be formed from T by removing the vertices v_1 and v_2 and adding the edges u_0u_1 and u_1u_2. So $T' = T - \{v_1, v_2\} + \{u_0u_1, u_1u_2\}$. Now observe that $\mathcal{C} = \{u_0, u_1, u_2\}$. One reason why u_0 is in $\mathcal{C}$ is because $u_1 \in N_{T'}(u_0)$

but $u_1 \notin N_T(u_0)$. The reason $u_1 \in \mathcal{C}$ is because u_1 is in T' but not in T.

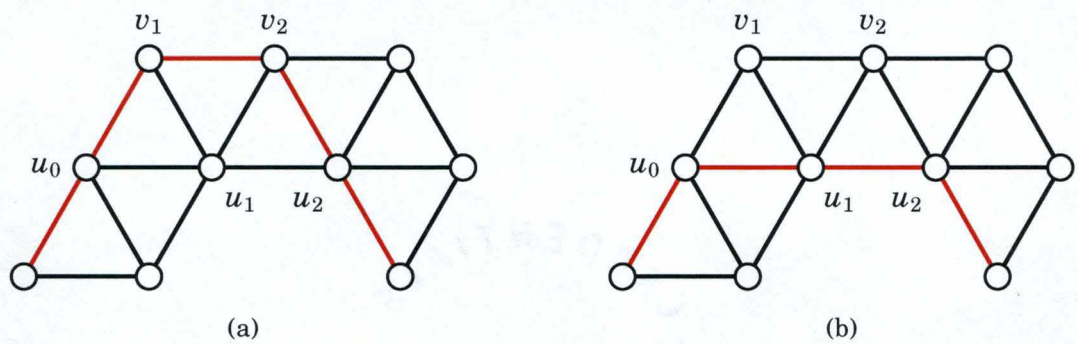


Figure 6.5 – An example of a tree T and T'.

Lemma 6.4. *Let G be a graph and let $T \subseteq G$ such that T is a tree. Let $X \subseteq V(G)$, $Y \subseteq E(G)$, and $T' = (T - X) + Y$. If $V(T) \cap \mathcal{C} \neq \emptyset$ and the vertices of $\mathcal{C}$ are connected in T', then T' is connected.*

Proof. Let $u \in V(T) \cap \mathcal{C}$ and $v \in V(T')$. We will show the vertex v is connected to u in T'. If $v \notin V(T)$, then $v \in \mathcal{C}$. In this case, the vertex v is connected to u by assumption. So we assume that $v \in V(T)$. Let $P : v = v_1, v_2, \ldots, v_k = u$ be the $v - u$ path in T. If P is a path in T', then we are done. Otherwise, let $e = v_i v_{i+1}$ where i is the smallest integer such that $e \notin E(T')$. Since $v \in V(T')$, it follows that $v_i \in V(T')$. Moreover, we have that $v_i \in \mathcal{C}$ since $v_{i+1} \in N_T(v_i)$ and $v_{i+1} \notin N_{T'}(v_i)$. By our assumptions, we have in T' that v is connected to v_i and that v_i is connected to u. Hence v is connected to u in T'. □

A **spanning tree** is a spanning subgraph that is also a tree. It is well known that every connected graph has a spanning tree. We have the following corollary of Lemma 6.4, which shows that the neighborhood in G of a vertex in a Steiner W-tree is limited if the vertex is not in W. It also extends Lemma 6A for the Steiner set case.

Corollary 6.1. *Let G be a connected graph. Let $W \subseteq V(G)$ be nonempty and let T be a Steiner W-tree. If $v \in V(T)$ and $v \notin W$, then the subgraph induced by $N_T(v)$ in G is disconnected.*

Proof. Assume, to the contrary, that $\langle N_T(v) \rangle$ is connected and let $k = |N_T(v)|$. Let F be a spanning tree of $\langle N_T(v) \rangle$. Let $T' = (T - v) + E(F)$ and observe $C = N_T(v)$. Thus, we can apply Lemma 6.4 to obtain that T' is connected. Since $|E(F)| = k - 1$, it follows that $|E(T')| < |E(T)|$. By our assumption that $v \notin W$, we also have $W \subseteq V(T')$. This contradicts that T is a Steiner W-tree. □

We have the following application of Corollary 6.1. Suppose G is a MOP and that v, W, and T satisfy the conditions of Corollary 6.1. Since the graph induced by $N(v)$ in G is a path, Corollary 6.1 implies that $N_T(v)$ cannot induce a subpath of $\langle N(v) \rangle$.

