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PLANAR GRAPHS: A HISTORICAL PERSPECTIVE

By

Rick Alan Hudson
B.A., University of Southern Indiana, 2002

A Thesis
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Graduate School of the University of Louisville
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For the Degree of

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Department of Mathematics
University of Louisville
Louisville, Kentucky

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A Thesis Approved on

July 20, 2004

By the following Thesis Committee

Dr. Richard M. Davitt, Thesis Director

Dr. André Kézdy

Dr. Dar-jen Chang

DEDICATION

This thesis is dedicated to my parents

David & Marilyn Hudson

who have always supported all of my endeavors.

ACKNOWLEDGEMENTS

I would like to thank the members of my thesis committee, Dr. Richard Davitt, Dr. André Kézdy, and Dr. Dar-jen Chang for their guidance and advice while preparing this thesis. I am also grateful to my parents, David and Marilyn Hudson, my greatest source of moral support. Without the encouragement of my family, I could not have achieved many of my goals. Finally, I would like to express my great appreciation to the many great teachers and professors who have touched my life through the gift of education. Specifically, I would like to thank Dr. Kathy Rodgers, Dr. David Kinsey, and Dr. William Wilding of the University of Southern Indiana for encouraging me to continue my education at the graduate level.

ABSTRACT

PLANAR GRAPHS: A HISTORICAL PERSPECTIVE

Rick Alan Hudson

July 20, 2004

The field of graph theory has been indubitably influenced by the study of planar graphs. This thesis, consisting of five chapters, is a historical account of the origins and development of concepts pertaining to planar graphs and their applications. The first chapter serves as an introduction to the history of graph theory, including early studies of graph theory tools such as paths, circuits, and trees. The second chapter pertains to the relationship between polyhedra and planar graphs, specifically the result of Euler concerning the number of vertices, edges, and faces of a polyhedron. Counterexamples and generalizations of Euler's formula are also discussed. Chapter III describes the background in recreational mathematics of the graphs of K_5 and $K_{3,3}$ and their importance to the first characterization of planar graphs by Kuratowski. Further characterizations of planar graphs by Whitney, Wagner, and MacLane are also addressed. The focus of Chapter IV is the history and eventual "proof" of the four-color theorem, although it also includes a discussion of generalizations involving coloring maps on surfaces of higher genus. The final chapter gives a number of measurements of a graph's closeness to planarity, including the concepts of crossing number, thickness, splitting number, and coarseness. The chapter concludes with a discussion of two other coloring problems – Heawood's empire problem and Ringel's earth-moon problem.

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CHAPTER I

A BRIEF INTRODUCTION TO THE HISTORY OF GRAPH THEORY

A **planar graph** is a graph that can be drawn in the plane without any edges crossing one another. The history of planar graphs begins with the history of graph theory in that many of the early graph representations were planar. Early graphs, formed by a set of nodes (or vertices) and a set of edges, can be traced as far back as Ancient Egypt in the context of games and Ancient Rome in the form of genealogical family trees (Kruja et al. 2002, 272-277). However, the true genesis of the field of mathematical graph theory is universally attributed to Leonhard Euler, who at the time of his discoveries in 1736, was serving as chief mathematician at the Academy at St. Petersburg.

Königsburg Bridge Problem

The problem of the Königsburg bridges can be traced back to Heinrich Kühn, a mathematics professor at the academic gymnasium in Danzig. The mayor of Danzig was Carl Leonhard Gottlieb Ehler, a close friend of Euler. The two men corresponded during the time period of 1735 to 1742. Through Ehler's letters, Kühn was able to communicate with Euler, and among their discussions in a letter dated March 9, 1736, was the question of the Königsburg bridges, although it is likely that Euler and Ehler had written about the problem previously (Sachs, Stiebitz and Wilson 1988, 134-135). The Prussian city of

Königsburg, known today as Kaliningrad, was situated on the Pregel River in such a way that an island in the river near one of its branchings formed four land masses. The four land areas were connected by a total of seven bridges, as shown in Figure 1. The problem was to attempt to start in any of the four areas of land and to create a route that crosses each bridge exactly once (Biggs, Lloyd and Wilson 1976, 1-8). On March 13, 1736, Euler wrote to Giovanni Jacobo Marinoni of Venice, who was serving as Court Astronomer to Kaiser Leopold I. In this letter, Euler remarked on the Königsburg problem,

“The question is so banal, but seemed to me worthy of attention in that neither geometry nor algebra, nor even the art of counting [ars combinatoria] was sufficient to solve it. In view of this, it occurred to me to wonder whether it belonged to the geometry of position [geometria situs], which Leibniz had once so much longed for” (Sachs, Stiebitz, and Wilson 1988, 135-136).

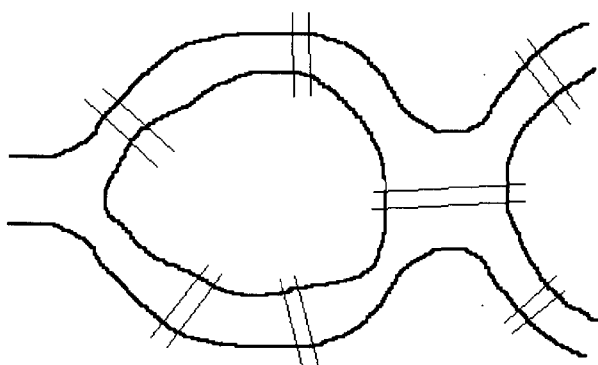


Figure 1. The Königsburg Bridges

By early April, Euler replied to Ehler’s previous letter by saying that the Königsburg question “bears little relationship to mathematics... and its discovery does not depend on any mathematical principle” (Sachs, Stiebitz, and Wilson 1988, 136). Euler proclaimed his ignorance of the new discipline of the geometry of position that Gottfried Wilhelm Leibniz, one of the founders of calculus, had mentioned in his work.

In a 1679 letter from Leibniz to Christiaan Huygens, Leibniz had described the need for such a geometry of position that has a “new characteristic, completely different from algebra, which will have great advantage to represent to the mind exactly and naturally though without figures, everything which depends on imagination” (Kruja et al. 2002, 277-281). Leibniz had been closely connected to the Academy at St. Petersburg, and it was through this relationship that Euler likely learned of Leibniz’s views.

Even though the letter to Ehler may make one believe Euler’s judgment was that the Königsburg bridge problem had no relationship to mathematics, he did in 1736, produce a paper on the problem. In the paper, he described Leibniz’s view that there were two branches of geometry: the one studied extensively dealing with distances, and a second branch that involves the geometry of position (Sachs, Stiebitz, and Wilson 1988, 136). Euler generalized the Königsburg bridge problem to any number of land regions and bridges, and stated a necessary and sufficient condition for a walk across every bridge to take place. The condition was that either no region or only two regions having an odd number of bridges allows for the walk to take place. Since the land masses of the city of Königsburg each were attached by an odd number of bridges, due to Euler’s conclusions, one can deduce that the bridges cannot be traversed in the desired manner (Lloyd 1975, 411-412).

One may note that although Euler’s work does contain the graph theoretic concepts of edges and vertices, he provided no graphical drawings in his paper. The article did contain two of Euler’s sketches of maps of Königsburg. Euler’s choice not to create a graphical depiction may have been influenced by Leibniz’s words cited above that stated that the geometry of position could be represented “without figures.” W.W.

Rouse Ball eventually was the first person to represent the Königsburg bridge problem graphically in a book on recreational mathematics in 1892 (Kruja et al. 2002, 277-281). One can create a figure similar to Ball's by replacing each land mass with a vertex, and then for each bridge, drawing an edge between the two vertices that represent the corresponding land masses that that bridge connects. A model of the Königsburg bridges in graph theoretic form, similar to the model presented by Ball, is shown in Figure 2. In modern graph theory, a sequence of vertices and edges in which each edge occurs only once is called an **Eulerian path** and the **valency**, or **degree**, of a vertex is the number of edges incident to the vertex. Thus, Euler's initial findings in graph theory can be succinctly stated in the following theorem: "If a connected graph has more than two vertices of odd valency, then it cannot contain an Eulerian path" (Biggs, Lloyd, and Wilson 1976,10).

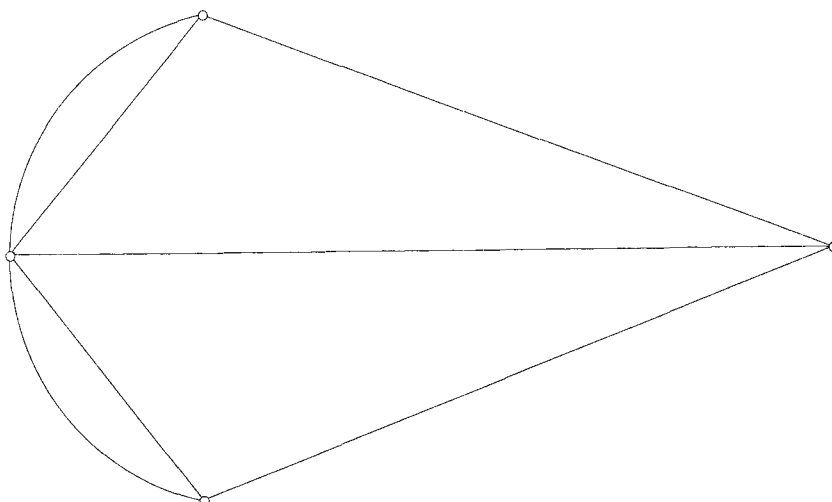


Figure 2. A Graphical Representation of the Königsburg Bridges

It is important to point out that Euler was only able to prove the necessary condition correctly. His corresponding proof for sufficiency was deficient (Lloyd 1975, 411-412). However, he was correct in assuming it, in that almost 140 years later, in 1873, a paper was published posthumously by Carl Hierholzer that detailed a proof of the sufficiency of Euler's argument. Thus, "if a connected graph has no vertices of odd valency, or two such vertices, then it contains an Eulerian path" (Biggs, Lloyd, and Wilson 1976, 10-11). It is likely that Hierholzer was unaware of Euler's discussion of the Königsburg bridge problem, because in an editorial note he did not make mention of it. However, Hierholzer did discuss *Vorstudien zur Topologie* by Johann Benedict Listing. Listing's very important work discussed contexts in geometry that rely on position rather than measure, such as screws, knots, links, and diagram tracing puzzles. Problems similar to that of the Königsburg bridges were then subsequently analyzed, including Coupy's 1851 publication, which translated Euler's earlier papers into French and described an application involving the bridges across the Siene River. Saalaschütz reported in 1876 that a new bridge had been built in Königsburg allowing an Eulerian walk to take place (Wilson 1978, 2).

