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FENICHEL'S THEOREMS
WITH APPLICATIONS IN DYNAMICAL SYSTEMS

By

Jeremy Warren Riley
B.S., Missouri Western State University, 2010

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Master of Arts

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FENICHEL'S THEOREMS WITH APPLICATIONS IN DYNAMICAL SYSTEMS

Submitted by

Jeremy Warren Riley

A Thesis Approved on

June 20th, 2012

(Date)

by the Following Reading and Examination Committee:

Dr. Changbing Hu, Thesis Director

Dr. Jiaxu Li

Dr. Ming Yu

DEDICATION

This paper is dedicated to my son

Cooper Warren Riley

With hard work and perserverence

you can accomplish anything you put your mind to

ACKNOWLEDGEMENTS

I would like to thank Dr. Changbing Hu for all of his time and assistance in the preparation of this paper. Thanks also goes to my committee members Dr. Jiaxu Li and Dr. Ming Yu for their time and consideration. I would also like to thank Dr. Thomas Riedel for his support, advice and friendship the past two years, I truly appreciate it. Also, thanks to all my fellow GTA's for their friendship and assistance, especially Chad Money, Erika Foreman, Charlie Suer, Lucas Hoots and Heather Hunt.

And finally, I cannot possibly express how much I thank my wife Melanie, without whom I would not be half the person I am today. Melanie, I love you.

ABSTRACT

FENICHEL'S THEOREMS
WITH APPLICATIONS IN DYNAMICAL SYSTEMS

Jeremy Warren Riley

May 12, 2012

Three main theorems due to Fenichel are fundamental tools in the exploration of geometric singular perturbation theory. This expository paper attempts to provide an introduction to the concepts stated in Fenichel's theorems and provide illustrative examples. The goal is to provide enough insight to gain a basic understanding of the usefullness of these theorems.

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INTRODUCTION

1.1 Introduction

There are many physical applications which give rise to mathematical models in the form of a system of ordinary differential equations. Some of these systems involve several processes which evolve on different time scales. The resulting equations have a specific structure which, through the application of the following theorems, can be readily understood. When studying such systems, simplifying assumptions may be of great help; if not to understand the full system, then at least to get a first insight into the system's behavior [6].

This expository paper focuses on some of the geometric constructs and theory for systems of differential equations of the form:

$$\begin{cases} x' = f(x, y, \varepsilon) \\ y' = \varepsilon g(x, y, \varepsilon) \end{cases} \tag{1.1}$$

where $' = \frac{d}{dt}$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^l$ and $\varepsilon \in \mathbb{R}$. The functions f and g are assumed to be C^∞ functions of x, y, and ε in $U \times I$, where U is an open subset of $\mathbb{R}^n \times \mathbb{R}^l$ and I is an open interval containing $\varepsilon = 0$.

We shall attempt to compile various hypothesis about the system (1.1), denoted with the letter H, and introduce several key concepts that will prove to be invaluable in the study of systems such as (1.1).

(H1) The functions f and g are assumed to be C^∞ on a set $U \times I$ where $U \subset \mathbb{R}^N$ is open, with $N = n + l$, and I is an open interval containing 0.

Note here we assume full smoothness on the nonlinear terms which is unnecessary but greatly simplifies the discussion. If less smoothness is present in a given problem the precise smoothness required can be easily retraced through the proofs [3].

System (1.1) can be rewritten with a change in time scale as:

$$\begin{cases} \varepsilon \dot{x} = f(x, y, \varepsilon) \\ \dot{y} = g(x, y, \varepsilon) \end{cases} \tag{1.2}$$

where $\cdot = \frac{d}{d\tau}$ and $\tau = \varepsilon t$. We refer to the time scale given by τ as slow, whereas the time scale for t is fast. Further, as long as $\varepsilon \neq 0$, the two systems are equivalent and are referred to as *singular perturbation* when $0 < \varepsilon \ll 1$. Hence, we refer to (1.1) as the *fast system* and (1.2) as the *slow system*. Each of the systems (1.1) and (1.2) has a naturally associated limit as $\varepsilon \to 0$. In (1.1) letting $\varepsilon \to 0$ we obtain the system

$$\begin{cases} x' = f(x, y, 0) \\ y' = 0 \end{cases} \tag{1.3}$$

According to (1.3) the variable x will vary while y will remain constant. Thus x is called the fast variable. If we let $\varepsilon \to 0$ in (1.2), the limit only makes sense if $f(x, y, 0) = 0$ [3], and is given by

$$\begin{cases} 0 = f(x, y, 0) \\ \dot{y} = g(x, y, 0) \end{cases} \tag{1.4}$$

One thinks of the condition $f(x, y, 0) = 0$ as determining a set on which the flow is given by $\dot{y} = g(x, y, 0)$. It is natural to attempt to solve x in terms of y from the equation $f(x, y, 0) = 0$ and plug it into the second equation of (1.4). Notice that this set is exactly the set of critical points for (1.3). Hence, we have created a

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"formal" picture that (1.3) has sets of critical points and that (1.4) "blows up" the flow on this set up to produce non-trivial behavior [3].

In either limiting case, one pays a price. On the set $f(x, y, 0) = 0$ the flow is trivial for (1.3). Whereas under (1.4) the flow is non-trivial on this set, but the flow is not defined off this set. The primary goal of geometric singular perturbation theory is to realize both these aspects (i.e., fast and slow) simultaneously. This seemingly contradictory aim will be accomplished within the phase space of (1.1) for $0 < \varepsilon \ll 1$.[3]

1.2   Fundamental Concepts

Sets of points that have special properties relative to an ordinary differential equation are important for studying the system dynamics. We will discuss several such sets in this section including fixed points, periodic orbits, invariant sets and steady state solutions. More importantly, we will discuss the stability of such sets and their respective impact on the dynamics of a given system. To better illustrate each type of set, we conclude the introduction with relevant examples discussing each.

The simplest such sets one can encounter are fixed points. For the equation $y' = h(y)$, where $y \in \mathbb{R}^k$, a *fixed point* is a point y at which the function h vanishes, that is a point y such that $h(y) = 0$. These types of points are extremely important in the fact that they represent equilibrium states of the system that is being modeled. The linear system $\dot{x} = Ax$ with matrix $A = Df(x_0)$ is called *the linearization of* (1.1) *at* x_0. If $x_0 = 0$ is an equilibrium point of (1.1), then $f(0) = 0$ and by Taylor's Theorem,

$$f(x) = Df(0)x + \tfrac{1}{2}D^2 f(0)(x, x) + \dots$$

It follows that the linear function $Df(0)x$ is a good first approximation to the

nonlinear function $f(x)$ near $x = 0$. It is reasonable to expect that the behavior of the nonlinear system (1.1) near the point $x = 0$ will be approximated by the behavior of its linearization at $x = 0$. Further, one can see that even if $x_0 \neq 0$, one can perform an elementary shift to bring x_0 to this position and the previous statements will still apply. A fixed point, or equilibrium point of (1.1), is called a *hyperbolic equilibrium point* if none of the eigenvalues of the matrix $Df(x_0)$ have zero real part.

A *steady state solution* of a given system is a solution which remains constant independent of time. That is, the first derivative remains zero for all $t \in \mathbb{R}$. These types of sets are better understood in the examples at the end of this section.

One may also be interested in studying sets of points that remain invariant relative to the governing system of equations. Here a set V is said to be an *invariant set* of the equation $y' = h(y)$ if $y(t_0) \in V$ for some $t_0 \in \mathbb{R}$ implies that $y(t) \in V$ for all $t \in \mathbb{R}$. The simplest types of invariant sets include fixed points and periodic orbits. We say an open set V is said to be *locally invariant* with respect to an open set W under the system $y' = h(y)$ if V is a subset of W and if any trajectory leaving V simultaneaously leaves W.

The invariant sets have a particularly important quality, namely they are manifolds. Let $\mathbb{R}^p$ denote the p-dimensional Euclidean space. A set of points in $\mathbb{R}^p$ is said to be *a smooth manifold of dimension* q, where $q \leq p$, if each point in the set has a neighborhood that is locally C^∞ diffeomorphic to an open subset of $\mathbb{R}^q$, as defined in Kaper[11].

Perko [18] defines a periodic orbit of the system (1.1) as any closed solution curve of (1.1) which is not an equilibrium point of (1.1). Here it is useful to describe the stability of a given set. A periodic orbit Λ is called *stable* if for each $\varepsilon > 0$ there is a neighborhood U of Λ such that for all $x \in U$, $d(\Lambda_x^+, \Lambda) < \varepsilon$; i.e., if for all $x \in U$ and $t \geq 0$, $d(\phi(t, x), \Lambda) < \varepsilon$. A periodic orbit Λ is called *asymptotically stable* if it

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is stable and if for all points x in some neighborhood U of Λ

$$lim_{t \to \infty} d(\phi(t, x), \Lambda) = 0$$

Note that stability can also be applied to fixed points as well and is easily determined by the linearization of such a point, and the subsequent analysis of the eigenvalues of the Jacobian matrix. Further, as we will see in the following sections, stability is a very important property of manifolds as well.