The next lemma gives a common necessary condition for geodetic sets and Steiner sets in MOP G. Recall that a neutral segment in G is a u–v path that lies on the hamiltonian cycle of G such that $u \neq v$, $\{u, v\} \subseteq D_3$, and $\mathrm{Arc}(u, v) \subseteq D_4$. Let $P : v_1, v_2, \ldots, v_{k+2}$ be a neutral segment of G with k vertices of degree four, $k \geq 0$. As we discussed in Chapter 2, it follows from Theorem 2.1 that the graph induced by $N[V(P)]$ is isomorphic to the odd zig-zag P^2_{2k+5}. we use the following notation in the next lemma (see Figure 6.6 for an example): We label the path induced by $N[V(P)] - V(P)$ with the sequence $u_0, u_1, \ldots, u_{k+2}$ such that $v_i u_{i-1}$ and $v_i u_i$ are edges in G, $1 \leq i \leq k + 2$. We also let $v_0 = u_0$.

Lemma 6.5. *Let G be a MOP and let $W \subseteq V$. If P is a neutral segment of G and W is a geodetic set or a Steiner set, then $V(P) \cap W \neq \emptyset$.*

Proof. Assume, to the contrary, that $V(P) \cap W = \emptyset$. Thus, if W is a geodetic set, then there exists a geodesic with ends in W containing the vertex v_1; if W is a

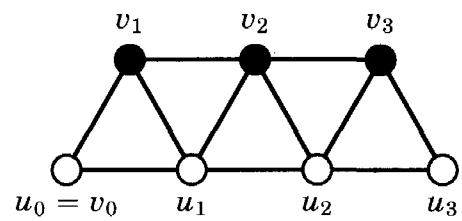


Figure 6.6 – A labeled odd zig-zag for the case $k = 1$.

Steiner set, then there exists a Steiner W-tree containing v_1. In either case, let such a geodesic or Steiner W-tree be T. First, we show that T has a particular type of subgraph, and then we produce a contradiction in both cases. We use the notation described above for the graph induced by $N[V(P)]$.

Since $v_1 \notin W$, the vertex v_1 is not a leaf in T. Observe that u_1 is adjacent in G to both the other two neighbors of v_1. From this observation and by virtue of Corollary 6.1, it follows that $N_T(v_1) = \{v_0, v_2\}$. Let s be the largest integer with $s \leq k+2$ such that $v_0, v_1, \ldots, v_s$ is a path contained in T. Since $v_s \in V(P)$, we have $v_s \notin W$ by assumption. Therefore, at least one of the edges $v_s u_{s-1}, v_s u_s$ is in T also.

Case 1: W is a geodetic set and T is a geodesic.

Suppose the edge $v_s u_s$ is in T. Then the path $v_0, v_1, \ldots, v_s, u_s$ in T is a v_0–u_s geodesic of length $s+1$. However the v_0–u_s path $v_0 = u_0, u_1, \ldots, u_s$ has length s. This is a contradiction. The case when $v_s u_{s-1}$ is in T is handled similarly.

Case 2: W is a Steiner set and T is a Steiner W-tree.

Let $T' = (T - \{v_1, v_2, \ldots, v_s\}) + \{u_0u_1, u_1u_2, \ldots, u_{s-1}u_s\}$. Observe that we have removed at least $s+1$ edges and added at most s edges to form T'. Therefore $|E(T')| < |E(T)|$. Note that $u_0 \in V(T) \cap \mathcal{C}$ and that $\mathcal{C} \subseteq \{u_0, u_1, \ldots, u_s\}$. Thus T' is connected by Lemma 6.4. Since $V(P) \cap W = \emptyset$, we have $W \subseteq V(T')$. This contradicts that T is a Steiner W-tree. □

Next consider the MOP G in Figure 6.7. Let $W = \{u, v, w\}$. First note that $D_2 \subseteq W$. There are two neutral segments in G, and w lies on both. However, the graph induced by $\{u, v, w, z\}$ is the unique Steiner W-tree of G, which shows that W is not a Steiner set. Therefore, the combination of the two necessary conditions we have for a Steiner set in a MOP is not a sufficient condition for a Steiner set in a MOP. On the other hand, this combination is a sufficient condition for geodetic sets in MOPs. We will establish some lemmas before we prove this result.