The Study of Circuits

As we will see in subsequent chapters, many of the problems in graph theory were inspired by puzzles, mysteries, and brainteasers. While the problem of the Königsburg bridges required a path that crossed each edge only once, a **circuit** is a path in which all the vertices and edges are distinct, except for the last vertex that is the same as the first. A circuit that passes through all the vertices of a graph is said to be a **Hamiltonian circuit**,

and graphs that have such a circuit are called **Hamiltonian**. A very famous puzzle involving Hamiltonian circuits is the knight's tour problem. In the game of chess, knights are allowed to move two units in one direction parallel to an edge of the board, followed by one unit in the perpendicular direction. The problem, which has been known for centuries, is to move a knight in such a way that it lands on each of the 64 squares of the chessboard once and only once and returns to its starting position (Biggs, Lloyd, and Wilson 1976, 21-22). This is a Hamiltonian problem, because one can represent each square on the board as a vertex, and for each move by a knight, an edge is placed between the starting and ending squares (or vertices). In the 1600s, specific solutions of the knight's tour problem were known to individuals such as De Montmort and De Moivre. However, the general problem was not studied extensively until the 18th century, when Euler examined it. Euler concluded that no solution is possible on similar boards that have an odd number of squares, and gave the 5 x 5 board as a classic example (Wilson 1999, 507-508). In a 1771 paper, Alexandre-Theophile Vandermonde described a method for finding a knight's tour, as well as a discussion of several other topological results (Wilson 1978, 3). A diagram of Vandermonde's graph of a knight's tour is shown in Figure 3.

The Hamiltonian circuit is named for Sir William Rowan Hamilton, one of the greatest mathematicians of his era. One of his most intriguing discoveries is that which he termed the "Icosian Calculus." The Icosian Calculus is a non-commutative algebra that involves paths on the graphical representation of the regular dodecahedron. He announced his discovery in a letter dated October 1856, and later published two papers on the subject. Hamilton used the graphical representation to form a game, which he

presented at a meeting of the British Association in Dublin in 1857. An interested purveyor of games and puzzles purchased the game for 25 pounds, and began marketing the game two years later. Unfortunately, the game was not as successful as one would have hoped (Wilson 1999, 509). The object of the game was to attempt to find paths and circuits on the graph of the dodecahedron, given specific initial conditions. The first problem was essentially to find a Hamiltonian circuit on the game board. A second version of the Icosian game featured a solid dodecahedron with pegs on each of the vertices, and was called “The Traveler’s Dodecahedron” or “A Voyage Round the World.” This game named the vertices for 20 significant places from around the world, and the player had to attempt to connect all of the pegs with a thread (Biggs, Lloyd, and Wilson 1976, 31-35).

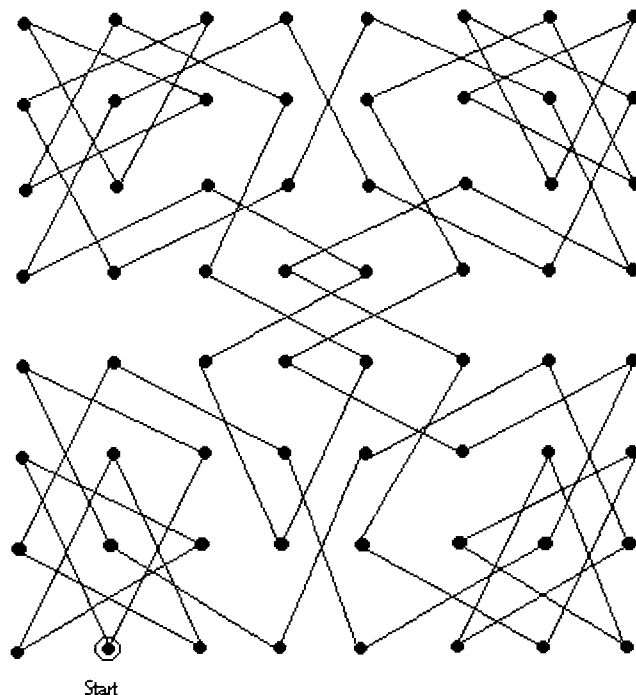


Figure 3. Vandermonde’s Solution to the Knight’s Tour

Although the games' renown caused these types of circuits to be named after Hamilton, he was not the first to publish on the subject. Thomas Penyngton Kirkman, an English clergyman, had discussed the problem in an 1856 paper on polyhedra. In his paper, Kirkman inquired if a circuit always existed in a given polyhedral graph. He made a claim for sufficiency, but unfortunately, his claim was false. His main contribution to the study of circuits was to identify a general class of graphs with no circuits. He did determine that polyhedra with an odd number of vertices and an even number of edges on each face do not have such a circuit (Biggs, Lloyd and Wilson 1976, 28-30). Unlike the Eulerian path problem, there have been no necessary and sufficient conditions found for deciding whether a general graph has a Hamiltonian circuit or not, although sufficient conditions were discovered by Dirac in 1952 and Ore in 1960 (Wilson 1999, 509).

Trees

Arthur Cayley introduced the word '**tree**' in an 1857 paper to describe a connected graph with no cycles. However, the concept of trees had been used at least ten years earlier by both von Staudt and Kirchhoff (Biggs Lloyd and Wilson 1995, 2176-2177). Cayley was motivated to research trees by a differential calculus problem. Using rooted trees and generating functions, Cayley attempted to determine the number of trees with a certain number of edges. His original paper dealt with **rooted trees**, in which one particular vertex is labeled as the root of the graph. Two years later, he advanced his study by examining rooted trees in which each of the branches is the same distance from the root. It was not until 1875, that Cayley solved the difficult problem of enumerating

unrooted trees in a paper presented to the British Association (Biggs, Lloyd, and Wilson 1976, 37-44).

Cayley's work was expanded by Camille Jordan in 1869, when he submitted a paper on connected "assemblages" of lines intersecting at vertices. He opened his paper by describing what we know today as isomorphic graphs. Two graphs, G and H , are said to be **isomorphic** if the vertices of H may be relabeled to be equivalent to those of the graph of G . In his discussion, Jordan introduced several new concepts relating to trees, such as the centroid, bicentroid, centre and bicentre, which significantly simplified Cayley's methods (Biggs, Lloyd, and Wilson 1995, 2176-2177). In the aforementioned 1875 paper, Cayley was able to utilize Jordan's work to solve the problem of unrooted trees, as well as to discuss some applications to the field of chemistry.

James Joseph Sylvester and William Kingdon Clifford worked closely with Cayley in his work on chemical molecules, especially in the study of invariants. One late night at 3 A.M., Sylvester ingeniously thought to represent the invariants by diagrams. In 1878, Sylvester wrote a note, followed by a paper, in an attempt to link these invariants to chemistry. Since a diagram can be drawn to represent every chemical molecule, Sylvester conjectured that a connection to his invariant theory must exist (Wilson 1978, 4). Clifford had used graphical notation to represent the connection, and in his note, Sylvester coined the word "graph." This is the first time that the word "graph" is used in mathematical literature in its modern sense as dealing with a collection of sets of vertices and edges (Biggs, Lloyd, and Wilson 1995, 2177). There is uncertainty as to whether Clifford or Sylvester was the first to use the word "graph," but it is clear that Sylvester was the first to use the word in print. Sylvester's paper originally appeared in the very first volume of

the *American Journal of Mathematics*, which Sylvester founded not long after accepting his professorship at Johns Hopkins University.

Another interesting tree-counting problem involved determining the number of ways, t_n , that n vertices can uniquely be joined to form a tree. For example, $t_4 = 16$ since sixteen unique trees can be drawn using four vertices. This type of problem is often referred to as a “labeling tree problem.” Although C.W. Borchard and Sylvester made progress on the subject, Cayley discovered the first solution in 1889:

$$t_n = n^{n-2}.$$

However, much like Euler’s “proof” of the Königsburg bridges result, Cayley’s work was less than perfect. A more thorough proof appeared in 1918 when Heinz Prüfer, a German mathematician, wrote a paper on the question that arose from considering a problem involving permutations (Biggs, Lloyd, and Wilson 1976, 51-54).

König’s 1936 Textbook

As the old saying goes, “From Königsburg to König’s book... So runs the graphic tale” (König 1990, 1). In 1936, ironically the 200th anniversary of Euler’s first letter on the Königsburg bridges, Dénes König of Budapest wrote the first comprehensive book on graph theory entitled *Theorie der endlichen und unendlichen Graphen* or *Theory of Finite and Infinite Graphs*. In his foreward, König discussed the two origins of graph theory: science and intellectual games,

“Just like most newer branches of mathematics graph theory has not been created as an end in itself but in connection with older parts of mathematics and the natural sciences.... Perhaps graph theory owes more to the contact of mankind with himself than to the contact of mankind with nature” (König 1990, 48).

The study of paths, circuits, and trees furnishes a foundation for the field of graph theory, and serves as an introduction to the other properties of planar graphs that will be discussed in the following chapters.

CHAPTER II

POLYHEDRA

The study of planar graphs has its foundations in the study of polyhedra. One of the most important developments concerning polyhedra was a formula and its generalizations that connect the number of faces, edges, and vertices of a convex polyhedron. This elegant and useful equation, first found by Euler, states that

$$V - E + F = 2 ,$$

where V is the number of vertices, E is the number of edges, and F is the number of faces of the polyhedron. A definition of a polyhedron in Euler's time was a solid that is bounded by planar faces (Lakatos 1976, 14). As this chapter will describe, modifications must be made to this seemingly simple definition in order to allow Euler's formula to be valid.

Early Studies of Polyhedra

Although polyhedra were most certainly studied by the Ancient Egyptians, today we credit the Greeks with discovering various mathematical properties of polyhedra. The Greeks were especially intrigued with **regular polyhedra**, those in which all of the faces are congruent regular polygons and all the polyhedral angles are congruent. They determined that only five regular convex polyhedra exist – the tetrahedron, cube,

octahedron, dodecahedron, and icosahedron. In his *Elements*, Euclid explains that the tetrahedron, cube and dodecahedron were known by the Pythagoreans, while the octahedron and icosahedron are attributed to Theaetetus. The five regular figures are commonly called the Platonic solids, because Plato described the construction of these five figures in his famous treatise *Timaeus*. Plato's work was based on the cosmology of Timaeus of Locri, a Pythagorean, who mystically associated four of the regular polyhedra with the four Empedoclean primal "elements" – fire (tetrahedron), air (octahedron), water (icosahedron), and earth (cube). The dodecahedron was similarly compared to the enveloping universe. All five of the Platonic solids occur in nature in the form of crystals or as skeletons of microscopic sea creatures (Eves 1990, 92).