Cycles of the system (1.1) correspond to periodic solutions of (1.1) since $\phi(\cdot, x_0)$ defines a closed solution curve of (1.1) if and only if for all $t \in \mathbb{R}, \phi(t + T, x_0) = \phi(t, x_0)$ for some $T > 0$. The minimal T for which this equality holds is called the period of the periodic orbit $\phi(\cdot, x_0)$. We demonstrate the notion of a periodic orbit in the following examples:

1.3 Examples

1.3.1 Periodic Orbits

Consider the following system:

$$\begin{cases} \dot{x} = -y + x(r^4 - 3r^2 + 1) \\[2mm] \dot{y} = x + y(r^4 - 3r^2 + 1) \end{cases} \tag{1.5}$$

where $r^2 = x^2 + y^2$. Here we convert the system to polar coordinates:

$$r\dot{r} = x\dot{x} + y\dot{y} = -xy + x^2(r^4 - 3r^2 + 1) + xy + y^2(r^4 - 3r^2 + 1)$$

which yields:

$$\dot{r} = (r^4 - 3r^2 + 1)r \tag{1.6}$$

From simple substitution we see that if $r = 1 \Rightarrow \dot{r} = -1 < 0$ and if $r = 2 \Rightarrow \dot{r} = 10 > 0$. We can also see that the only equilibrium point of the system

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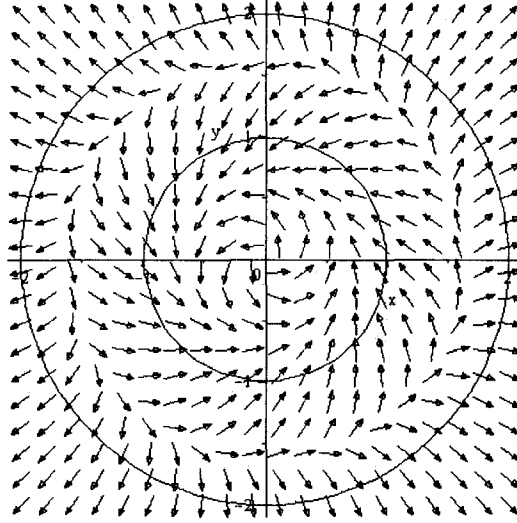
Figure 1.1 – By creating the annular region $A_1 = \{x \in \mathbb{R}^2 | 1 < |x| < 2\}$ and over-laying onto the direction field one can easily verify the hypothesis of the Poincare' Bendixson Theorem holds for this system

lies at the origin, hence there exists no equilibium point in the annular region $A_1 = \{x \in \mathbb{R}^2 | 1 < |x| < 2\}$. Further, if we pick any point x_0 in the annulus then Γ^- trajectory stays in the annulus. These facts satisfy the hypotheses of the Poincare'-Bendixson Theorem stated here:

THEOREM 1.1. *Suppose that $f \in C^1(E)$ where E is an open subset of $\mathbb{R}^2$ and that (1.1) has a trajectory Γ with Γ^+ contained in a compact subset F of E. Then if $\omega(\Gamma)$ contains no critical point of (1.1), $\omega(\Gamma)$ is a periodic orbit of (1.1)*

We continue our analysis of (1.5) by observing the following

$$n^2 - 3n + 1 = 0$$

$$\Rightarrow n = \frac{3 \pm \sqrt{5}}{2}$$

$$\Rightarrow r = \sqrt{n} = \sqrt{\frac{3 \pm \sqrt{5}}{2}}$$

From here we observe

$$0 < \sqrt{\tfrac{3-\sqrt{5}}{2}} < 1 < \sqrt{\tfrac{3+\sqrt{5}}{2}} < 2$$

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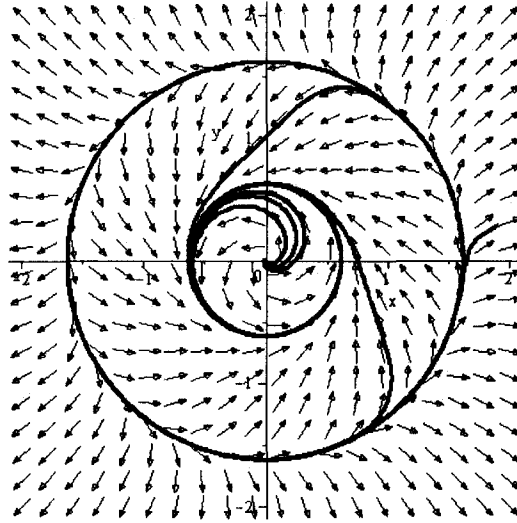
Figure 1.2 – Graphical representation of the system (1.5) with several trajectories. Here we can easily see the manifolds and their given stabilities

$$\Rightarrow \dot{r} = 0 \text{ when } r = \sqrt{\frac{3 \pm \sqrt{5}}{2}}$$

Therefore when $0 < r < \sqrt{\frac{3-\sqrt{5}}{2}} \Rightarrow \dot{r} > 0$, which implies the origin is an unstable focus. Hence, the origin is not part of an ω-limit set for any point in the closure of $A_2 = \{x \in \mathbb{R}^2 | 0 < |x| < 1\}$. We have therefore satisfied the hypothesis of the Poincare'-Bendixson Theorem, and can conclude the existence of a periodic orbit in the region A_2. In this system we have two unstable manifolds; the orgin, which is an unstable focus, and an unstable circular periodic orbit centered at the origin with radius $r = \sqrt{\frac{3+\sqrt{5}}{2}}$. Finally, there is a stable circular periodic orbit centered at the origin with radius $r = \sqrt{\frac{3-\sqrt{5}}{2}}$. Figure 1.2 graphically depicts our situation; one can easily see the instability of the origin and the outer periodic orbit, that is any initial trajectory within a small neighborhood of these manifolds eventually leaves such a neighborhood.A similar observation can be made of the inner periodic orbit's stability, that is all initial trajectories in a neighborhood of this manifold stay within said neighborhood.

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1.3.2 Fixed Points and Steady State Solutions

Consider the following simple linear system:

$$\begin{cases} x' & = -x + 2y \\ y' & = \varepsilon(x - y) \end{cases} \tag{1.7}$$

which has eigenvalues $\lambda_1 = \frac{\sqrt{(1+6\varepsilon+\varepsilon^2)}-1-\varepsilon}{2}$ and $\lambda_1 = \frac{-(\sqrt{(1+6\varepsilon+\varepsilon^2)}+1+\varepsilon)}{2}$ this yield the corresponding eigenvectors:

$$v_1 = \begin{vmatrix} \frac{2}{\frac{1}{2}-\frac{1}{2}\varepsilon+\frac{1}{2}\sqrt{1+6\varepsilon+\varepsilon^2}} \\ 1 \end{vmatrix}.$$

,

$$v_2 = \begin{vmatrix} \frac{2}{\frac{1}{2}-\frac{1}{2}\varepsilon-\frac{1}{2}\sqrt{1+6\varepsilon+\varepsilon^2}} \\ 1 \end{vmatrix}.$$

This allows us to diagonalize the system by means of a linear transformation.

$$P = \begin{vmatrix} \frac{2}{\frac{1}{2}-\frac{1}{2}\varepsilon+\frac{1}{2}\sqrt{1+6\varepsilon+\varepsilon^2}} & \frac{2}{\frac{1}{2}-\frac{1}{2}\varepsilon-\frac{1}{2}\sqrt{1+6\varepsilon+\varepsilon^2}} \\ 1 & 1 \end{vmatrix}.$$

,

$$P^{-1} = \begin{vmatrix} \frac{1}{8}\frac{(1-\varepsilon+\sqrt{1+6\varepsilon+\varepsilon^2})(-1+\varepsilon+\sqrt{1+6\varepsilon+\varepsilon^2})}{\sqrt{1+6\varepsilon+\varepsilon^2}} & \frac{1}{2}\frac{1-\varepsilon+\sqrt{1+6\varepsilon+\varepsilon^2}}{\sqrt{1+6\varepsilon+\varepsilon^2}} \\ \frac{-1}{8}\frac{(1-\varepsilon+\sqrt{1+6\varepsilon+\varepsilon^2})(-1+\varepsilon+\sqrt{1+6\varepsilon+\varepsilon^2})}{\sqrt{1+6\varepsilon+\varepsilon^2}} & \frac{1}{2}\frac{-1+\varepsilon+\sqrt{1+6\varepsilon+\varepsilon^2}}{\sqrt{1+6\varepsilon+\varepsilon^2}} \end{vmatrix}.$$