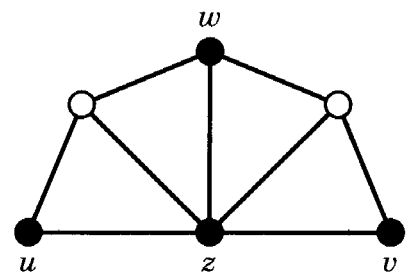


Figure 6.7 – The set $W = \{u, v, w\}$ is a geodetic set but not a Steiner set.

Lemma 6.6. *Let G be a MOP and let $e = uv$ be a chord of G. If $W \subseteq V(G)$ such that $D_2 \subseteq W$, then $\{u, v\} \cap I(W) \neq \emptyset$.*

Proof. Since e is a chord of G, there exist vertices $w_1, w_2 \in D_2$ such that $w_1 \in \mathrm{Arc}(u, v)$ and $w_2 \in \mathrm{Arc}(v, u)$. Furthermore, the set $\{u, v\}$ separates w_1 and w_2. Since $D_2 \subseteq W$, it follows that $I(w_1, w_2) \cap \{u, v\} \neq \emptyset$. Hence $I(W) \cap \{u, v\} \neq \emptyset$. $\square$

We use Theorem 6.1 in the next lemma and theorem.

Lemma 6.7. *Let G be a MOP and let $W \subseteq V(G)$ such that $D_2 \subseteq W$. If u and v are consecutive vertices on the hamiltonian cycle of G such that $u \in I(W)$ and $v \in D_4$, then $v \in I(W)$.*

Proof. Note that the graph induced by $N(v)$ is a path. Since uv is an outer edge of G, u and v have only one common neighbor. Thus the vertex u is a leaf in $\langle N(v)\rangle$, which implies that there exist neighbors x and y of v such that neither is adjacent to u and that the edge vy is an outer edge of G (see Figure 6.8). It follows that either $y \in D_2$ or that the edge xy is a chord of G. By our assumptions and Lemma 6.6, it follows there exists a vertex w in $\{x, y\} \cap I(W)$. Hence $\{u, w\} \subseteq I(W)$. Observe that path u, v, w is a u–w geodesic. Therefore, $v \in I(u, w) \subseteq I(I(W)) = I(W)$ by Theorem 6.1, and the proof is complete. □

Lemma 6.8. *Let G be a MOP and let $W \subseteq V(G)$ such that $D_2 \subseteq W$. If u and v are vertices of G such that* $\mathrm{Arc}(u, v) \subseteq D_4$ *and* $\mathrm{Arc}[u, v] \cap I(W) \neq \emptyset$, *then* $\mathrm{Arc}(u, v) \subseteq I(W)$.

Proof. It suffices to assume $\mathrm{Arc}(u, v) \neq \emptyset$. Next, without loss of generality, suppose $u \in I(W)$. Let w be the vertex in $\mathrm{Arc}(u, v)$ that is adjacent to u on the hamiltonian cycle. It follows from Lemma 6.7 that $w \in I(W)$. Thus, it suffices to assume $\mathrm{Arc}(u, v) \cap I(W) \neq \emptyset$. Similarly, we can apply Lemma 6.7 as many times as necessary to obtain that $\mathrm{Arc}(u, v) \subseteq I(W)$. □

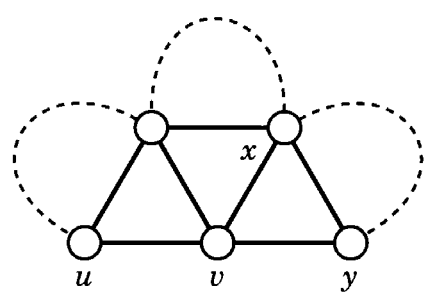


Figure 6.8 – The MOP G in the proof of Lemma 6.8.

Theorem 6.2. *Let G be a MOP and let $W \subseteq V(G)$. Then W is a geodetic set of G if and only if both the following conditions hold.*

(1) $D_2 \subseteq W$.

(2) *If P is a neutral segment of G, then $V(P) \cap W \neq \emptyset$.*

Proof. If W is a geodetic set, then Lemma 6A and Lemma 6.5 guarantee these conditions must be satisfied. Now assume both conditions (1) and (2) hold, and let v be a vertex of G. We need to show that $v \in I(W)$. The cases in the proof are based on the degree of v.