It is believed by some that Euclid's *Elements*, written circa 320 B.C.E., was meant to serve as an introduction to the study of the five Platonic solids (Biggs, Lloyd, and Wilson 1976, 75). However, in his classic text on the history of mathematics, Howard Eves (1990, 149) states,

"The frequently stated remark that Euclid's *Elements* was really intended to serve merely as a drawn-out account of the five regular polyhedra appears to be a lopsided evaluation. More likely, it was written as a beginning text in general mathematics."

No matter what Euclid's intentions in writing the *Elements* were, it is certain that he made no reference to the Eulerian formula relating the vertices, edges, and faces of polyhedra. In fact, no evidence has been found that the Greeks were aware of this rather simple relationship. It is possible that the Greeks had discovered the formula, but that the results were lost over time. Another likely explanation is that the Greeks never made this connection, because their geometry was primarily concerned with measurement, and

consequently they were not interested in this topologically-based formula (Biggs, Lloyd, and Wilson 1976, 75).

Almost 2000 years after Euclid penned the *Elements*, the co-founder of analytic geometry, René Descartes, did not notice the formula either. However, while in Paris in 1675-1676, Leibniz copied a manuscript that Descartes had written circa 1639. It was later rediscovered and published in 1860, by Foucher de Careil. In this work, Descartes stated the following relationship

$$\text{Number of plane angles} = 2\phi + 2\alpha - 4,$$

where ϕ is the number of faces and α is the number of solid angles (or point-like vertices). Later, he also stated that the number of plane angles equals twice the number of edges. By combining these two statements, one can easily deduce Euler's Polyhedral Formula, although Descartes apparently did not make this connection. Presumably, like the Greeks, Descartes was concerned primarily with measurement, congruence, and similarity and not topological properties (Lakatos 1976, 6). Due to Descartes' near discovery, the formula for polyhedra is sometimes called the Euler-Descartes Formula.

Euler's Polyhedral Formula

Euler first conjectured his formula connecting the numbers of vertices, edges, and faces in a discussion with Christian Goldbach, with whom Euler had communicated for several years. The letter, dated November 1750, was written partly in Latin and partly in German. Euler's intentions were to determine properties of solid figures analogous to properties concerning plane figures that had previously been established. A key "invention" of Euler was the concept of vertices and edges. He was the first to notice that

the character of a polyhedron could be described not only in terms of its number of faces, but also by the number of lines and points on the surface of the polyhedron. He created the term ‘acies’ (edge), instead of using the word ‘latus’ (side), which referred to the boundaries of polygons, rather than polyhedra (Lakatos 1976, 6).

Biggs, Lloyd, and Wilson’s (1976, 76-77) classic account of the early history of graph theory contains a partial translation of Euler’s letter to Goldbach. In his letter, Euler stated many conclusions about the properties of polyhedra. For example, using basic facts from plane geometry, he noted that the total number of plane sides was equal to the total number of plane angles. He also found that the number of edges was equal to half the number of plane sides, given that two sides intersect to form one edge. Thus, the number of plane sides must always be an even number. Because each face must have at least three sides, Euler concluded that the number of plane sides is greater than or equal to three times the number of faces. Similarly, the number of plane angles must be greater than or equal to three times the number of vertices. While he was able to give “satisfactory proof(s)” for the previous statements, there were several other combinatorial formulas that he had been unable to prove at the time of his letter. The first of these propositions was the famous Euler formula for a polyhedron:

$$V - E + F = 2 .$$

He also stated the impossibility of the following claims:

$$E + 6 > 3F$$

$$E + 6 > 3V$$

$$F + 4 > 2V$$

$$V + 4 > 2F .$$

An argument that proves the second of the above inequalities appears in Chapter III. The other inequalities are easily proved using modern graph theory. In addition, Euler believed that no solid could be formed from faces that each had more than five sides or whose vertices are formed by six or more plane angles. He also conjectured that the sum of the measures of the plane angles is equal to $360(V - 2)$ degrees, where V is the number of vertices, as described above.

Euler's quest for knowledge on the subject did not end with his letter to Goldbach in 1750. Two years later he wrote two papers on the polyhedral formula. In the first paper, he described how he had verified it for several families of solids, such as prisms, pyramids, etc. In the second paper, Euler described a "proof by dissection." This method consisted of "slicing" away tetrahedral parts of a given polyhedron in such a way that the value of $V - E + F$ does not change. Finally, he concluded that a single tetrahedron remained, for which we know that $V - E + F = 4 - 6 + 4 = 2$. Albeit a creative method, it was not widely accepted, because there was uncertainty that his slicing procedure could always be performed, and that the slicing method might not always result in a non-degenerate polyhedron. Another criticism of Euler's proof was that he failed to specify the class of polyhedra for which it holds; others discovered several counterexamples to Euler's result. One matter that Euler seemed to overlook in his treatment of polyhedra is the property of convexity. A solid figure is said to be **convex** if any two points of the figure can be connected by a line that lies entirely within the figure (Biggs, Lloyd, and Wilson 1976, 77-78). Euler failed to recognize that his formula did not hold for all non-convex polyhedra, such as, for example, a polyhedron with a hole drilled from one face to another.

The issue of Euler's less than perfect proof was thought to be resolved in 1794 when Adrien-Marie Legendre presented a proof using metrical properties of spherical polygons. The proof was quite different than Euler's, but used ideas similar to those of Descartes from 150 years earlier. Legendre, unlike Descartes, had the luxury of knowing what he was attempting to solve (Biggs, Lloyd, and Wilson 1976, 78). Legendre appears to have been perplexed about what kind of polyhedra Euler's formula satisfies. Although he gave a rather general definition of polyhedra, he still gave a proof that does not apply to all non-convex polyhedra. However, in a fine print note to his work, he restricted his statements to convex polyhedra only (Lakatos 1976, 28).

In 1809, Louis Poinot wrote a paper describing four non-convex regular polyhedra. In addition to the five Platonic solids, Poinot inquired whether these were the only nine regular polyhedra (Biggs, Lloyd, and Wilson 1976,78). In the fifteenth century, Kepler had introduced two "star-shaped" or "stellated" regular polyhedra. Poinot found two other such stellated polyhedra. These four polyhedra are commonly called the small stellated dodecahedron, the great stellated dodecahedron, the great dodecahedron, and the great icosahedron. Graphical illustrations of these stellated polyhedra and other polyhedra can be found in Lyusternik (1966, 157-158). Poinot also included a discussion of Legendre's vague statements about convexity. He believed that Euler's formula held for all polyhedra, convex or not, for which a point exists in the interior that can project the polyhedron onto a sphere in such a way that the faces of the polyhedron do not overlap when projected (Lakatos 1976, 65).

By 1813, Augustin-Louis Cauchy was able to answer Poinot's question on the number of regular polyhedra by utilizing Euler's polyhedral formula in an innovative

way. Cauchy found that the only possible regular polyhedra were the five Platonic solids, the four non-convex figures found by Poincaré and another non-convex polyhedron, the octahedron composed of two intersecting tetrahedra (Lyusternik 1966, 159). While Cauchy described a manner of projecting a polyhedron onto a particular plane, Euclid, Euler, Legendre and others had only considered polyhedra as solid figures. Figure 4 gives a projection of the cube onto the plane that is similar to the projection described by Cauchy. The vertices and edges of the cube are related to one another on the plane as they are in the 3-dimensional cube. One may notice that the face closest to the point P is essentially “lost” on the plane. However, one may consider the unbounded face on the exterior of the planar representation as the “lost” face. Cauchy began his paper by

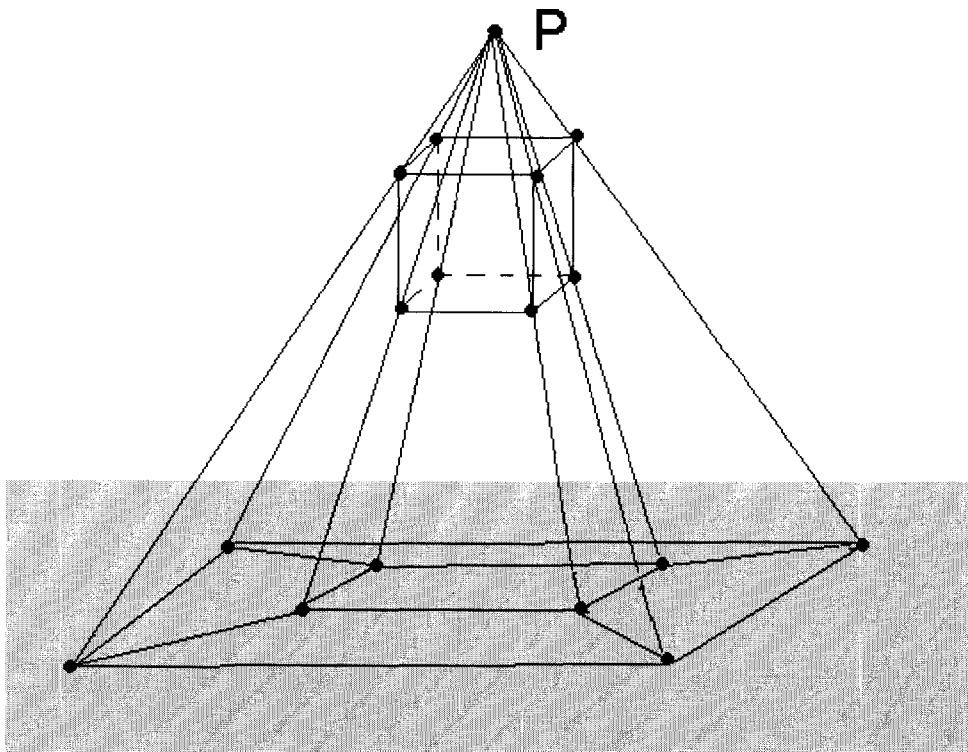


Figure 4. Cube Projected onto the Plane

developing a generalization to Euler's formula by allowing extra vertices and edges inside the polyhedron to create a set of n separate polyhedra. Cauchy considered the equation

$$V - E + F = n + 1.$$

Euler's formula then is a direct consequence of the above statement when we consider the entire polyhedron without extra vertices or edges in the interior, i.e., when $n = 1$.