From here we may compute $B = P^{-1}AP$ where B has the form

$$B = \begin{vmatrix} \lambda & 0 \\ 0 & \mu \end{vmatrix}.$$

with

$$\lambda = \frac{2(-\frac{1}{8}\frac{(1-\varepsilon+\sqrt{1+6\varepsilon+\varepsilon^2})(-1+\varepsilon+\sqrt{1+6\varepsilon+\varepsilon^2})}{\sqrt{1+6\varepsilon+\varepsilon^2}} + \frac{1}{2}\frac{(1-\varepsilon+\sqrt{1+6\varepsilon+\varepsilon^2})\varepsilon}{\sqrt{1+6\varepsilon+\varepsilon^2}})}{\frac{1}{2}-\frac{1}{2}\varepsilon+\frac{1}{2}\sqrt{1+6\varepsilon+\varepsilon^2}}$$
$$+ \frac{1}{4}\frac{(1-\varepsilon+\sqrt{1+6\varepsilon+\varepsilon^2})(-1+\varepsilon+\sqrt{1+6\varepsilon+\varepsilon^2})}{\sqrt{1+6\varepsilon+\varepsilon^2}}$$
$$- \frac{1}{2}\frac{(1-\varepsilon+\sqrt{1+6\varepsilon+\varepsilon^2})\varepsilon}{\sqrt{1+6\varepsilon+\varepsilon^2}}$$

and

$$\mu = \cfrac{2\left(\frac{1}{8}\frac{(1-\varepsilon+\sqrt{1+6\varepsilon+\varepsilon^2})(-1+\varepsilon+\sqrt{1+6\varepsilon+\varepsilon^2})}{\sqrt{1+6\varepsilon+\varepsilon^2}} + \frac{1}{2}\frac{(-1+\varepsilon+\sqrt{1+6\varepsilon+\varepsilon^2})\varepsilon}{\sqrt{1+6\varepsilon+\varepsilon^2}}\right)}{\frac{1}{2}-\frac{1}{2}\varepsilon-\frac{1}{2}\sqrt{1+6\varepsilon+\varepsilon^2}}$$
$$-\frac{1}{4}\frac{(1-\varepsilon+\sqrt{1+6\varepsilon+\varepsilon^2})(-1+\varepsilon+\sqrt{1+6\varepsilon+\varepsilon^2})}{\sqrt{1+6\varepsilon+\varepsilon^2}}$$
$$-\frac{1}{2}\frac{(-1+\varepsilon+\sqrt{1+6\varepsilon+\varepsilon^2})\varepsilon}{\sqrt{1+6\varepsilon+\varepsilon^2}}$$

Further it can be shown that $\mu \leq 0 \leq \lambda$ for all $0 < \varepsilon < 1$. From these facts we can verify that the origin is a saddle point for this system. In the $\varepsilon = 0$ limit the nullcline $\{(x,y)|f(x,y,0)=0\}$ illustrated in Figure (1.3) consists of the parts

$$\mathcal{M}_0^0 = (0,0)$$

$$\mathcal{M}_0^1 = \{(x,y)|y = \frac{1}{2}x\}$$

One can see that $\mathcal{M}_0^0$, as a saddle point, is by definition unstable. Now through phase plane analysis we see that the steady state $\mathcal{M}_0^1 = \{(x,y)|y = \frac{1}{2}x\}$ is stable, that is, any point within a neighborhood of $\mathcal{M}_0^1$ is attracted to $\mathcal{M}_0^1$ as $t \to \infty$. The notion is made clear in Figure(1.4).
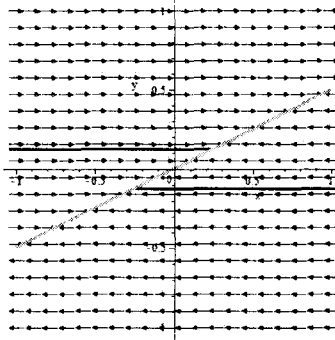
Figure 1.3 – In the $\varepsilon = 0$ limit the nullcline $\{(x,y)|f(x,y,0) = 0\}$ consists of the parts $\mathcal{M}_0^0 = (0,0)$ a saddle point, and $\mathcal{M}_0^1 = \{(x,y)|y = \frac{1}{2}x\}$ the steady state solution.
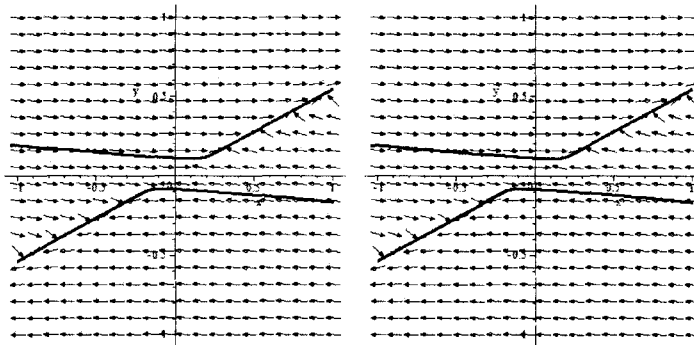
Figure 1.4 – Graphical representation of the system (1.7) with initial conditions $(x(0) = 0.5, y(0) = -0.125)$ and $(x(0) = -0.5, y(0) = 0.125)$. The system on the left represents $\varepsilon = 0.1$ while the system on the right represents $\varepsilon = 0.01$. Note that under perturbation the system maintains the trajectories and stability of the manifolds $\mathcal{M}_\varepsilon^1$ and $\mathcal{M}_\varepsilon^0$ respectively.

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CHAPTER 2
FENICHEL'S FIRST THEOREM

Having given a strong introduction of the various fundamental concepts and terminology one studies in order to appropriately analyze the lower dimensional systems introduced in the first chapter, we are now able to begin our discussion of Fenichel's theories for the general system (1.1). We will begin by looking at Fenichel's first theorem for compact manifolds with boundary.

The set of critical points $f(x, y, 0) = 0$ for (1.3) is formed by solving n equations in $\mathbb{R}^N$, where $N = n + l$ and thus is expected to be, at least locally, an l-dimensional manifold. It is natural to expect it to have a parametrization by the variable y. Thus, we shall assume that we are given an l-dimensional manifold, possibly with boundary, $\mathcal{M}_0$ which is contained in the set $\{f(x, y, 0) = 0\}$. Fenichel's first theorem asserts the existence of a manifold that is a perturbation of $\mathcal{M}_0$. It will be connected with the flow of (1.1) when $\varepsilon \neq 0$ [3]. We shall use the notation $x \cdot t$ to denote the application of the flow after time t to an initial condition x, and we say a set M is locally invariant under the flow of (1.1) if it has a neighborhood V so that no trajectory can leave M without also leaving V. That is, a set M is locally invariant if for all $x \in M$, $x \cdot [0, t] \subset V$ implies that $x \cdot [0, t] \subset M$ [3].

We make another hypothesis concerning (1.1) (in addition to (H1) mentioned earlier) before stating Fenichel's first theorem.

(H2) The set $\mathcal{M}_0$ is a compact manifold, possibly with boundary, and is normally hyperbolic relative to (1.3).

The set $\mathcal{M}_0$ will be referred to as the critical manifold, and we are now in a position to state the first theorem proved by Fenichel, under the hypotheses (H1) and (H2).

THEOREM 2.1. *FENICHEL'S FIRST THEOREM [4]*

If $\varepsilon > 0$, but sufficiently small, there exists a manifold $\mathcal{M}_\varepsilon$ that lies within $\mathcal{O}(\varepsilon)$ of $\mathcal{M}_0$ and is diffeomorphic to $\mathcal{M}_0$. Moreover it is locally invariant under the flow of (1.1) and C^r, including in ε, for any $r < +\infty$.

The manifold $\mathcal{M}_\varepsilon$ will be referred to as the slow manifold. It should be noted that the only association to the flow is through the statement that the perturbed manifold $\mathcal{M}_\varepsilon$ is locally invariant [3]. This seems to be a weak statement, but in fact is not, as it entails that we can restrict the flow to this manifold, which is lower dimensional, in order to find interesting structures [3]. The fact that the manifold is *locally* invariant as opposed to *invariant* is due to the possible presence of the boundary and the resulting possibility that trajectories may fall out of $\mathcal{M}_\varepsilon$ by escaping through the boundary [3].

One may simplify our notation by restricting our attention to the case that $\mathcal{M}_0$ is given as the graph of a function of x in terms of y. We follow Jones' [3] description concerning this restriction in the following manner. We assume there is a function $h^0(y)$, defined for $y \in K$, with K being a compact domain in $\mathbb{R}^l$ so that

$$\mathcal{M}_0 = \{(x, y) | x = h^0(y)\}.$$

This is a natural assumption as it can always be satisfied for $\mathcal{M}_0$ *locally*. In fact, on account of normal hyperbolicity mention in hypotheses 2 (H2) the matrix

$$D_x f(\hat{x}, \hat{y}, 0)$$

is invertible for any $(\hat{x}, \hat{y}) \in \mathcal{M}_0$ and hence x can locally be solved for y by the Implicit Function Theorem. We are thus assuming that such a solution can be made *globally* over $\mathcal{M}_0$.

Thus consider $x = h^0(y)$ wherein $y \in K$ and make the following hypothesis.

(H3) The set $\mathcal{M}_0$ is given as the graph of the C^∞ function $h^0(y)$ for $y \in K$. The set K is a compact, simply connected domain whose boundary is an $(l-1)$-dimensional C^∞ submanifold.

Under the hypotheses (H1)-(H3), Jones [3] restates Fenichel's first theorem in terms of the graph of a function.

THEOREM 2.2. *If $\varepsilon > 0$ is sufficiently small, there is a function $x = h^\varepsilon(y)$, defined on K, so that the graph*

$$\mathcal{M}_0 = \{(x,y) | x = h^\varepsilon(y)\}$$

is locally invariant under (1.1). Moreover, h^ε is C^r, for any $r < +\infty$, jointly in y and ε.