Case 1: $deg(v) = 2$.

We are assuming that $D_2 \subseteq W$. Therefore $D_2 \subseteq I(W)$.

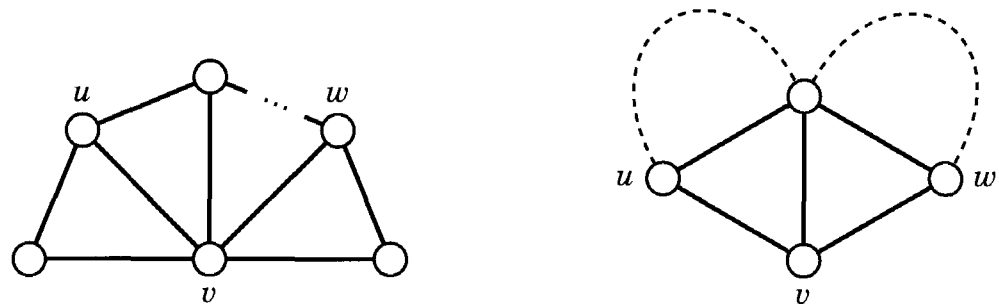
Case 2: $deg(v) \geq 5$.

Note that the graph induced by $N(v)$ is a path. Therefore, since $deg(v) \geq 5$, there exist neighbors u and w of v such that both uv and vw are chords of G and that $d(u, w) = 2$ (see Figure 6.9a). It follows from Lemma 6.6 that both $\{u, v\}$ and $\{w, v\}$ have a nonempty intersection with $I(W)$. Thus, it suffices to assume $\{u, w\} \subseteq I(W)$. Now observe the path u, v, w is a u–w geodesic. Therefore, $v \in I(u, w) \subseteq I(I(W)) = I(W)$ by Theorem 6.1.

Case 3: $deg(v) = 4$.

Let u and w be vertices of G such that $v \in \mathrm{Arc}(u, w) \subseteq D_4$ and none of u and w is in D_4. If, without loss of generality, u is not of degree 3, then by the above analysis we have $u \in I(W)$. In this case, Lemma 6.8 implies $v \in I(W)$. So we may assume that both u and w have degree 3. Hence the path on the hamiltonian cycle with vertex set $\mathrm{Arc}[u, w]$ is a neutral segment in G. Condition (2) guarantees that $\mathrm{Arc}[u, w] \cap W \neq \emptyset$, which implies that $\mathrm{Arc}[u, w] \cap I(W) \neq \emptyset$. It follows from Lemma 6.8 that $v \in I(W)$.

Case 4: $deg(v) = 3$.



(a) $deg(v) \geq 5$.

(b) $deg(v) = 3$.

Figure 6.9 – Illustrations of the graph induced by $N[v]$ in the proof of Theorem 6.2.

Let u and w be the two neighbors of v on the hamiltonian cycle (see Figure 6.9b). Observe that the path u, v, w is a u–w geodesic, which implies $v \in I(u, w)$. If both u and w are in $I(W)$, then $v \in I(u, w) \subseteq I(W)$ by Theorem 6.1. So we may assume, without loss of generality, that $u \notin I(W)$, and hence $u \notin W$. By the above analysis, it must be the case that u is of degree 3. Thus the edge uv forms a neutral segment of G. Since $u \notin W$, condition (2) guarantees that $v \in W$. Hence $v \in I(W)$ as desired. □

To see an example of an application of Theorem 6.2, consider the MOP G in Figure 6.10, where each vertex is labeled with its degree. Note that G has two neutral segments, namely the paths a, b, c and c, d, e. Since the vertex c is in both neutral segments, it follows from Theorem 6.2 that $W = D_2 \cup \{c\}$ is the unique minimum geodetic set of G. Hence $g(G) = 4$.

Now observe that the combination of the necessary conditions we have for Steiner sets in MOPs is a sufficient condition for geodetic sets in MOPs. As a consequence, we have the following theorem.