Cauchy's greatest contribution however, was his insight that connected the study of Euler's formula to what we know today as the study of planar graphs. Through his projection method described above, Cauchy showed that Euler's formula is not only a formula about polyhedra, but is also a theorem concerning planar graphs (Biggs, Lloyd, and Wilson 1976, 79-83).

Cauchy's proof of the formula involved a method known as triangulation and requires the use of a **plane graph**, which is a planar graph drawn in such a way that is embedded in a plane. Notice that given a plane graph without loops or multiple edges, one can add edges between vertices without changing the value of $V - E + F$, because each additional edge also produces exactly one additional face. If by adding vertices in such a way that all of the faces are bounded by exactly three edges, the graph is called **triangulated**. Once a graph is triangulated, one can then remove triangles one by one. This can be accomplished either by removing a boundary edge or by removing two boundary edges and their common vertex. (A **boundary edge** is an edge that is on the infinite face of the graph.) In either case, a face also disappears, so the value of $V - E + F$ remains constant. One continues this method of triangle removal until a single triangle remains, and we know that $V - E + F = 1$ for a triangle. Thus, our original graph

must also have the property that $V - E + F = 1$. The technique is invalid if one chooses to remove a triangle in the interior. Figure 5 shows a graph undergoing the process of triangulation. When removing triangles 1, 2, 3, 4, and 9, a single edge is successively deleted from the graph. For triangles 5, 6, 7, 8, 10, and 11, two edges and one vertex are deleted from the graph at each stage. After the process of triangulation is complete, we are left with one last triangle, labeled 12. Notice that at each step, a triangle is removed on the exterior of the bounded region. An excellent representation of the process of triangulation and more specific instructions can be found in Lakatos (1976, 7-12).

While Cauchy was working on the paper described above, Simon-Antoine-Jean Lhuilier, a mathematics professor in Geneva, was proposing and studying several exceptions to Euler's formula. Although he published a paper on the subject in 1811, he also sent a lengthy memoir of his work to the French mathematician, J.D. Gergonne. Gergonne had founded his own journal, but Lhuilier's notes were so long that they could not be published in its pages. Gergonne took it upon himself to edit Lhuilier's work and also added his own commentary. Biggs, Lloyd, and Wilson (1976, 83-88) point out that although it was known at the time that only five regular convex polyhedra existed, the work of Lhuilier on this subject is notable for three reasons. First, his derivation of Euler's formula does not concern notions of congruence or other metrical properties. Secondly, Lhuilier noticed that the regular polyhedra can be paired in a manner that anticipates the study of duality (see Chapter III). Finally, he established a relationship between regular subdivisions of the plane and regular polyhedra with infinitely many small faces. Lhuilier determined that both contexts could be explained by triangles joined six by six, squares joined four by four or hexagons joined three by three.

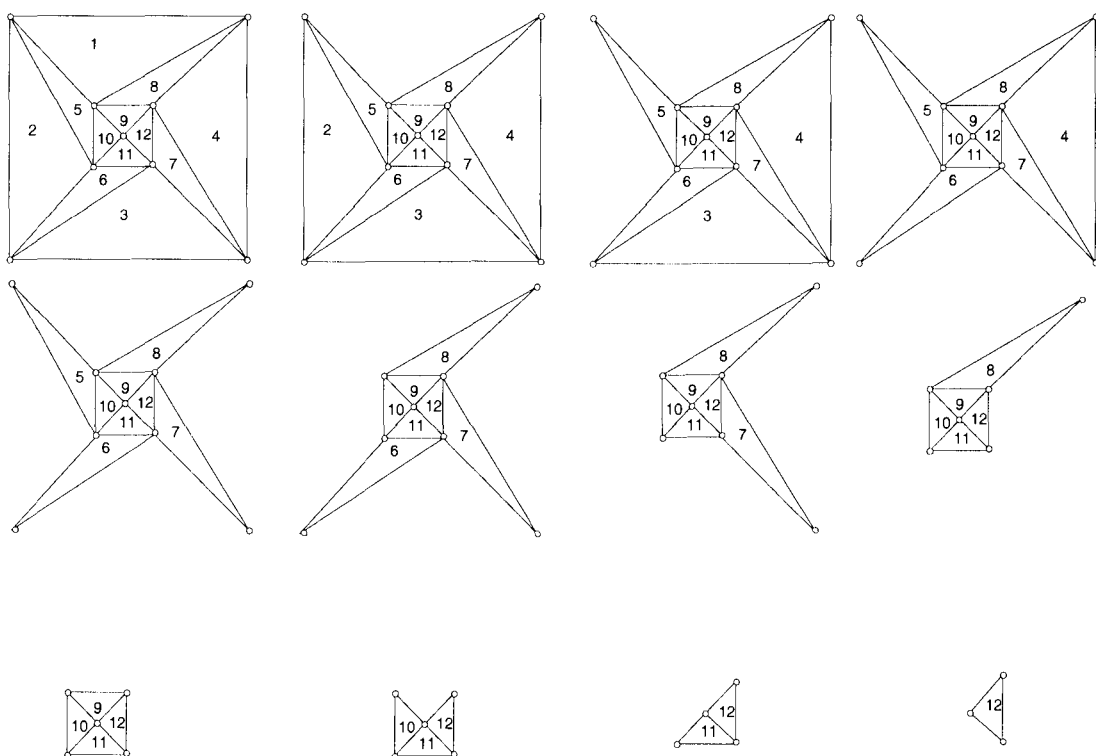


Figure 5. Cauchy's Method of Triangulation

In addition, Lhuillier gave three “counterexamples” to Euler’s formula, and made generalizations concerning each of them. However, Gergonne noted that he too had recognized the first two of these counterexamples long before reading Lhuillier’s memoir. First, Lhuillier considered polyhedra that contain internal cavities. An example, shown in Figure 6, is described by a cube contained within a cube, forming a box-like entity (Lakatos 1976, 13). Lhuillier found that for polyhedra that contain n internal closed polyhedral surfaces, an equation similar to Euler’s formula,

$$V - E + F = 2n + 2,$$

suffices to describe the relationship between vertices, edges, and faces.

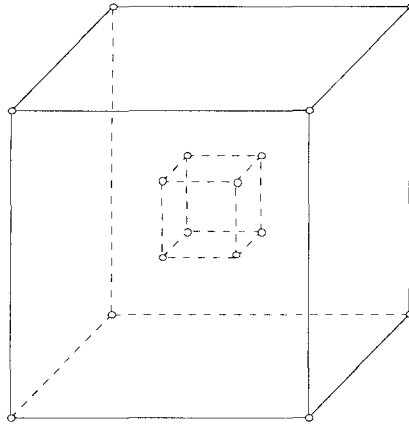


Figure 6. Lhuilier’s First Counterexample: A Cube with a Cubic Cavity

The second counterexample occurs when a polyhedron is ring-shaped, i.e., is a single surface with an opening passing through it. For instance, a cube with a rectangular tunnel drilled through the center is an example of a ring-shaped polyhedron, as shown in Figure 7. This type of figure is sometimes referred to as the “picture frame” (Lakatos 1976, 19-21). As with the case of the cavities, Lhuilier found an equation relating the number of vertices, edges, and faces; for a polyhedron pierced with n distinct openings,

$$V - E + F = -2n + 2.$$

(It is common notation to let $\eta = -2n + 2$.) For example, when a polyhedron is pierced with one distinct opening, the figure is topologically equivalent to a torus, also known as an anchor ring. In this case, $\eta = -2n + 2 = -2(1) + 2 = 0$. The implications of this counterexample are quite interesting and historically significant, as they initiated Listing’s subsequent investigation of the subject in the early 1860s, and eventually helped lead to the development of topology as a separate branch of mathematics.

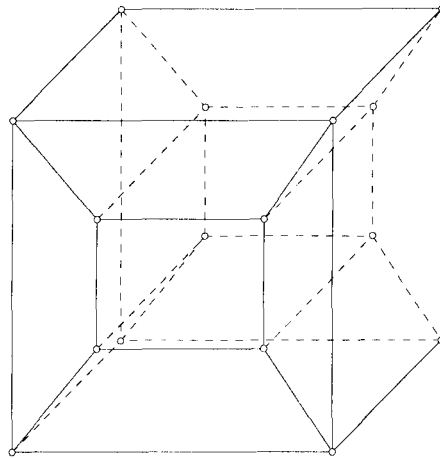


Figure 7. Lhuilier's Second Counterexample: The Picture Frame

Finally, Lhuilier's third exception can be exemplified by polyhedra with "indentations" in their faces. For example, one could consider a cube with a smaller cube indented into the top face of it, as shown in Figure 8. In this case, there are 16 vertices, 24 edges, and 11 faces for which $V - E + F = 3$ (Lakatos 1976, 34-35). Lhuilier and Gergonne were very confident that they had discussed all the possible exceptions to Euler's formula. Their certainty in this was so great that Gergonne remarked erroneously, "...the specified exceptions ... seem to be the only ones that can occur..." (Lakatos 1976, 27).

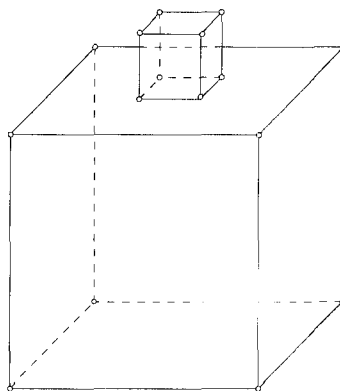


Figure 8. Lhuilier's Third Counterexample: A Smaller Cube Indented on a Cube

In editing Lhuilier's paper, Gergonne attempted to describe a way of determining which kinds of polyhedra are **Eulerian**, or in other words, those that make Euler's polyhedral formula true. He argued that if one face of a polyhedron was transparent, and if a person could look through this face and see all other faces, then the polyhedron is Eulerian. Lakatos (1976, 59-60) compared Gergonne's method to the situation where when one can take a photograph of the interior, the resulting photograph produces a two-dimensional figure for which $V - E + F = 1$. Hence, when we include the transparent face in our formulation, Euler's formula follows. Jacob Steiner independently discovered Gergonne's method in 1826.