An equation on $\mathcal{M}_\varepsilon$ can easily be calculated using Theorem 2.1. We substitute the function $h^\varepsilon(y)$ into (1.1) and see that the y equation will decouple from that of the x equation. Hence, we obtain an equation for the variation of the variable y. Since y parametrizes the manifold $\mathcal{M}_\varepsilon$, this equation will suffice to describe the flow on $\mathcal{M}_\varepsilon$. It is given in Jones [3] by

$$y' = \varepsilon g(h^\varepsilon(y), y, \varepsilon). \tag{2.1}$$

In the alternative slow scaling we can recast (2.1) as

$$\dot{y} = g(h^\varepsilon(y), y, \varepsilon), \tag{2.2}$$

Which has the advantage that a limit exists as $\varepsilon \to 0$, given by

$$\dot{y} = g(h^0(y), y, 0), \tag{2.3}$$

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which naturally describes a flow on the critical manifold $\mathcal{M}_0$, and is exactly the second equation (1.2). Using this theorem and this resulting equation (2.2), the problem of studying (1.1), at least on $\mathcal{M}_\varepsilon$, is reduced to a *regular singular pertur-bation problem* [3].

2.1   An example from enzyme kinetics

We illustrate the use of Fenichel's first theorem with an example from enzyme kinetics. Consider the following basic enzymatic reactions proposed by Michaelis and Menten [9] involving a substrate, or molecule, S reacting with an enzyme E to form a complex SE which in turn is converted into a product P. Then, schematically we have

$$S + E_{k_{-1}}^{k_1} \to SE, \qquad SE \to^{k_2} P + E.$$

Let

$$s = [S], \ e = [E], \ c = [SE], \ p = [P]$$

Where [ ] denotes concentration. The law of mass action states that the rate of a reaction is proportional to the product of the concentration of the reactants. Hence, we have the system of nonlinear differential equations

$$\begin{cases} \frac{ds}{dt} = -k_1 e s + k_{-1} c, & \frac{de}{dt} = -k_1 e s + (k_{-1} + k_2)c, \\ \frac{dc}{dt} = k_1 e s - (k_{-1} + k_2)c, & \frac{dp}{dt} = k_2 c, \\ s(0) = s_0, & e(0) = e_0, \quad c(0) = c_0, \quad p(0) = p_0. \end{cases} \qquad (2.4)$$

From (2.4), we have

$$\frac{de}{dt} + \frac{dc}{dt} = 0 \quad or \quad e(t) + c(t) \equiv 0 \qquad (2.5)$$

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by (2.5) we have the following

$$
\begin{cases}
\frac{ds}{dt} = -k_1 e_0 s + (k_1 s + k_{-1})c, \\[2mm]
\frac{dc}{dt} = k_1 e_0 s - (k_1 s + k_{-1} + k_2)c, \\[2mm]
s(0) = s_0, \qquad c(0) = 0
\end{cases}
\tag{2.6}
$$

With the nondimensionalization

$$
\begin{cases}
\tau = k_1 e_0 t, \quad u(\tau) = \frac{s(t)}{s_0}, \quad v(\tau) = \frac{c(t)}{e_0} \\[2mm]
\lambda = \frac{k_2}{k_1 s_0}, \quad K = \frac{k_{-1}+k_2}{k_1 s_0}, \quad \varepsilon = \frac{e_0}{s_0}
\end{cases}
\tag{2.7}
$$

the system (2.6) becomes

$$
\begin{cases}
\frac{du}{d\tau} = -u + (u + K - \lambda)v \\[2mm]
\varepsilon \frac{dv}{d\tau} = u - (u + K)v \\[2mm]
u(0) = 1, \quad v(0) = 0
\end{cases}
\tag{2.8}
$$

where $0 < \varepsilon \ll 1$ and from (2.7), $K > \lambda$. Here $v(\tau)$ changes rapidly in dimensionless time $\tau = O(\epsilon)$. After that $v(\tau)$ is essentially in a steady state, or $\epsilon \frac{dv}{d\tau} \approx 0$, i.e., the v-reaction is so fast it is more or less in equilibrium at all times. This is Michaelis and Menten's pseudo-steady state hypothesis [9]. To conclude we use the method of matched asymptotic expansions to determine the manifolds for the Michaelis-Menten kinetics model.

Going back to (2.8) we have the following

$$
\begin{cases}
\frac{dx}{dt} = f(x,y) = -x + (x + K - \lambda)y \\[2mm]
\varepsilon \frac{dy}{dt} = g(x,y) = x - (x + K)y \\[2mm]
K > 0, \quad \lambda > 0
\end{cases}
\tag{2.9}
$$

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2.1.1 Outer solution

We will look for a strait forward expansion of the form

$$
\begin{cases}
x(t, \varepsilon) = x_0(t) + \varepsilon x_1(t) + O(\varepsilon^2) \\
y(t, \varepsilon) = y_0(t) + \varepsilon y_1(t) + O(\varepsilon^2)
\end{cases}
\tag{2.10}
$$

Using this expansion in (2.8), and equating the leading order terms of the order ε^0, we find

$$
\begin{cases}
\frac{dx_0}{dt} = -x_0 + (x_0 + K - \lambda)y_0 \\
0 = x_0 - (x_0 + K)y_0
\end{cases}
\tag{2.11}
$$

From the second equation

$$
y_0 = \frac{x_0}{x_0 + K}
$$

This concentration y_0 corresponds to a quasi-equilibrium for the substrate concentration x_0, in which the creation of the complex by the binding of the enzyme with the substrate is balanced by the destruction of the complex by the reverse reaction and the decomposition of the complex into the product and enzyme.

Substituting this result into the first equation, we get a first order ODE for $x_0(t)$:

$$
\frac{dx_0}{dt} = \frac{\lambda x_0}{x_0 + K}
$$

The solution of this equation is given by

$$
x_0(t) = K \log x_0(t) = C - \lambda t
\tag{2.12}
$$

Where C is a constant of integration. This solution is valid near $t = 0$ because no choice of C can satisfy the initial conditions x_0 and y_0.

2.1.2 Inner solution

There is a short initial layer, for time $t = O(\varepsilon)$, in which x, y adjust from their initial values that are compatible with the outer solution found above.

We introduce the inner variables

$$\tau = \tfrac{t}{\varepsilon}, \quad X(\tau, \varepsilon) = x(t, \varepsilon), \quad Y(\tau, \varepsilon) = y(t, \varepsilon)$$

The inner equations are

$$\begin{cases} \frac{dX}{d\tau} = \varepsilon[-x + (x + K - \lambda)y] \\[2mm] \frac{dY}{d\tau} = x - (x + K)y \\[2mm] X(0, \varepsilon) = 1, \quad Y(0, \varepsilon) = 0 \end{cases} \tag{2.13}$$

We look for an inner expansion

$$X(\tau, \varepsilon) = X_0(\tau) + \varepsilon X_1(\tau) + O(\varepsilon^2)$$
$$Y(\tau, \varepsilon) = Y_0(\tau) + \varepsilon Y_1(\tau) + O(\varepsilon^2)$$

The leading order inner equations are

$$\begin{cases} \frac{dX_0}{d\tau} = 0 \\[2mm] \frac{dY_0}{d\tau} = X_0 - (X_0 - K)Y_0 \\[2mm] X_0(0) = 1, \quad Y_0(0) = 0 \end{cases} \tag{2.14}$$

The solution is

$$X_0 = 1$$
$$Y_0 = \frac{1}{1+K}(1 - e^{-(1+K)\tau})$$
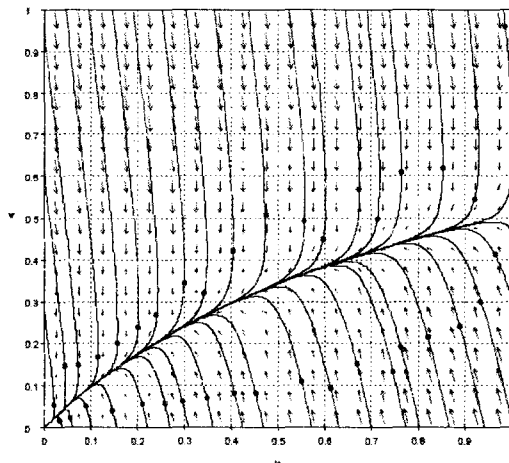
Figure 2.1 – Phase portrait of the system (2.8) with $\varepsilon = 0.2$, $K = 1.0$, and $\lambda = 0.5$. One can see the formation of the manifold $\mathcal{M}_\varepsilon$ via the trajectories plotted.

2.1.3 Matching

We assume that the inner and outer expansions are both valid for intermediate times of the order $\varepsilon \ll t \ll 1$. We require that the expansions agree asymptotically in this regime, where $\tau \to \infty$ and $t \to 0$ as $\varepsilon \to 0$. Hence, the matching condition is

$$\lim_{\tau \to \infty} X_0(\tau) = \lim_{t \to 0^+} x_0(t),$$

$$\lim_{\tau \to \infty} Y_0(\tau) = \lim_{t \to 0^+} y_0(t)$$

This condition implies that

$$x_0(0) = 1, \quad y_0(0) = \tfrac{1}{1+K},$$

which is satisfied when $C = 1$ in the outer solution. Therefore,

$$x_0(t) + K \log x_0(t) = 1 - \lambda t$$

The slow manifold for the enzyme system is the curve

$$y = \tfrac{x}{x+k}.$$

This is precisely the manifold guaranteed by Fenichel's theorem. Trajectories rapidly approach the slow manifold in the initial layer. They then move more slowly along the slow manifold and approach the equilibrium $x = y = 0$ as $t \to \infty$. The inner layer corresponds to the small amount of enzyme "loading up" on the substrate. The slow manifold corresponds to the enzyme working at full capicity in converting substrate into product.