Theorem 6.3. *Let G be a MOP and let $W \subseteq V(G)$. If W is a Steiner set, then W*

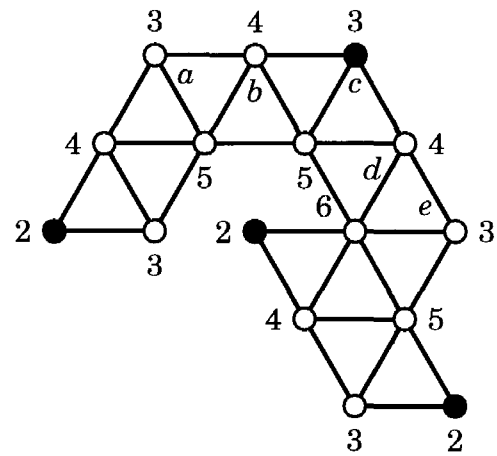


Figure 6.10 – The set of colored vertices is the unique minimum geodetic set.

is a geodetic set.

For a graph G, the **geodetic number** $g(G)$ is the minimum cardinality of a geodetic set. Douthat and Kong [8] have shown that finding the geodetic number of a chordal graph is NP-complete. The problem of finding a hamiltonian cycle in a graph is known to be NP-complete. However Vo [20] gives a simple linear time algorithm to find the unique hamiltonian cycle in an outerplanar graph. Therefore, in conjunction with Theorem 6.2, one can obtain a linear time algorithm to find the geodetic number of a MOP.

6.3 Some differences in Geodetic Sets and Steiner Sets

In this section, we discuss some differences in geodetic sets and Steiner sets. We do not have a characterization of Steiner sets for MOPs. However, the first theorem in this section shows that for some MOPs, every Steiner set must contain some set of vertices of degree four. Theorem 6.2 guarantees that for every MOP, there exists a geodetic set containing no vertices of degree 4. The **Steiner number** $\mathrm{st}(G)$ of G is defined as the minimum cardinality of a Steiner

set of G. The other main reults of this section focus on geodetic numbers and Steiner numbers in MOPs. Since every Steiner set of a MOP is a geodetic set, we have $g(G) \leq \mathrm{st}(G)$ for every MOP G. The last result shows that $\mathrm{st}(G) - g(G)$ can be arbitrarily large for the family of MOPs.

Given a graph and a Steiner W-tree for some set of vertices W, the next lemma shows a way to construct another Steiner W-tree.

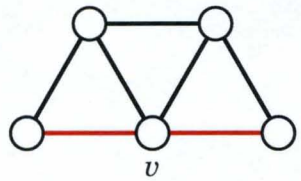
Lemma 6.9. *Let G be a graph, $W \subseteq V(G)$ be nonempty, and T be a Steiner W-tree. Suppose $uv \in E(G) - E(T)$. If P is a u–v path in T and e is any edge of P, then the graph $T' = (T - e) + uv$ is also a Steiner W-tree.*

Proof. Clearly T and T' have the same size. Observe that the graph $T + uv$ contains a cycle with the edge e. It follows that T' is connected and that $V(T) = V(T')$. Therefore, T' must be a Steiner W-tree. □

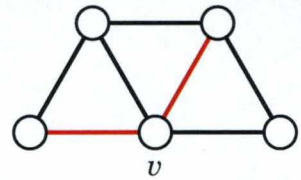
One consequence of Lemma 6.9 is that for any triangle in a graph, if two edges are in a Steiner tree, then either can be traded for the third edge to form another Steiner tree.

Next suppose G is a MOP and that W and T satisfy the conditions of Lemma 6.9. Let v be a vertex of G of degree 4. If $v \in V(T) - W$, then, as a consequence of Corollary 6.1, the graph induced by $N_T(v)$ is disconnected. Up to symmetry, Figure 6.11 displays the three possibilities for the edges in $\langle N_T(v) \rangle$ by coloring them red. For convenience, we refer to v as a vertex of type $2a$, type $2b$, or type 3, as indicated Figure 6.11.

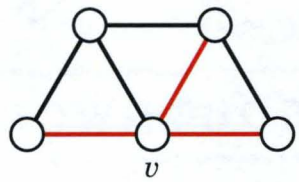
Let H be the MOP in Figure 6.12. Since H has no vertices of degree 3, the MOP H has no neutral segments. It follows from Theorem 6.2 that $D_2(H) = D_2$ is the unique minimum geodetic set. It can be shown that the Steiner interval $S(D_2) = V(H) - \{v\}$, which implies that D_2 is not a Steiner set for H. The next theorem shows that if a MOP G contains a subgraph isomorphic to H such



(a) The vertex v is of type $2a$.