Almost twenty years after the work of Lhuilier, F.C. Hessel, motivated by the work of Steiner, independently rediscovered many of Lhuilier's counterexamples and published a paper on the subject in 1832. Shortly after Hessel submitted his manuscript, he discovered that Lhuilier had already established many of his findings. It is ironic that both Lhuilier and Hessel's discoveries of the cavity or nested cubes counterexample were due in part to mineralogical specimens in which a double crystal was present that had a clear outer covering, but also had an inner crystal that was not translucent (Lakatos 1976, 13). Most importantly, Hessel's paper offered a new type of counterexample that had eluded Lhuilier and Gergonne. Hessel's new counterexamples are often called twintetrahedra, because they are formed by taking two tetrahedra and fusing either an edge from each solid together, (cf. Figure 9) or a vertex from each solid together (cf. Figure 10) (Lakatos 1976, 15). It is obvious that in these cases involving fused edges and vertices, $V - E + F = 3$.

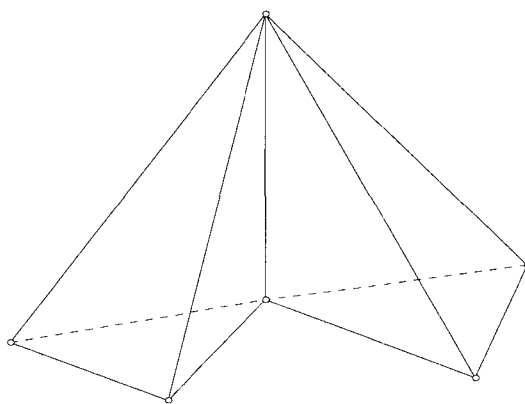


Figure 9. Hessel's Counterexample: A Fused-Edge Twintetrahedron

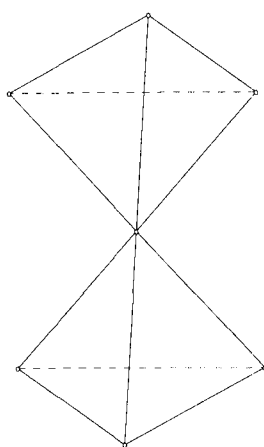


Figure 10. Hessel's Counterexample: A Fused-Vertex Twintetrahedron

In the fall of 1847, Karl Georg Christian von Staudt, a German mathematician at Erlangen, published a book entitled *Geometrie der Lage*. The book was a compilation of many years of the author's work on the study of the geometry of position. *Geometrie der Lage* developed the idea of projective geometry without reference to length or angle measure. The text, with no figures or diagrams and minimal use of notation and formulas, was not easily readable. Even the distinguished mathematician Felix Klein stated that "von Staudt's presentation was completely inaccessible to him" (Mulder 1988, 28). However, von Staudt's work turned out to be very influential, as he gave a correct

hypothesis for the general case in which Euler's formula holds. Von Staudt's theorem states:

“Let P be a polyhedron with vertex-set V , edge-set E , and set of faces F . If P satisfies the following conditions:

- (i) every vertex of P is joint to every other vertex by an edge or by a line consisting of edges put together,
 - (ii) the surface of P is divided into two parts by any closed line, consisting of edges put together, that does not pass more than once through any vertex,
- then we have $|V| + |F| = |E| + 2$ ” (Mulder 1988, 29).

In modern terms, von Staudt was essentially stating that Euler's formula in its general form is valid for polyhedra that are simply connected with simply connected faces.

This definition of suitable polyhedra can account for the exceptions to Euler's formula exemplified by such “monsters” as the nested cubes and the picture frame. In the late nineteenth century, Jonquières emerged as one of the greatest advocates of Euler's formula. He argued that the nested cubes counterexamples offered by Lhuillier and Hessel were not truly polyhedra, but instead represented two distinct polyhedra. Implicitly, Jonquières redefined an **Eulerian polyhedron** as “a surface consisting of a system of polygons.” (Lakatos 1976, 14). In the case of the nested cubes, there are two surfaces on the polyhedron. In an attempt to account for the picture frame, he added implicitly “through any arbitrary point in space, there will be at least one plane whose cross-section with the polyhedron will consist of one single polygon” (Lakatos 1976, 21). In the case of the picture frame, if you choose a point on the interior of the tunnel, this point will have no plane that yields a polygonal cross-section of the polyhedron. Obviously, this new definition of an Eulerian polyhedron was proposed to account for the earlier counterexamples such as the nested cubes and the picture frame. Similarly, Möbius offered a definition in 1865 that restricts the twintetrahedra described by Hessel from

being considered polyhedra. In his definition, Möbius states that a Eulerian polyhedron is
a

“system of polygons arranged in such a way that (1) exactly two polygons meet at every edge and (2) it is possible to get from the inside of any polygon to the inside of any other polygon by a route which never crosses any edge at a vertex”
(Lakatos 1976, 15).

Like Jonquières, Matthiesen, who authored a paper on polyhedra in 1863, was confident that one could retain past definitions of polyhedra that still satisfied Euler’s formula. He allows that polyhedra may have hidden faces and edges, and if these hidden entities are counted, then Euler’s formula would also be valid for solids with tunnels and cavities. Although Matthiesen was a staunch supporter of this idea, he was not its originator, as Hessel previously had discussed this concept in his 1832 paper (Lakatos 1976, 38-39).

Crelle, in 1826-27, extended Cauchy’s work on triangulation by noticing that this method also can be applied to polyhedra with “bent faces,” although Crelle did insist that only straight edges be used. During the 1860s, several notable mathematicians, including Cayley (1861), Listing (1861), and Jordan (1866) independently found that the triangulation procedure also can be applied to polyhedra with curved edges (Lakatos 1976,89).

Finally, a Frenchman, Jules Henri Poincaré, would settle the long debate concerning Euler’s formula. Poincaré used many of the ideas that Listing had set forth in his works *Vorstudien zur Topologie* and *Der Census Räumliche Complexe* (*The Census of Spatial Complexes*). In these writings, Listing defined objects that he called “complexes,” because they had been created from smaller pieces. He examined their properties and in his most significant theorem related the numbers of vertices, edges, faces, and

subcomplexes (Mulder 1988, 33). In a collection of papers dating from 1895-1904, Poincaré demonstrated a way of creating these complexes from basic cells, which he called 0-cells (vertices) and 1-cells (edges). Using the work of Gustav Robert Kirchhoff in the field of physics, Poincaré modified Kirchhoff's techniques that described electrical networks by using matrices in place of linear equations (Wilson 1999, 516).

In 1858, Listing and Möbius independently discovered the one-sided surface known today as the "Möbius strip." Its two-fold counterpart, the Klein bottle, was described in 1910 in a paper by Heinrich Tietze. In Tietze's discussion of this object, he referred to the Klein bottle as a "closed two-fold one-sided surface" (Biggs, Lloyd, and Wilson 1976, 124-129). The Klein bottle contains a crosscap, created by removing the interior of a disk on the surface of a sphere and then identifying opposite points on the boundary (White & Beineke 1978, 17). Surfaces such as the Möbius strip and the Klein bottle are called **non-orientable**, because there exists a closed Jordan curve on each of these surfaces such that the rotation direction is not preserved as one goes around the curve once (Aigner 1987, 18). In layman's terms, an orientable surface is one that is two-sided, like a sphere or torus, while a non-orientable surface "has only one side." One can consider an orientable surface to be topologically equivalent to a sphere to which a certain number of handles have been attached. A handle can be produced by removing two disjoint disks from the surface of a sphere and joining the two disks with a truncated cylinder. Similarly, a non-orientable surface can be formed by adding a certain number of crosscaps to the surface of a sphere. Suppose S_h describes an orientable surface with h handles and N_k describes a non-orientable surface with k crosscaps. In a 1923 paper, Brahana proved that every surface is topologically equivalent to either S_h for some $h \geq 0$

or N_k for some $k > 0$ (White and Beineke 1978, 17). If we consider the value of $V - E + F$ of a surface S to be **Euler's characteristic**, $e(S)$, then the Euler-Poincaré formula can be stated as follows for both orientable and non-orientable surfaces, respectively

$$\begin{aligned} e(S_h) &= 2 - 2h & h &\geq 0 \\ e(N_k) &= 2 - k & k &\geq 1 \end{aligned}$$

(Aigner 1987, 20). Poincaré's work was considered to be "an instant success" (Wilson 1999, 516) and was later extended by the efforts of Oswald Veblen during a series of lectures he delivered to the American Mathematical Society (Biggs, Lloyd and Wilson 1976, 135-136).

Although Poincaré's work was more topological in nature, it also contained an abundance of applications in graph theory. Poincaré laid a basis for later work determining what kinds of graphs may be embedded on surfaces not homeomorphic to the sphere. As Chapter IV will discuss, in 1968, the Heawood conjecture regarding map coloring on surfaces was also proved using the Euler-Poincaré formula. In general, many direct consequences of Euler's polyhedral formula are utilized throughout the field of graph theory, and some of these consequences will be discussed in the next chapter regarding characterizing planar graphs.

CHAPTER III

CHARACTERIZATION OF PLANAR GRAPHS

One of the most significant studies in the early history of planar graphs was the attempt to characterize which graphs are planar and which are not. As with many graph theory problems, the characterization question's roots can be found in recreational mathematics, as well as a number of applications to electric circuit boards. Two planarity problems, one by Möbius and another by Dudeney, would prove to be especially significant in leading up to Kuratowski's characterization of planar graphs in the late 1920s. Kuratowski's Theorem would inspire and enlighten many other individuals to discover other characterizations of planar graphs.

Puzzles of Planarity

One of the earliest questions concerning planarity was presented by Möbius during a lecture in about the year 1840, where Möbius presented the following problem:

“There was once a king with five sons. In his will he stated that after his death the sons should divide the kingdom into five regions so that the boundary of each region should have a frontier line in common with each of the other four regions. Can the terms of the will be satisfied?” (Wilson 1999, 516-517).

This question can be rephrased using the geometric dual of the land regions. The **geometric dual** of a map G , denoted G^* , can be formed by replacing each face of G with a vertex and connecting vertices if the faces of G share an edge (cf. G and G^* in Figure

13). Then, the question of Möbius can be posed in a graph theoretic context as the problem of asking if each of the five sons has a road connecting his capital city to all his other brothers' capital cities in such a way that no two roads intersect. The problem of the five princes can be solved if the complete graph of five vertices, denoted K_5 , as shown in Figure 11, is a planar graph. A **complete graph on n vertices**, denoted by K_n , is a graph that has any two vertices in the graph connected with an edge.