CHAPTER 3
FENICHEL'S SECOND THEOREM

Fenichel's first theorem is a bit limited in its use as it only gives us an understanding of the dynamics of the system (1.1) on a very local level for small $\varepsilon > 0$. Through satisfaction of the given hypothesis the theorem guarantees the existence of the slow manifold and gives an approximation for the flow on this slow manifold. If, however, one's goal is a more global understanding of the system (1.1), in particular addressing the interaction between the slow manifold and the surrounding phase space, then we require a slightly more strict theorem to apply [6]. In general, the interaction described takes place via the stable and unstable manifolds. These are precisely the objects of concern in Fenichel's second theorem. Consider equation (1.1). Suppose that for $\varepsilon = 0$ the normally hyperbolic critical manifolds $\mathcal{M}_0 \subset \{f(u, v, 0) = 0\}$ has an $l+m$ dimensional stable manifold $W^s(\mathcal{M}_0)$, with $m + n = k$. That is, suppose the Jacobian $\frac{\partial f}{\partial u}(u, v, 0)|\mathcal{M}_0$ has m eigenvalues λ with $Re(\lambda) < 0$ and n eigenvalues with $Re(\lambda) > 0$. Then the following theorem holds:

THEOREM 3.1. *FENICHEL'S SECOND THEOREM [4]*

Suppose $\mathcal{M}_0 \subset \{f(u, v, 0) = 0\}$ is compact, possibly with boundary, and normally hyperbolic, and suppose f and g are smooth. Then for $\varepsilon > 0$ and sufficiently small, there exist manifolds $W^s(\mathcal{M}_\varepsilon)$ and $W^u(\mathcal{M}_\varepsilon)$, that are $\mathcal{O}(\varepsilon)$ close and diffeomorphic to $W^s(\mathcal{M}_0)$ and $W^u(\mathcal{M}_0)$, respectively, and that are locally invariant under the flow of (1.1)

The manifolds $W^s(\mathcal{M}_\varepsilon)$ and $W^u(\mathcal{M}_\varepsilon)$ are still "stable" and "unstable" manifolds as suggested, but in a slightly different sense. Now the manifold $\mathcal{M}_\varepsilon$ is no longer simply a set of fixed points. Instead, solutions in $W^s(\mathcal{M}_\varepsilon)$ decay to $\mathcal{M}_\varepsilon$ at an exponential rate in forward time, and likewise solutions in $W^u(\mathcal{M}_\varepsilon)$ decay to $\mathcal{M}_\varepsilon$ at an exponential rate in backward time [6]. It is important to note that local invariance implies that the solutions only decay to $\mathcal{M}_\varepsilon$ as long as they stay in a neighborhood of the compact, possibly bounded $\mathcal{M}_\varepsilon$.

The manifolds $W^s(\mathcal{M}_\varepsilon)$ and $W^u(\mathcal{M}_\varepsilon)$ have respective dimensions $l + m$ and $l + n$, so that one can conclusively say that the stability properties of $\mathcal{M}_0$ are inherited by $\mathcal{M}_\varepsilon$. When $mn > 0$, the conclusion of Fenichel's first theorem can be concluded from this one by taking the intersections of $W^s(\mathcal{M}_\varepsilon)$ and $W^u(\mathcal{M}_\varepsilon)$ [6].

3.1 A Simple Linear Example in Three Dimensions

Consider the following example:

$$
\begin{cases}
x' = x \\
y' = -y \\
z' = \varepsilon z
\end{cases}
\tag{3.1}
$$

Let ϕ_t, which maps the points $(x(t_0), y(t_0), z(t_0))$ to their images $(x(t_0 + t), y(t_0 + t), z(t_0 + t))$ after time t, denote the flow of (3.1).We will begin with the fast system (when $\varepsilon = 0$), and conclude with the geometry of the full system.

3.1.1 When $\varepsilon = 0$

When $\varepsilon = 0$, the z-axis is an invariant manifold of (3.1), it consists entirely of fixed points. Also, this fast subsystem is $f_x(x, y, z, 0) = x$ and $f_y(x, y, z, 0) = -y$. Thus, for each z the Jacobian of f at the equilibrium point $(x = 0, y = 0)$ has
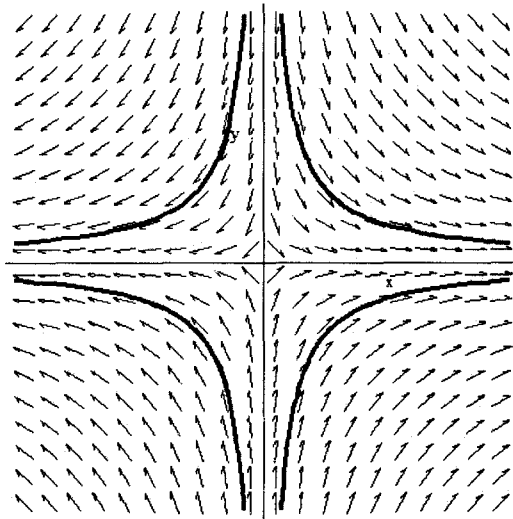
Figure 3.1 – The phase space of the fast subsystem (3.1) with $0 < \varepsilon \ll 1$. The slow subsystem is the stable z-axis. The phase space for the full system may be obtained by crossing the fast and slow subsystems.

eigenvalues -1 and +1. Hence, as neither value has zero real part $\mathcal{M}_0$ is also normally hyperbolic by construction.

The (x, z) plane is the set of points that approach $\mathcal{M}_0$ in backward time at an exponential rate. Likewise, the (y, z) plane is the set of points which approach $\mathcal{M}_0$ in forward time at an exponential rate. Hence, we conclude that the (x, z) plane is the unstable manifold of $\mathcal{M}_0$, and the (y, z) plane is the stable manifold of $\mathcal{M}_0$. We label them $W^U(\mathcal{M}_0)$ and $W^S(\mathcal{M}_0)$, respectively. Let $F_z^{0,U}$ and $F_z^{0,S}$ denote the lines $\{(x, y, z) | y = 0\}$ (the x-axis for all z) and $\{(x, y, z) | x = 0\}$ (the y-axis for all z), respectively. These lines define the fast unstable and stable fibers over $\mathcal{M}_0$, and the union of these fibers over all z are the manifolds $W^U(\mathcal{M}_0)$ and $W^S(\mathcal{M}_0)$, respectively.

In the three-dimensional (x, y, z) phase space, each plane defined by $z = $ constant is an invariant set of (3.1) with $\varepsilon = 0$. We denote these planes by Π_z. The dynamics on Π_z are given for every z by the fast system $x' = x$, $y' = -y$ (see figure 3.1). In the two dimensional (x, y) plane, the families of vertical ($x = $ constant)

and horizontal ($y =$ constant) share a special property. For any given amount of fast time τ, all of the points on the vertical line $x = x(0)$ flow toward the vertical line $x = x(\tau)$; and likewise, all of the points on the horizontal line $y = y(0)$ flow to the horizontal line $y = y(\tau)$. Hence, we may conclude that both are invariant families of the lines in the full system (3.1).

The jump from a two-dimensional system to a three-dimensional system allows us to introduce new features important for understanding general systems. On Π_z, each horizontal line of constant y is a fast unstable fiber, which we denote $F^{0,U}_{(y,z)}$. The basepoint of $F^{0,U}_{(y,z)}$ is the point $(x = 0, y)$ on the y-axis. This basepoint evolves according to the contracting fast component $y' = -y$. The point $(0, y(0))$ is the basepoint of the fiber $F^{0,U}_{(y(0),z)}$ on which our initial condition lies, and the image of this point after fast time τ is $(0, y(\tau))$. Moreover, the fiber $F^{0,U}_{(y(\tau),z)}$ is precisely the image of the initial fiber $F^{0,U}_{(y(0),z)}$, and $(0, y(\tau))$; i.e.,

$$F^{0,U}_{(y(\tau),z)} = \phi_\tau F^{0,U}_{(y(0),z)}$$

This allows us to conclude that the evolution of any initial condition in Π_z is decomposed into two components: one corresponding to exponential expansion along the unstable fibers and the second corresponding to exponential contraction in the y-direction of the basepoints of the fibers. See figures (3.1) and (3.3). Taking the union over all z of the fibers $F^{0,U}_{(y,z)}$, one can obtain a family $\mathcal{F}^{0,U}$ that is invariant with respect to (3.1) and that is normally transverse to $W^S(\mathcal{M}_0)$. Each point on $W^S(\mathcal{M}_0)$ is the basepoint of a fiber from this family, and all basepoints lie on $W^S(\mathcal{M}_0)$. This family foliates the entire plane, that is, since there is one line for every z and the lines completely fill out the plane.