(b) The vertex v is of type $2b$.



(c) The vertex v is of type 3.

Figure 6.11 – Up to symmetry, the three possibilities for the graph induced by $N_T[v]$.

that the vertices corresponding to u, v, w have degree 4, then every Steiner set for G must include at least one of these vertices. On the other hand, we can construct minimum geodetic sets in every MOP without any vertices of degree 4 by choosing all vertices of degree 2 and exactly one vertex of degree 3 on every neutral segment.

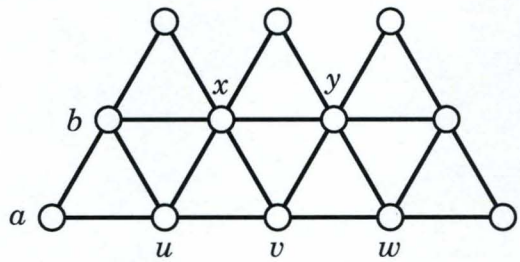


Figure 6.12 – Every Steiner set of H must contain a vertex from the set $\{u, v, w\}$.

Theorem 6.4. *Let G be a MOP containing a subgraph isomorphic to the MOP H in Figure 6.12 such that the vertices corresponding to u, v, w have degree 4 in G. If W is a Steiner set for G, then $W \cap \{u, v, w\} \neq \emptyset$.*

Proof. Assume, to the contrary, that W is a Steiner set for G and that $W \cap \{u, v, w\} = \emptyset$. We will use the labels of vertices in the MOP H in Figure 6.12 for the subgraph of G that is assumed to be isomorphic to H. Let T be a Steiner W-tree of G containing the vertex v. First we show that is suffices to assume that v is of type $2b$ in T.

Suppose v is of type $2a$ in T. We show there exists a Steiner W-tree such that v is of type 3. Observe that the set $\mathrm{Arc}(x, y)$ must contain a vertex of degree 2 in G, which must be in W by Lemma 6A. Furthermore, the set $\{x, y\}$ separates v and every vertex in $\mathrm{Arc}(x, y)$. We assume, without loss of generality, that the vertex y is in T. Therefore, T contains a v–y path. Note $vy \in E(G) - E(T)$ since v is of type $2a$. Let e be any edge of the v–y path not incident to v. It follows from Lemma 6.9 that G contains a Steiner W-tree such that v is of type 3.

Now suppose v is of type 3. Without loss of generality, we assume $N_T(v) = \{u, y, w\}$ (see Figure 6.11). Consider the triangle induced by $\{v, y, w\}$. It follows from Lemma 6.9 that G contains a Steiner W-tree such that v is of type $2b$.

Since a Steiner W-tree containing v exists, we have proven a Steiner W-tree exists such that v is of type $2b$. So we assume that v is of type $2b$ in T. Without loss of generality, we assume $N_T(v) = \{u, y\}$. Next we consider the possibilities for $N_T(u)$. Consider the case when u is of type $2a$. By the same argument used above, at least one of the vertices b, x must be in T. Suppose b is in T, which implies there exists a u–b path in T. Let e be the edge of this path incident to b, and observe e is not incident to u or v. By Lemma 6.9, we can trade the edge e for the edge ub to form another Steiner W-tree such that u is of type 3 and v is of type $2b$. The case when x is in T is handled similarly. Thus, it suffices to assume that v is of type $2b$ and either u is of type $2b$ or type 3. We produce a contradiction in both cases.

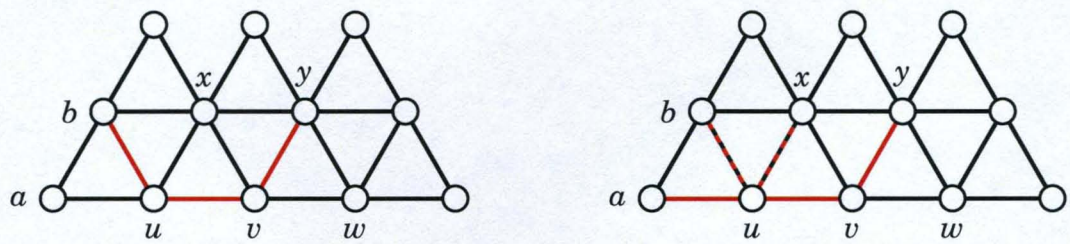
Case 1: v is of type $2b$ and u is of type $2b$.