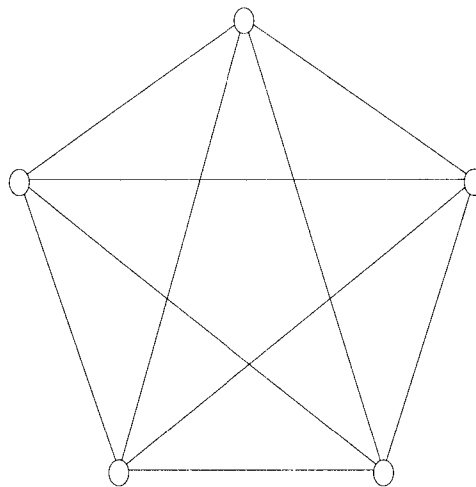


Figure 11. The Complete Graph of Five Vertices, K_5

A graph G is called a **maximal planar graph** if adding an edge between any two nonadjacent vertices of G results in a nonplanar graph. In general, maximal planar graphs have faces consisting entirely of triangles, because if a graph has a face with more than three bounding edges, then a diagonal can be drawn. Thus, the graph would remain planar. Using the consequences of Euler's polyhedral formula, one can prove the following theorem: If G is a maximal planar graph with V vertices and E edges where $V \geq 3$, then $E = 3V - 6$.

Proof: Since G is a maximal planar graph, then all its faces are bounded by 3 sides. Consequently, each edge is formed by two faces connecting. Thus,

$3F = 2E$, where F is the number of faces of G . This implies $F = \frac{2}{3}E$. By Euler's

polyhedral formula, we know that $V - E + F = 2$. Thus by substitution,

$V - \frac{1}{3}E = 2$, and by solving for E , we find $E = 3V - 6$. Q.E.D.

A direct consequence of the preceding theorem is that for any planar graph, $E \leq 3V - 6$, which was stated by Euler in 1750 in the aforementioned letter to Goldbach. (Hartsfield and Ringel 1994, 152-153). Notice that the graph K_5 has 5 vertices and $\binom{5}{2} = 10$ edges.

Because $10 = E > 3V - 6 = 9$, the graph K_5 is not planar. Thus, the terms in the will in Möbius's problem cannot be satisfied.

In 1917, another significant puzzle about planarity first appeared in a book entitled *Amusements in Mathematics* by Henry Ernest Dudeney of England:

“There are some half-dozen puzzles, as old as the hills, that are perpetually cropping up, and there is hardly a month in the year that does not bring inquiries as to their solution. Occasionally, one of these, that one had thought was a distinct volcano, bursts into eruption in a surprising manner. I have received an extraordinary number of letters respecting the ancient puzzle that I have called ‘Water, Gas, and Electricity.’ It is much older than electric lightning or even gas, but the new dress brings it up to date. The puzzle is to lay on water, gas, and electricity, from W, G, and E, to each of the three houses A, B, and C, without any pipe crossing one another. Take your pencil and draw lines showing how this should be done. You will soon find yourself landed in difficulties.” (Dudeney 1958, 73).

Dudeney's “utilities problem” has been described in a number of equivalent ways, such as the bad neighbors problem (also known as the houses and wells problem), the Corsican vendetta problem, or the Persian caliph's problem (Kullman 1979, 299-300).

Although Dudeney claims this puzzle is “as old as the hills,” no substantiation of this claim has been found, as Dudeney’s 1917 book offers the first written record of it being posed. American Sam Loyd, Jr. claims that his father “brought out” the problem in 1900, but Loyd did not claim that his father originated the problem (Biggs, Lloyd, and Wilson 1976, 142). Many historians of mathematics, however, believe that the origins date back at least to the early 1800s.

Like the problem of the king and his five sons, it is impossible to draw a planar graph to represent the utilities. The utilities problem is one that involves a complete bipartite graph. The **complete bipartite graph**, $K_{m,n}$, has two sets of vertices, one with m vertices and the other with n vertices, where each vertex in the first set is connected to every vertex in the second set by an edge. Using an argument similar to that used in establishing the theorem above for K_5 , one can conclude that if G is a planar bipartite graph with V vertices and E edges where $V \geq 3$, then $E \leq 2V - 4$. The only difference in the proof is that instead of using triangles, the region with the fewest possible sides is a quadrilateral. Using this argument, one can show that $K_{3,3}$ (cf. Figure 12), the graph described by the utilities problem, is not planar, in that $9 = E > 2V - 4 = 8$ (Hartsfield and Ringel 1994, 153).

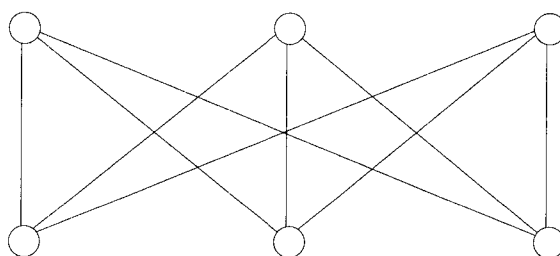


Figure 12. The Complete Bipartite Graph $K_{3,3}$

In a book published in 1926 and reprinted in 1967, Dudeney (1967, 153-154) reflected on the problem of the utilities:

“I think I receive, on an average, about ten letters a month from unknown correspondents respecting this puzzle which I published some years ago under the above title [Water, Gas, and Electricity]. They invariably say that someone has shown it to them who did not know the answer, and they beg me to relieve their minds by telling them whether there is, or is not, any possible solution. As many of my readers may have come across the puzzle and be equally perplexed, I will try to clear up the mystery for them in a more complete way than I have done in *Amusements in Mathematics*.”

In the “Answers” section of the book, Dudeney explains further that the solution can only be found by means of a trick. He admits that if a householder allows one of the utility companies to pass a pipe through his house, then the problem can be solved easily. By connecting the three utilities to two houses, he argued by elimination that there is no position one can place the third house to be able to draw lines to all three utilities without utilizing the trick.

The importance of the graphs of K_5 and $K_{3,3}$ utilized in solving the problems of Möbius and Dudeney will be discussed later in this chapter in detailing the work of Kazimierz Kuratowski, as well as many others. The notation of using the letter K to describe complete graphs and complete bipartite graphs is attributed to Frank Harary. When they met at a conference in Rome in 1973, Kuratowski questioned Harary as to why he had chosen to use the letter K to describe these graphs. Harary replied, “Well, the K in K_5 stands for Kazimierz and the K in $K_{3,3}$ for Kuratowski!” (Harary 1981, 218).

Kuratowski’s Theorem

Harary’s tribute to Kuratowski, described above, is well deserved, as he is widely recognized as the first person to correctly characterize planar graphs. Kuratowski, the son

of a famous Warsaw lawyer, was born in 1896. His first mathematical paper was published in 1918, and he graduated from Warsaw University one year later. In his 1921 doctoral dissertation and in subsequent works, Kuratowski studied the topology of the plane intently (Krasinkiewicz 1981, 221-222). Subsequently, as an associate professor at the Lwow Polytechnical University, Kuratowski developed and published his now famous characterization of planar graphs. In 1929, Kuratowski announced (and one year later he published) a proof of his famous theorem: “A graph is planar if and only if it does not contain a subgraph homeomorphic to either K_5 or $K_{3,3}$ ” (Kennedy, Quintas and Syslo 1985, 356). Two graphs are **homeomorphic** if one can be obtained from the other by a sequence of subdivisions of edges. Kuratowski initially announced his results to the Polish Mathematical Society on June 21, 1929. In 1930, his paper “Sur le problème des courbes gauches en topologie” or “On the topological problem of non-planar curves” containing his proof appeared in *Fundamenta Mathematica*. At the same time Kuratowski was working on this paper, the American mathematicians Orrin Frink and Paul A. Smith were independently working on proving the same theorem. An abstract of Smith and Frink’s work was published in the *Bulletin of the American Mathematical Society*, but, after Kuratowski’s proof appeared in *Fundamenta Mathematica*, their paper was rejected by the *Transactions of the American Mathematical Society*, the journal they had hoped to publish it in (Biggs, Lloyd, and Wilson 1976, 147-148).

Similar to the controversy surrounding the discovery of calculus by Newton and Leibniz, in regards to the theorem that is widely known today as Kuratowski’s, there are some mathematicians who believe credit for being the first to prove it should be given to a second individual. In a footnote to his paper, Kuratowski explained that P.S.

Aleksandrov had told him that L.S. Pontryagin had proved the result earlier, but had never published it. Pontryagin was a second-year student at Moscow State University in the winter of 1927-1928 when Aleksandrov attempted to give a proof by Kuratowski of the planar characterization that was invalid, because he originally had used only one of the two forbidden subgraphs. Pontryagin noticed the error and corrected it, but never published his findings, because Aleksandrov wanted him to extend his work further. Kuratowski admitted that he had initially thought that only K_5 would prevent planarity, but only later discovered the need to include $K_{3,3}$ as well. It is not certain whether Aleksandrov communicated Pontryagin's findings to Kuratowski or whether Kuratowski discovered his error on his own. It is difficult for one to assess now, given that almost all of Kuratowski's correspondence was destroyed during World War II. R. Engelking, a close associate of Kuratowski, believed that Kuratowski later regretted crediting Pontryagin with the theorem because Pontryagin never published his work (Kennedy, Quintas, and Syslo 1985, 361-363). During the 1960s, a number of authors began to refer to the theorem as the Pontryagin-Kuratowski theorem, especially in the Soviet Union. The origin of this usage began with A. A. Zykov's Russian translation of a book by C. Berge in 1962. Although Pontryagin may have given a proof of the theorem prior to Kuratowski, it was certainly Kuratowski who first published the finding, and he continues to be the single person credited with first establishing it according to most sources. The article by Kennedy, Quintas, and Syslo (1985, 356-368) gives a more thorough account of the origins of Kuratowski's Theorem and the addition of Pontryagin's name to the theorem in the mathematical literature.