In a completely analogous way, we may conclude the following for the stable fibers. Each vertical line defined by constant x is a fast stable fiber $F^{0,S}_{x,z}$. Its basepoint $(x, 0, z)$ evolves according to the fast expanding component $x' = x$, and we once again have the desired invariance property:
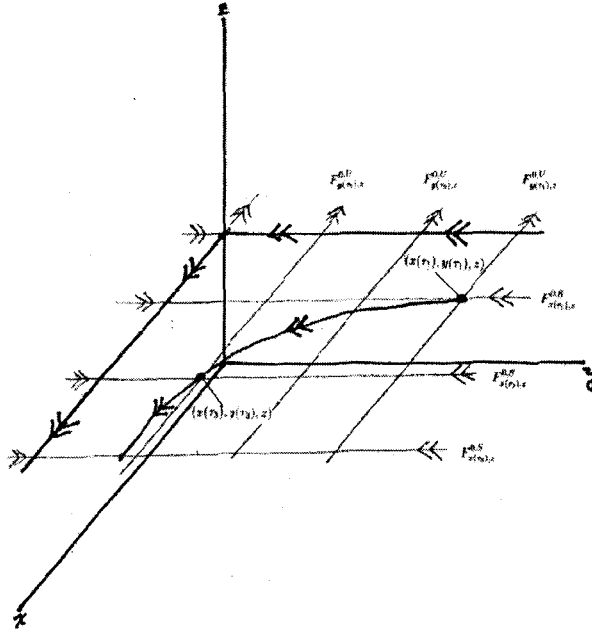
Figure 3.2 – Illustration of the invariant fibers and an arbitrary trajectory for the fast subsystem when $\varepsilon = 0$ and a given z value.

$$F^{0,S}_{(x(\tau),z)} = \phi_\tau F^{0,s}_{(x(0),z)}$$

The union of the fibers $F^{0,S}_{x,z}$ over all x and z is a family $\mathcal{F}^{0,S}$ is invariant with respect to (3.1) when $\varepsilon = 0$. All basepoints lie on $W^U(\mathcal{M}_0)$, and conversely each point on $W^U(\mathcal{M}_0)$ is the basepoint of a fiber from this family.

3.1.2 When $0 < \varepsilon \ll 1$

The phase space of this example may be obtained directly by crossing that of the fast and slow subsystems. The geometric structures present when $\varepsilon = 0$ persist in the full system (3.1) when $0 < \varepsilon \ll 1$. The z-axis is still an invariant manifold denoted $\mathcal{M}_\varepsilon$. Initial conditions on $\mathcal{M}_\varepsilon$ remain on it in both forward and backward time, and contract at a weak exponential rate toward the origin as $t \to \infty$. The (x, z) plane and the (y, z) plane are now the perturbed unstable and stable manifolds, $W^S(\mathcal{M}_\varepsilon)$ and $W^U(\mathcal{M}_\varepsilon)$ respectively. Each horizontal line of constant
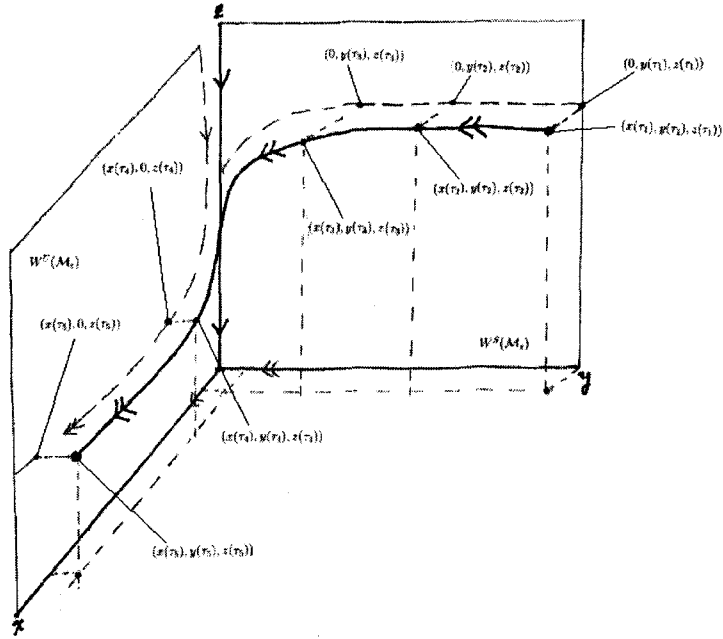
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Figure 3.3–An illustration tracing the evolution of an arbitrary trajectory into components along the fast fibers and slow manifolds in (3.1).

z value in the (y, z) plane (or vertical line in the (x, z) plane) is a persistent fast stable (or unstable) fiber $F_z^{\varepsilon, S}$ ($F_z^{\varepsilon, U}$), and its basepoint is the point $(0, 0, z)$ on $\mathcal{M}_\varepsilon$. It is important to note that the individual stable (or unstable) fibers are no longer invariant on their own, as they were in the fast $\varepsilon = 0$ system. Rather, they are invariant as a family of fibers. Also, as there exists one fiber of each type for each z, we say these families foliate $W^S(\mathcal{M}_\varepsilon)$ and $W^S(\mathcal{M}_\varepsilon)$, respectively. In this example (and in general), the exisistence of the perturbed slow manifolds $\mathcal{M}_\varepsilon$ may be viewed as a consequence of the persistence and transverse intersection of the local stable and unstable manifolds.

The fibers not in the (x, z) and (y, z) planes also persist. In the three-dimensional phase space, each line parallel to the y-axis is again a fast stable fiber $F_{(x,z)}^{\varepsilon, S}$ with basepoint $(x, 0, z)$ on $W^U(\mathcal{M}_\varepsilon)$, and similarly each line parallel to the x-axis is again a fast unstable fiber $F_{(y,z)}^{\varepsilon, U}$ with basepoint $(0, y, z)$ on $W^S(\mathcal{M}_\varepsilon)$. These

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fibers again form two invariant families:

$$\mathcal{F}^{\varepsilon,U} \equiv \bigcup_{(y,z)} F^{\varepsilon,U}_{(y,z)} \quad \text{and} \quad \mathcal{F}^{\varepsilon,S} \equiv \bigcup_{(x,z)} F^{\varepsilon,S}_{(x,z)}$$

We may now also express the stable and unstable manifolds of $\mathcal{M}_\varepsilon$ as $\mathcal{W}^S(\mathcal{M}_\varepsilon) \equiv \bigcup_{z \in \mathcal{M}_\varepsilon} F^{\varepsilon,S}_{0,z}$ and $\mathcal{W}^U(\mathcal{M}_\varepsilon) \equiv \bigcup_{z \in \mathcal{M}_\varepsilon} F^{\varepsilon,U}_{0,z}$.

One can trace the evolution of an arbitrary initial condition, not on $\mathcal{M}_\varepsilon$, $W^S(\mathcal{M}_\varepsilon)$ and $W^U(\mathcal{M}_\varepsilon)$, in the following manner (see figure 3.3). Fix an arbitrary, nonzero, $\mathcal{O}(1)$ values of $x(0), y(0)$ and $z(0)$. The given initial condition lies on the fibers $F^{\varepsilon,U}_{y(0),z(0)}$ and $F^{\varepsilon,S}_{x(0),z(0)}$. The orbit through this point evolves so that at any time τ, the point $(x(\tau), y(\tau), z(\tau))$ lies on the unstable and stable fibers, $F^{\varepsilon,U}_{\phi_\tau(y(0),z(0))}$ and $F^{\varepsilon,S}_{\phi_\tau(x(0),z(0))}$, respectively. This orbit moves inward toward $\mathcal{M}_\varepsilon$ at an exponential rate along the fibers of the family $\mathcal{F}^{\varepsilon,S}$ and outward away from $\mathcal{M}_\varepsilon$ at an exponential rate along the fibers of the family $\mathcal{F}^{\varepsilon,U}$. Its slow component evolves according to the motion of the fiber's basepoint $(0,0,z)$. Therefore, the system dynamics can be decomposed into fast and slow components in a natural way. First the fast components, in which the dynamics are governed by the exponential rates of growth and decay along the fast stable and unstable fibers. Then the slow components, governed by the motion of the basepoints of fibers along the slow manifold.

CHAPTER 4
FENICHEL'S THIRD THEOREM

A normally hyperbolic critical manifold $\mathcal{M}_0$ is by definition filled with critical points, each of which has corresponding stable and/or unstable manifolds $W^S(v_0)$ and $W^U(v_0)$. Suppose, for our system (1.1), that the $\varepsilon = 0$ Jacobian $\frac{\partial f}{\partial u}(u, v, 0)|_{\mathcal{M}_0}$ has m eigenvalues λ with $Re(\lambda) < 0$ and n eigenvalues with $Re(\lambda) > 0$, as in Fenichel's second theorem. Then $W^S(\mathcal{M}_0)$ and $W^U(\mathcal{M}_0)$ are the unions

$$W^S(\mathcal{M}_0) = \bigcup_{v_0 \in \mathcal{M}_0} W^S(v_0), \quad W^U(\mathcal{M}_0) = \bigcup_{v_0 \in \mathcal{M}_0} W^U(v_0).$$

That is, the manifolds $W^S(v_0)$ and $W^U(v_0)$ form collections of fibers for $W^S(\mathcal{M}_0)$ and $W^U(\mathcal{M}_0)$ respectively, with basepoints $v_0 \in \mathcal{M}_0$ [6].