Then we must have $N_T(u) = \{b, v\}$. Let $T' = (T - u - v) + bx + xy$. Observe $\mathcal{C} = \{b, x, y\}$ (see Figure 6.13).

Case 2: v is of type $2b$ and u is of type 3.

There are two possibilities for $N_T(u)$. Either way we let $T' = (T - u - v) + ab + bx + xy$. Observe $\mathcal{C} = \{a, b, x, y\}$ (see Figure 6.13)

In both cases, we clearly have $|E(T')| < |E(T)|$. Since $u, v \notin W$, we have $W \subseteq V(T')$. It follows from Lemma 6.4 that T' is connected, which contradicts that T is a Steiner W-tree. □



(a) Case 1. (b) Case 2.

Figure 6.13 – Case 1 and Case 2 of Theorem 6.4

To motivate the conditions of the Theorem 6.4, consider the two MOPs in Figure 6.14. The MOP G_1 has three consecutive vertices of degree 4 on the hamiltonian cycle, but two of them are not in D_4'. The MOP G_2 has two consecutive vertices of degree 4 and they are both in D_4'. It can be shown that for each of these MOPs, the set of vertices of degree 2 is a Steiner set. Therefore, the vertices of degree 4 are not required in every Steiner set.

The rest of this section focuses on geodetic numbers and Steiner numbers. Next we show sharp bounds for the geodetic number and Steiner number of a MOP based on the order of the MOP. We clearly have $g(G) \geq 2$ and $\mathrm{st}(G) \geq 2$ for any nontrivial connected graph G. Since the even zig-zags have no neutral segments, Theorem 6.2 shows that the even zig-zags have geodetic number 2 and Steiner number 2. Thus, the even zig-zags form an infinite family of MOPs

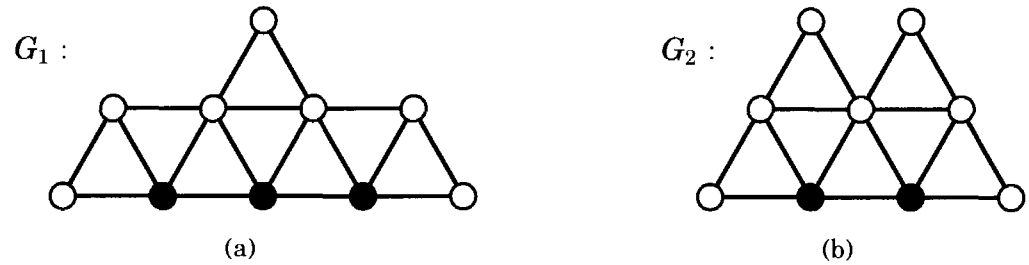


Figure 6.14 – D_2 forms a Steiner set, so no vertices of degree 4 are required.

meeting these bounds. Now we give the upper bound for the geodetic number of a MOP.

Theorem 6.5. *If G is a MOP of order n, $n \geq 4$, then $g(G) \leq \lceil \frac{n}{2} \rceil$. Furthermore, the fan F_k for $k \geq 2$ achieves this bound.*

Proof. Suppose $k \geq 2$. First we show that F_k achieves the bound. The case for $k = 2$ is easy, so we assume $k \geq 3$. Note $n = k + 2$ and that F_k has $k - 1$ vertices of degree 3. It follows from Theorem 6.2 that we can form a minimum geodetic set of F_k by taking all vertices of degree 2 and $\lfloor \frac{k-1}{2} \rfloor$ vertices of degree 3. Hence $g(F_k) = 2 + \lfloor \frac{n-3}{2} \rfloor = \lceil \frac{n}{2} \rceil$.