The importance of Kuratowski's Theorem is not its applications to graph theory, but the fact that planar graphs can be characterized by the exclusion of a finite number of subgraphs (Thomassen 1981, 225). This concept was extended in the late 1970s when Glover, Hueke and Wang produced a list of 103 "forbidden subgraphs" of graphs that can be embedded in the projective plane. Robertson and Seymour generalized the concept in 1985 by proving that a finite list of forbidden subgraphs exists for surfaces of any genus, although the list may be lengthy (Biggs, Lloyd, and Wilson 1995, 2178). Kuratowski's theorem also yields a useful characterization for nonplanar graphs: all nonplanar graphs must have a subdivision of K_5 or $K_{3,3}$. Many proofs of Kuratowski's Theorem can be transformed into planarity testing algorithms that are calculable in polynomial time, such as the algorithm described by Hopcroft and Tarjan in 1974. However, one of the most important aspects of Kuratowski's Theorem is that it can be used to determine other criteria for planarity, such as those described by Whitney, Wagner, and MacLane (Thomassen 1981, 225-226).

Whitney's Duality Theorem

Prior to Kuratowski's characterization of planar graphs, Dénes König stated in 1916, that a characterization involving duality might be necessary to further progress on proving the four-color conjecture that will be addressed in Chapter IV. One may note that in a geometric dual, a one-to-one correspondence exists between the edges of G and the edges of G^* . Also, if G is a connected graph, then $G = (G^*)^*$ (Biggs, Lloyd, and Wilson 1976, 148-149).

Hassler Whitney, a young American who studied physics in Germany, became interested in the four-color problem in the late 1920s. He wrote about a dozen papers on graph theory between 1930 and 1935. Whitney established an important combinatorial relationship between the graphs of geometric duals. He formulated an abstract notion of duality, often known as combinatorial duality that is equivalent to the concept of geometric duality (Harary 1969, 114-115). Whitney's first account of combinatorial duality appeared in 1931, and a more thorough treatment of it was published in 1932. Most notably, Whitney offered his characterization of planar graphs: A graph is planar if and only if it has a combinatorial dual (Biggs, Lloyd, and Wilson 1976, 148-157). Defining V , E , and F as usual and P as the number of components of a graph G , the **rank** or **cocycle rank**, $R = m^*(G)$, is defined by

$$R = m^*(G) = V - P.$$

The **nullity** or **cycle rank**, $m(G)$, is given by

$$m(G) = E - R = E - V + P.$$

The **relative complement**, GVH , of a subgraph H of G is created by deleting the edges of H in G . One calls G^* a **combinatorial dual** of a graph G if a one-to-one correspondence exists between the edges of G and G^* in such a way that for any choice Y and Y^* of corresponding subsets of edges,

$$m^*(G \setminus Y) = m^*(G) - m(<Y^*>),$$

where $<Y^*>$ is the subgraph of G^* with edge set Y^* (Harary 1969, 114-115). A basic example is shown in Figure 13, where $m^*(G \setminus Y) = 5 - 2 = 3$, $m^*(G) = 5 - 1 = 4$, and $m(<Y^*>) = 4 - 5 + 2 = 1$, and so the equation for combinatorial duality holds. In Figure

13, the one-to-one correspondence of the edges of G and G^* is shown through the use of corresponding subscripts.

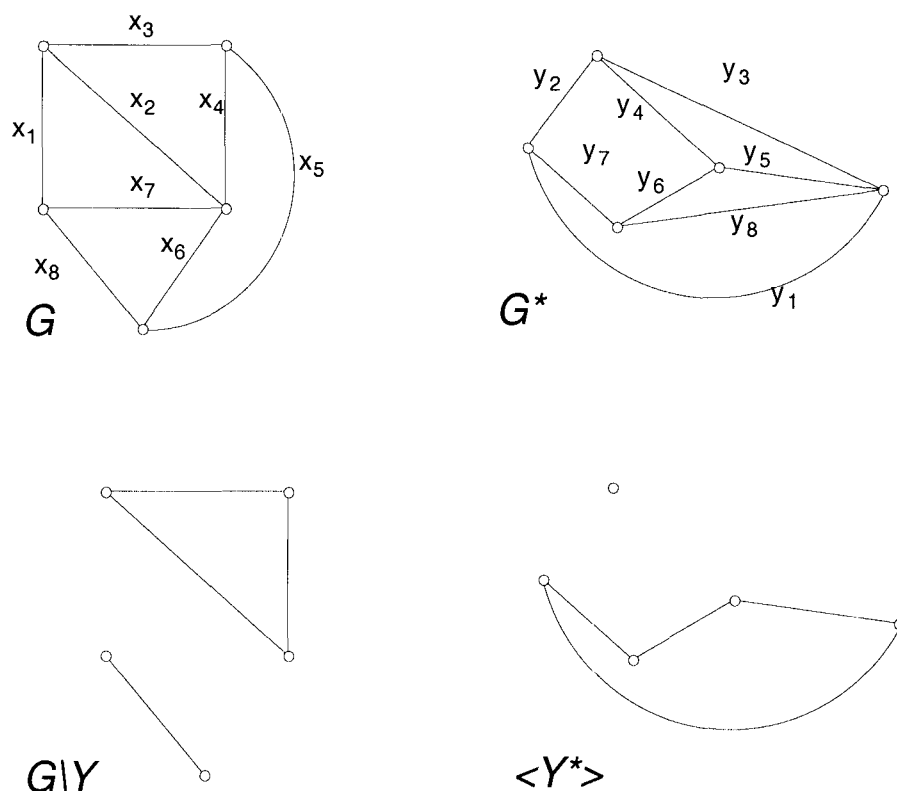


Figure 13. A Graph: G , the Geometric Dual of G : G^* ,
a Relative Component of G : $G \setminus Y$, and the Subgraph of G^* with Edge Set Y^* : $\langle Y^* \rangle$

Although Whitney gave an independent proof of his characterization theorem, at the end of his 1932 paper, he stated Kuratowski's Theorem and noted that neither K_5 nor $K_{3,3}$ had a combinatorial dual. It follows that any graph that has a Whitney dual must be nonplanar. One year later, in 1933, Whitney established the converse result and thus gave a new proof of Kuratowski's theorem (Biggs, Lloyd, and Wilson 1976, 157).

The Work of Wagner

K. Wagner found yet another characterization of planar graphs that sometimes appears in textbooks as a corollary to the theorems of Kuratowski and Whitney. Before discussing Wagner's theorem, it is necessary to establish some additional terminology. An **edge contraction** (cf. Figure 14) of a graph G can be obtained by deleting two adjacent vertices, u and v , and adding a new point, w , that is adjacent to those vertices that were adjacent to either u or v . A graph G is said to be **contractible** from a graph H if it can be obtained from H through a sequence of edge contractions. A **minor** of a graph is a subgraph of a contraction. In 1937, Wagner found the following important characterization of planar graphs: "A graph is planar if and only if it does not have a subgraph contractible to K_5 or $K_{3,3}$." (Harary 1969, 112-113). Wagner's theorem was later discovered independently by Harary and Tutte and described by them in a 1965 article.

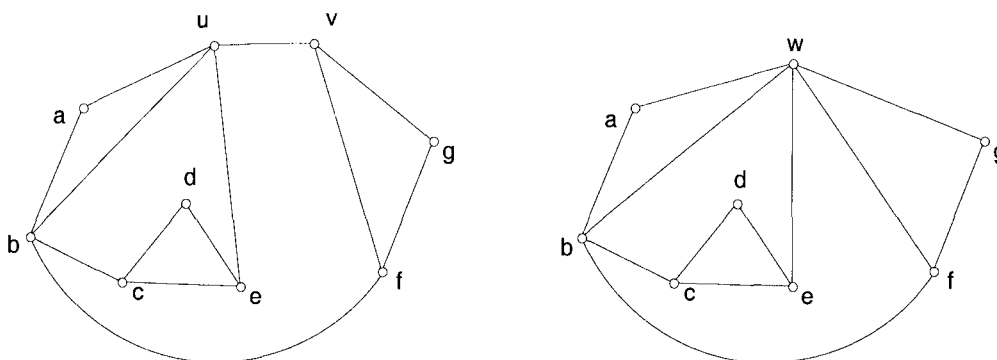


Figure 14. An Example of an Edge Contraction

Wagner's characterization of planar graphs was not his first contribution to the study of planar graphs. In 1936, he established that any planar graph G can be drawn in the plane where all the edges of G are straight lines. This important theorem is often

credited to Fáry who independently proved it (Hartsfield and Ringel 1994, 168-170).

However, Fáry's proof was not published until 1948, while Wagner's proof appeared in 1936.

MacLane's Characterization

S. MacLane offered another criterion for planar graphs in 1937. There are a number of ways that MacLane's characterization of planar graphs may be described. Succinctly, the theorem can be stated as "A non-separable graph is planar if and only if it has a set of circuits with the property that each edge of the graph lies in exactly two of the circuits" (Biggs, Lloyd, and Wilson 1976, 210). A graph is called **non-separable** if it is connected and cannot be disconnected by removing a single vertex. When such a vertex exists, it is called a **cut vertex**. For a more detailed version of MacLane's theorem involving the notion of a cycle basis, consult Harary (1960, 127-128) and Aigner (1986, 66-67).

Although the four characterizations listed above are certainly not the only characterizations of planar graphs. For example, in 1989, Walter Schnyder gave a characterization of planar graphs involving order dimension of posets (See Schnyder 1989, 232-343). Many of the origins and applications of the characterizations described in this chapter are related to the study of the four-color conjecture and other map coloring problems, the topic of Chapter IV. The geometric dual is particularly useful, where, as in the problem of the five princes, maps can be transformed into graphs.

CHAPTER IV

COLORING MAPS AND SURFACES

The four-color problem is arguably the most influential question in the development of graph theory. Aigner (1986, vi-vii) believed that “the 4-color problem almost alone permitted an entire discipline, graph theory, to arise as rarely occurs to this extent.” Many mathematicians of the nineteenth and twentieth centuries have sought to prove the conjecture. Perhaps part of the interest in the problem is its simplicity, because the concept can be understood by almost anyone. However, it is notoriously difficult to solve. The four-color “theorem” is known today as the following: “The countries (faces) of any map can be colored with four colors in such a way that neighboring countries are differently colored” (Wilson and Watkins 1990, 228). Equivalently, by the use of duality, the problem can be stated in a graph theoretic terminology as “the vertices of any connected planar graph can be colored with four colors in such a way that adjacent vertices are differently colored” (Wilson and Watkins 1990, 229). The four-color conjecture remained unproved for over 120 years, and its “proof” required the use of computer analysis, creating controversy among mathematicians. Generalizations of the four-color theorem also emerged for graphs on surfaces of higher genus. As this chapter will demonstrate, the four-color theorem and its generalizations have an interesting and unique history.