We have previously concluded that a compact critical manifold $\mathcal{M}_0$ and its stable and unstable manifolds perturb to analagous objects $\mathcal{M}_\varepsilon$, $W^S(\mathcal{M}_0)$ and $W^U(\mathcal{M}_0)$ respectively, when ε is suffeciently small. The question now is, whether the individual stable and unstable manifolds $W^S(v_0)$ and $W^U(v_0)$ also perturb to analagous objects. Fenichel's third theorem serves to answer this question.

Although the critical points $v_0 \in \mathcal{M}_0$ do not generally perturb to fixed points, the answer to the above question is yes. The individual stable and unstable manifolds do in fact, perturb to analogous objects as we will see in the theorem to follow. We will also include a corollary which will serve to take some of the technical aspects out of the theorem, thus making it a little more approachable (though we will make no attempt to apply it).

It is important to note that while the manifolds $W^{U,S}(v_0)$ are invariant, their counterparts $W^{U,S}(v_\varepsilon)$ are not. This is clear as their basepoint v_ε itself in not invariant under the flow of (1.1). However, as with the fibers in example (3.1), the whole families $\{W^U(v_\varepsilon)|v_\varepsilon \in \mathcal{M}_\varepsilon\}$ and $\{W^S(v_\varepsilon)|v_\varepsilon \in \mathcal{M}_\varepsilon\}$ are invariant to a certain degree [6].

To state this invariance we use the notation $x \cdot t$ to denote the application of a flow after time t to an initial point x. Similarly, $V \cdot t$ denotes the application of the flow after time t to a set V, and $x \cdot [t_1, t_2]$ is the resulting trajectory if the flow is applied over the interval $[t_1, t_2]$. However, in order to avoid difficulties we restrict ourselves to a neighborhood Δ of $\mathcal{M}_\varepsilon$ in which the linear terms of (1.1) are dominant, and consider only trajectories in $W^U(\mathcal{M}_\varepsilon)$ that have not left Δ in forward time (over our given time interval), and trajectories in $W^S(\mathcal{M}_\varepsilon)$ that have not left Δ in backward time [6]. For precision we offer the following definition from Jones (1995) [3].

DEFINITION 4.1. *The forward evolution of a set $V \subset \Delta$ restricted to Δ is given by the set*

$$V_\Delta \cdot t := \{x \cdot t | x \in V \, and \, x \cdot [0, t] \subset \Delta\}$$

We now are prepared to state Fenichel's third theorem.

THEOREM 4.1. *FENICHEL'S THIRD THEOREM [4]*

Suppose $\mathcal{M}_0 \subset \{f(u, v, 0) = 0\}$ is compact, possibly with boundary, and normally hyperbolic, and suppose f and g are smooth. Then for every $v_\varepsilon \in \mathcal{M}_\varepsilon$, $\varepsilon > 0$ and sufficiently small, there are an m-dimensional manifold $W^S(v_\varepsilon) \subset W^S(\mathcal{M}_\varepsilon)$ and an n-dimensional manifold $W^U(v_\varepsilon) \subset W^U(\mathcal{M}_\varepsilon)$, that are $\mathcal{O}(\varepsilon)$ close and diffeomorphic to $W^S(v_0)$ and $W^U(v_0)$ respectively. The families $\{W^{U,S}(v_\varepsilon)|v_\varepsilon \in \mathcal{M}_\varepsilon\}$ are invariant in the sense that

$$W_\Delta^S(v_\varepsilon t \subset W^S(v_\varepsilon \cdot t)$$

if $v_\varepsilon \cdot s \in \Delta$ for all $s \in [0, t]$, and

$$W_\Delta^U(v_\varepsilon)t \subset W^U(v_\varepsilon \cdot t)$$

if $v_\varepsilon \cdot s \in \Delta$ for all $s \in [t, 0]$

The order in which the flow after time t is applied to a base point and the fiber of a basepoint is constructed does not matter. In the unperturbed setting of (1.1) with $\varepsilon = 0$, the decay in forward time of points in $W^S(\mathcal{M}_0)$ to $\mathcal{M}_0$ is clearly the basepoint v_0 of their fiber, where as the decay rate as $t \to \infty$ is exponential, since all associated eigenvalues have nonzero real part [6].

The Fenichel fibers of Fenichel's third theorem offer an analagous matching between points in $W^S(\mathcal{M}_\varepsilon)$ and $\mathcal{M}_\varepsilon$ (similarly for the unstable version). If associated to a point $x \in W^S(\mathcal{M}_\varepsilon)$ there is a base point $x^+ \in \mathcal{M}_\varepsilon$, i.e. $x \in W^S(x^+)$, then the exponential decay is inherited from the unperturbed case [6].

We will now include a corollary, which is attractive in its less technical nature. Note that the inclusion of this corollary is for informative purposes only, as we make no attempt to apply it directly.

Corollary There are constants $k_S, \alpha_S > 0$ so that if $x \in W^S(x^+) \bigcap \Delta$, then

$$||x \cdot t - x^+ \cdot t|| \leq k_S e^{-\alpha_S t}$$

for all $t \geq 0$ for which $x \cdot [0, t] \subset \Delta$ and $x^+ \cdot [0, t] \subset \Delta$.

Similarly, there are constants $k_U, \alpha_U > 0$ so that if $x \in W^U(x^-) \bigcap \Delta$, then

$$||x \cdot t - x^- \cdot t|| \leq k_U e^{\alpha_U t}$$

for all $t \leq 0$ for which $x \cdot [t, 0] \subset \Delta$ and $x^- \cdot [t, 0] \subset \Delta$.

4.1 A Classic Example

For our final example we consider the classic Rosenzweig-MacArthur predator-prey model presented in Rinaldi and Muratori (1992) [19] and given in rescaled form by Hek (2010)[6].

$$\begin{cases} \dot{u} = u\left(1 - u - \dfrac{av}{u+d}\right) \\ \dot{v} = \varepsilon v\left(\dfrac{au}{u+d} - 1\right) \end{cases} \tag{4.1}$$

Here u the number of prey, and v the number of predators are both non-negative. They have been scaled with the constant predator-free carrying capacity of the prey. Our parameter $\varepsilon > 0$ is the ratio between the death rate of the predator and the growth rate of the prey, and a, d determine the impact of predation on the prey. If the prey reproduce faster than the predators and the predator is aggressive but in comparison not very efficient, then ε becomes a small parameter ($0 < \varepsilon \ll 1$) under these assumptions, described in detail in Rinaldi and Muratori [19], this system represents a slow-fast predator-prey model of the form (1.1) with

$$\begin{cases} f(u, v, \varepsilon) = u(1 - u) - \dfrac{auv}{u+d} \\ g(u, v, \varepsilon) = v\left(\dfrac{au}{u+d}\right) \end{cases} \tag{4.2}$$

Note that g is not well defined here if $u = -d$, but this represents a nonbiological value.

4.1.1 Fast, $\varepsilon = 0$ Manifolds

In the $\varepsilon = 0$ limit, the nullcline $\{(u, v)| f(u, v, 0) = 0, u \geq 0, v \geq 0\}$ consists of two parts, namely $\mathcal{M}_0^0 := \{(u, v)|u = 0, v \geq 0\}$ and $\mathcal{M}_0^1 := \{(u, v)|v = \frac{1}{a}(1 - u)(u + d), u, v \geq 0\}$. These critical manifolds represent all possible prey equilibria in the case of a constant (but arbitrary) predator population. The manifolds are normally hyperbolic everywhere with the exceptions $(0, \frac{d}{a}) \in \mathcal{M}_0^0 \bigcap \mathcal{M}_0^1$ and $(\bar{u}, \bar{v}) =$

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$$\left(\frac{1-d}{2}, \frac{(1+d)^2}{4a}\right) \in \mathcal{M}_0^1 \text{ for } d < 1.$$

4.1.2 Slow, $\tau = \varepsilon t$ Manifolds

On $\mathcal{M}_0^1$ the flow with respect to time $\tau = \varepsilon t$ can be found by writing $\mathcal{M}_0^1$ as a graph of a function in the slow variable v. If $0 < d < 1$ then this cannot be done in a global way, so we write $\mathcal{M}_0^1$ as the union of two hyperbolic parts $\mathcal{M}_0^+$ and $\mathcal{M}_0^-$, and a third small part around the nonhyperbolic fold point $(\bar{u}, \bar{v})$ as

$$\mathcal{M}_0^1 = \mathcal{M}_0^+ \bigcup \mathcal{M}_0^- \bigcup B(\bar{v}, \delta)$$

Here

$$\mathcal{M}_0^{\pm} := \left\{ (u,v) \mid u = u_{\pm}(v) := \frac{1}{2}\left(1 - d \pm \sqrt{(1+d)^2 - 4av}\right), \ u \geq 0, \ 0 \leq v \leq \bar{v} - \frac{\delta}{2} \right\}$$
(4.3)

and $B(\bar{v}, \delta)$ with $0 < \delta \ll 1$ is an open neighborhood within $\mathcal{M}_0^1$ of $(\bar{u}, \bar{v})$. It is important to note that if $d \geq 1$ we write $\mathcal{M}_0^1 = \mathcal{M}_0^+$ so that the flow may be written as $v' = v\left(\dfrac{u(a-1)-d}{u_{\pm}(v)+d}\right)$ on $\mathcal{M}_0^{\pm}$. The flow on $\mathcal{M}_0^1$ has equilibrium points $v = 0$ which is repelling on $\mathcal{M}_0^+$ and $v = \frac{d(a-1-d)}{(a-1)^2}$ which is a repellor on $\mathcal{M}_0^-$ if $a > \frac{1+d}{1-d}$, and an attractor on $\mathcal{M}_0^+$ if $1 < a < \frac{1+d}{1-d}$.