Now let G be a MOP that is not a fan. We show that $g(G) \leq \frac{n}{2}$. Let $u, v \in D_2$ such that $\mathrm{Arc}(u, v) \cap D_2 = \emptyset$. We claim there exists a vertex of degree at least 4 in $\mathrm{Arc}(u, v)$. Suppose this is not the case. Then observe $\mathrm{Arc}(u, v) \subseteq D_3$. Let P be the path on the hamiltonian cycle of G with vertex set $\mathrm{Arc}[u, v]$ and let $H = \langle N[V(P)] \rangle$. By Theorem 2.1, it follows that H is a fan. Since u and v have degree 2 in G, it follows that $G = H$. Thus, G is a fan, which is a contradiction. Therefore, we have that the number of vertices of G with degree 2 is at most the number of vertices of G with degree at least 4, that is $|D_2| \leq |\bigcup_{i \geq 4} D_i|$. Note that Theorem 6.2 guarantees that there exists a minimum geodetic set of G containing no vertex of degree 4 or higher and at most half of the vertices of degree 3. This implies that $g(G) \leq |D_2| + |\frac{D_3}{2}|$. Because $n = |D_2| + |D_3| +$

$\left|\bigcup_{i\geq 4} D_i\right|$, we obtain $g(G) \leq \frac{n}{2}$. $\square$

Recall that $\kappa(G)$ denotes the connectivity of a graph G. Chartrand and Zhang [7] showed that if G is a connected noncomplete graph or order n, then $\mathrm{st}(G) \leq n - \kappa(G)$. Let G be a MOP of order at least 4. Since $\kappa(G) = 2$, we have $\mathrm{st}(G) \leq n - 2$. This can also be proven directly since for every chord uv of G, the set $W = V(G) - \{u, v\}$ is a Steiner set for G. The next theorem shows that the Steiner number of the fan F_k for $k \geq 2$ meets this upper bound.

Theorem 6.6. *If G is a MOP of order n, $n \geq 4$, then* $\mathrm{st}(G) \leq n-2$. *Furthermore, the fan F_k ($k \geq 2$) achieves this bound.*

Proof. Note $\mathrm{st}(F_k) \leq n - 2$ by the comment above. We show the fan F_k achieves the bound. The case for $k = 2$ is easy, so we assume $k \geq 3$. Let w be the central vertex of G and let u and v be the vertices of degree two in G. Suppose there exist a Steiner set W such that $|W| \leq n - 3$. We can assume that $w \notin W$, for otherwise the graph $\langle W \rangle$ is connected, which implies that $S(W) = W$. This contradicts that W is a Steiner set. Also, by Lemma 6A, we have $\{u, v\} \subseteq W$. Hence there are at least two vertices of degree 3 not in W. Let x and y be two such vertices. Since wx is a chord of G, it follows that $\langle W \rangle$ is disconnected. However, the graph $\langle W \cup \{w\} \rangle$ is connected. Therefore, the size of a Steiner W-tree is $|W|$. Now suppose that T is a Steiner W-tree of G containing the vertex x. Then it must be the case that $V(T) = W \cup \{x\}$, and hence $w, y \notin V(T)$. However, the edge wy is a chord of G, which implies that T is not connected, a contradiction. Thus, if W is a Steiner set, then $|W| \geq n - 2$, and this completes the proof. $\square$

A connected graph G is **near self-centered** if $\mathrm{diam}\, G = \mathrm{rad}\, G + 1$. Observe that every fan has radius 1 and diameter 2. Since MOPs are 2-connected, the fans are 2-connected, near self-centered graphs. Chartrand and Zhang [7]

showed for any integers a, b such that $3 \leq a \leq b$, there exists a 2-connected, near self-centered graph G with $g(G) = a$ and $\mathrm{st}(G) = b$. We don't know if a similar result holds for MOPs, but the next result shows the difference between $\mathrm{st}(G)$ and $g(G)$ for a MOP G can be arbitrarily large.

Corollary 6.2. *The set* $\{\mathrm{st}(G) - g(G)\colon G \text{ is a MOP}\}$ *is the set of nonnegative integers.*

Proof. Let G be a MOP. By Theorem 6.3, every Steiner set for G is a geodetic set. Thus $\mathrm{st}(G) - g(G) \geq 0$. For $k \geq 2$, consider the fan F_k and let $n = k + 2$ be the order of F_k. By Theorem 6.5 and Theorem 6.6, we have $\mathrm{st}(F_k) - g(F_k) = n - 2 - \lceil \frac{n}{2} \rceil = \lfloor \frac{n}{2} \rfloor - 2$. Let m be a nonnegative integer and let $k = 2m + 2$. Then $n = 2m + 4$. Therefore, $\mathrm{st}(F_k) - g(F_k) = \lfloor \frac{2m+4}{2} \rfloor - 2 = m + 2 - 2 = m$, and the proof is complete. $\square$

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