These manifolds are more easily understood in a geometric context, as in figure 4.1. The biolgical meaning of this flow is the following, assume that for a constant predator population $v_1 \neq \bar{v}$ the prey is in an equilibrium state $(u_1, v_1) \in \mathcal{M}_0^+$. If the size of the predator population now begins to change (slowly) to a new value v_2, a new equilibrium $(u_2, v_2) \in \mathcal{M}_0^{\pm}$ will form, according to the predator-prey interaction on the nullcline $\mathcal{M}_0^{\pm}$. It is only if v passes the value $\bar{v}$ that this would not be a continuous process [6].

A fold point like our point $(\bar{u}, \bar{v})$ is nonhyperbolic and therefore needs more attention. In general, and in the case of our example, fold points that are important
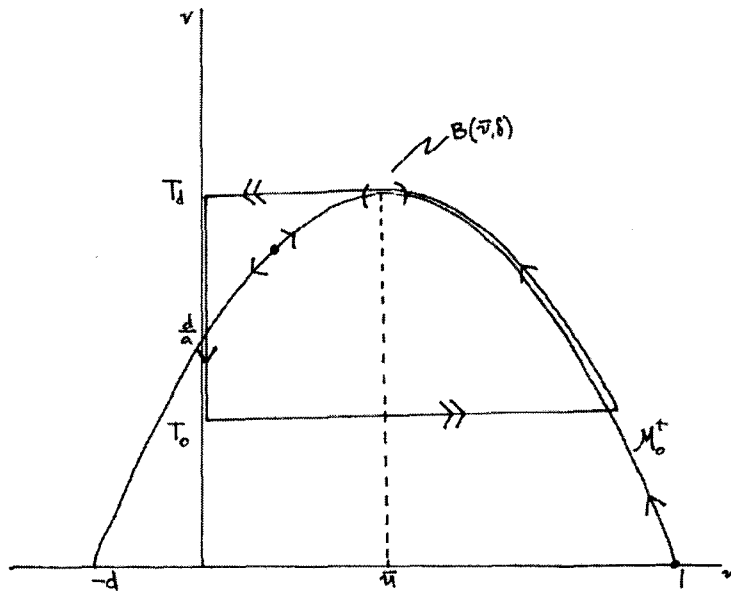
Figure 4.1 – An illustration of the configuration of the slow manifolds for the system (4.1).

to the dynamics are known as jump points [6]. It is here that the flow jumps off the slow manifold and starts to follow the fast vector field. Such jumping behavior is part of the mechanics behind the classical relaxation oscillations [6].

As can be seen in figure 4.1 given an initial point (u_0, v_0) our system follows the fast fiber to the manifold $\mathcal{M}_0^+$ on which it will travel until reaching the critical point $(\bar{u}, \bar{v})$. It is here that the system then jumps back to the manifold made of the v-axis at the point T_d (or touchdown point). Here it travels along the v-axis until it reaches the point T_o (take-off point) at which it jumps back onto a fast fiber traveling back toward $\mathcal{M}_0^+$.

We may calculate the points T_d, T_o in the follwing manner; the point T_d is found at the fold point $(\bar{u}, \bar{v}) = \left(\dfrac{1-d}{2}, \dfrac{(1+d)^2}{4a} \right)$, hence $T_d = \dfrac{(1+d)^2}{4a}$. We turn to Rinaldi and Muratori [19], for the derviation of the calculation of T_o. They derive
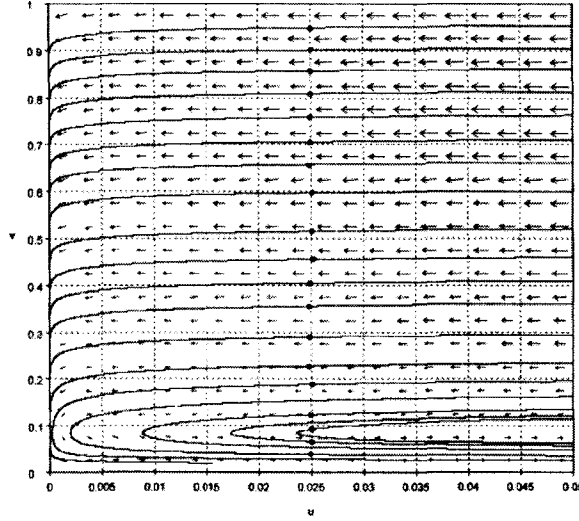
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Figure 4.2 – Phase plane for system 4.1 with several trajectories, here $a = 6, d = 0.05, \varepsilon = 0.1$. Here, one can see the interaction of the fast fibers (horizontal lines) and the slow manifold at $u = 0$. It is this interaction that causes our trajectories to turn.

the following equation:

$$\int_{v_{min}}^{v_{max}} \frac{f(0,v)}{vg(0,v)} dv = 0 \tag{4.4}$$

Where $v_{max} = T_d$, $v_{min} = T_o$. Hence, we solve

$$\int_{v_{min}}^{\frac{(1+d)^2}{4a}} \frac{1 - \frac{av}{d}}{v(-1)} dv = 0 \tag{4.5}$$

for v_{min}, which gives

$$\frac{a}{d}v - \ln v \Big|_{y_{min}}^{\frac{(1+d)^2}{4a}} = 0$$

$$\left(\frac{(1+d)^2}{4d} - \ln \frac{(1+d)^2}{d} \right) - \left(\frac{a}{d}y_{min} - \ln y_{min} \right) = 0$$

$$\ln y_{min} = \frac{a}{d}y_{min} + \ln \frac{(1+d)^2}{d} - \frac{(1+d)^2}{4d}$$

which can be solved numerically or by graphical inspection of the intersection of the graphs $f(y_{min}) = \ln y_{min}, g(y_{min}) = \frac{a}{d}y_{min} + \ln \frac{(1+d)^2}{d} - \frac{(1+d)^2}{4d}$.

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CHAPTER 5
FUTURE CONSIDERATIONS

Based on the work presented in the previous chapters one can now (hopefully) see the power and applicative nature of Fenichel's theorems. It is important to note that this discussion of singular perturbation theory is not complete. We have covered but a small fraction of the literature on geometric singular perturbation techniques and have not mentioned other widely used methods to study singular perturbed systems.

Fenichel's theorems only consider slow manifolds and their *local* stable and unstable manifolds. To unravel the *global* geometry of the stable and unstable manifolds and to keep track of their intersections it is important to understand behavior of orbits and manifolds as they pass near a slow manifold. The basic idea in studying such behavior (in forward time), is to take a disk $\mathcal{D}$ that transversely intersects the stable manifold $W^S(\mathcal{M})$ and use the fact that any point $q = q(0)$ in the intersection $W^S(\mathcal{M}) \cap \mathcal{D}$ satisfies $\lim_{t \to \infty} ||q(t) - \mathcal{M}|| = 0$ by the definition of the stable manifold. Conclusions about the fate of other points in $\mathcal{D}$ at $t \to \infty$ can then be drawn. An analogous conclusion can be drawn for the unstable manifold $W^U(\mathcal{M})$. Two well known lemmas based on this idea are the Lambda lemma for maps (see Guckenheimer and Holmes [5]) and the Exchange Lemma (see Jones and Kopell (1994) [10]) for flows.

We conclude by offering the reader a list of current publications which make use of Fenichel's Theorems.

The role of environmental persitence in pathogen transmission (Breban 2012

[2])

Modeling herbivore population dynamics in the Amboseli National Park, Kenya (Mose et. al 2012 [15])

spatial aggregation methods involving several time scales (Auger et. al. [1])

Persistence of traveling wave solution ina bio-reactor model with nonlocal delays (Yang et. al. 2010 [20])

Reduced models of networks of coupled enzymatic reactions (Kumar and Josic 2011 [12])

Existence and uniqueness of generalized stationary waves for viscous gas flow through a nozzle with discontinuous cross section (Hong et. al. 2012 [8])

Existence of travelling fronts in a diffusive vector disease model with spatio-temporal delay (Peng et. al. 2010 [17])

Approximate aggregation of a two time scales periodic multi-strain SIS epidemic model (Marv et. al. 2012 [14])

Transition state theory in liquids beyond planar dividing surfaces (Hernandez et. al. 2010 [7])

Bistable wavefronts in a diffusive and competitive LotkaVolterra type system with nonlocal delays (Lin and Li 2008 [13])

A probabilistic model of thermal explosion in polydisperse fuel spray (Nave et. al., 2010 [16])

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CURRICULUM VITAE

Jeremy W. Riley
2803 Gleeson Lane Louisville, KY 40299
jwrile01@louisville.edu

Education:

B.S. Mathematics
Missouri Western State University
St. Joseph, MO
2008-2010

M.A. Mathematics
Unversity of Louisville
Louisville, KY
2010-2012

Experience:

Graduate Teaching Assitant
University of Louisville
Louisville, KY
2010-2012

Presentations:

Missouri Sectional MAA Conference
Student Presentation
"Chaotic Waves and Their Embedded Song Lyrics"
2010