Origins of the Four-Color Theorem

At least two common myths that can be traced back to W.W.R. Ball's 1892 edition of *Mathematical Recreations and Essays* surround the origins of the four-color problem. Some individuals have attributed the problem's origins to the five princes problem of Möbius. However, the problem of the five princes and its proof simply show that five countries cannot have pairwise borders. Möbius demonstrated the impossibility, but did not generalize the concept to maps with more than five countries. No publication of Möbius exists on the problem, and it is unlikely that Francis Guthrie had heard of Möbius' work when he made his now famous conjecture (Holton and Purcell 1979, 11). A second common myth is that cartographers had known of the four-color property for many years. Kenneth May (1965, 346) studied a number of atlases in the Library of Congress to conclude that this myth is false. In his research, he found "no tendency to minimize the number of colors used." In fact, he found that the use of four colors was rare, and moreover that most of the maps that were colored with four colors only required three. Although problems relating to map coloring can be found in some books on the history of cartography, none of them mention the four-color property. The result would probably not interest mapmakers because even prior to the development of printing, it was easy to use many colors. Furthermore, since the invention of printing, colors may be applied one on top of another to create many additional colors, and hatching and shading may also be used to differentiate regions of maps. It is widely believed today that the four-color problem cannot be directly attributed to Möbius or to cartographers.

The first written record of the four-color problem can be found in a letter from Augustus De Morgan to Sir William Rowan Hamilton in October 1852. In the letter, De

Morgan, a professor at University College in London, explained that one of his students had suggested that “if a figure be anyhow divided and the compartments differently colored so that the figures with any portion of common boundary line are differently colored, four colours may be wanted, but not more...” (Holton and Purcell 1979, 11). De Morgan attempted to draw a map that needed five colors, but was unsuccessful. Uncertain of the veracity of the statement, De Morgan inquired of Hamilton if he could provide an explanation. Hamilton, obviously not interested, replied, “I am not likely to attempt your ‘quaternion of colors’ very soon” (Biggs, Lloyd, and Wilson 1976, 92).

The student referred to in De Morgan’s letter was Frederick Guthrie. He had learned of the problem from his brother Francis Guthrie, who later served as a professor of mathematics at the South African University in Cape Town. Although both brothers had studied under De Morgan, Francis had stopped attending classes; thus it was Frederick who proposed the four-color problem to De Morgan. The conjecture was discovered by Francis while coloring a map of the counties of England. He easily showed that the use of four colors was necessary, but was unable to produce a valid proof for sufficiency (Holton and Purcell 1979, 11).

Besides a few letters written by De Morgan during the 1850s, there would be little written on the four-color problem over the next twenty years. One person De Morgan corresponded with was William Whewell. A review of a book by Whewell appeared in the April 14, 1860, edition of *Athenæum* and included an anonymous reference to the four-color problem. It is likely that Charles Sanders Pierce, an American logician and philosopher, read the review and was inspired to begin studying the problem (Fritsch and Fritsch 1998, 11-20). He attempted to prove the four-color conjecture to the members of

the mathematical society at Harvard during the 1860s, but did not publish the work. In a paper dated 1869, Pierce connected the map-coloring problem to his “logic of relatives” (Biggs, Lloyd, and Wilson 1976, 92).

The problem was reintroduced to the mathematical world during the summer of 1878, when Arthur Cayley asked the London Mathematical Society whether the problem had been solved. The following year, Cayley wrote a short analysis of the problem in the *Proceedings of the Royal Geographic Society* (Fritsch and Fritsch 1998, 13). In his paper, Cayley questioned whether a sufficient, finite number existed for all maps, and suggested that there might be maps that require an extremely large number of colors (Biggs, Lloyd, and Wilson 1976, 93).

The First “Proof”

After Cayley had revived interest in the map-coloring problem, one of his students, Sir Alfred Bray Kempe, discovered a unique, although flawed, “proof” of the four-color theorem. Kempe, a London barrister, announced his findings without proof in the British journal, *Nature*. His first published “proof” of the theorem appeared in the newly founded *American Journal of Mathematics* in 1879 (Mitchem 1981, 110). William Edward Story, an associate editor of the journal attached a few addenda to Kempe’s article detailing some special cases that Kempe had not mentioned. Story then presented the paper with the addenda to the Scientific Association at Johns Hopkins University in November 1879. Among those present at the meeting was C.S. Pierce who was then a visiting faculty member at Johns Hopkins. Pierce addressed the four-color problem at the next meeting of the association in December. He explained his previous work on the

four-color problem, and stated that he believed Kempe's proof could be improved by using rules of logic. However, he did not refute Kempe's arguments, and so in 1879, the four-color theorem was thought to be proved (Fritsch and Fritsch 1998, 15-16).

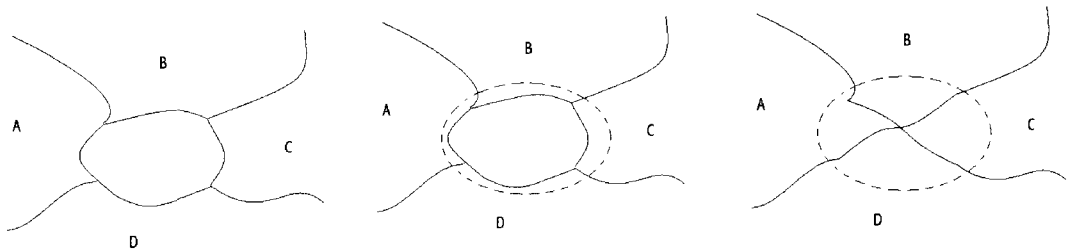


Figure 15. Using a Patch to Reduce a Map

Although Kempe's work contained a crucial flaw that will be discussed later, many concepts that he introduced would serve as integral contributions to further work on the four-color problem and its eventual solution. One of the significant ideas contributed by Kempe was the process of reducing and developing maps. One **reduces** a map by placing a patch (shown in Figure 15 by a dotted line) around a single region, and then joining the boundaries of the neighboring regions (in Figure 15 – regions A, B, C, and D) in such a way that the neighboring regions meet at a point in the interior of the patch. By consecutively patching one region after another, the map can eventually be reduced to one single region with no boundary lines or points of intersection. The opposite of reducing is **developing**, which can be demonstrated by removing the patches in reverse order. In this way, the original map can be developed one region at a time by removing successive patches. Kempe's basic goal was to try to show through mathematical induction that if a map can be colored with four colors at any step in the process of development, then it can be colored with four colors at the next step. His basis was trivial, since a map with a single region is not only four-colorable, but also one-

colorable. If the patch that is to be removed borders less than four regions, then it is obvious that the map developed by removing the patch can be four-colored as well. Thus, Kempe restricted his cases to those patches with more than three neighboring regions. Using an unnecessarily long proof, Kempe showed that every map contains a country with less than six neighbors, so he restricted his cases further to those which had patches with only four or five neighbors (MacKenzie 1999, 18-19).

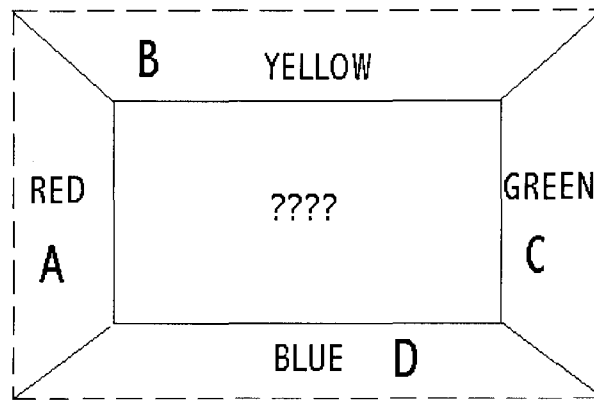


Figure 16. Kempe's Chain Argument

Kempe's argument considered a map in which all but one of the regions had been colored with four colors. Figure 16 shows a part of the plane containing the uncolored region of the map and its neighboring regions. One may assume that the uncolored region borders a region labeled with each of the four colors, because otherwise one could color the uncolored region with an unused color. One may begin by considering only the red and green regions of the entire map. Obviously, one of two cases must occur. In the first case, regions A and C are connected by means of a red-green chain of regions, or in the second case regions A and C are not connected through a link of red-green chains. The

latter case is easier to remedy, given that one may interchange the red and green regions that are connected to region A. Thus, A would be colored green and the uncolored region could be colored red. When a red-green chain of regions does connect A and C, interchanging the two colors would be unproductive, because the uncolored region would still be adjacent to both a red region and a green region. Therefore, further methods must be applied. Using the fact that our map is on a plane, we know that if there is a red-green chain connecting region A to region C, then there cannot be a blue-yellow chain connecting region B to region D. Thus, we can interchange the regions of the blue-yellow chain that includes region B. Now, the uncolored region may be colored yellow, because both regions B and D are blue. This kind of argument is known in modern graph theory as “the method of Kempe-chains” (Biggs, Lloyd, and Wilson 1976, 94-95). Kempe demonstrated a similar, but more complicated method for a patch that had five neighbors, and unfortunately it was this argument that would later be proved to be faulty.

Following Kempe’s “proof” of the four-color conjecture, a number of other individuals presented supposed “proofs” of the “theorem.” One notable “proof” was written by the British physicist Peter Guthrie Tait in 1880. Tait’s explanation, which appeared in the *Proceedings of the Royal Society of Edinburgh*, was basically nothing but a reformulation of the problem. Lewis Carroll, the famous author, devised a game involving four-coloring maps. Subsequently, a headmaster at a boys’ school in England assigned the four-color problem to his class in 1886 with the stipulation that “no solution may exceed one page, 30 lines of manuscript, and one page of diagrams.” Today, of course, this task is known to be insurmountable, since the widely known proof of Appel and Haken requires over 100 pages and still involves hundreds of hours of computer

time. An 1889 article in the *Journal of Education* contained yet another “proof” of the four-color theorem by Frederick Temple, the Bishop of London, who later was appointed Archbishop of Canterbury (Mitchem 1981, 110). Although many individuals thought that their proofs were infallible, none of them would stand up to strict scrutiny, including the infamous “proof” of Kempe (MacKenzie 1999, 22).

Heawood Discovers Kempe’s Mistake

For over ten years, Kempe’s “proof” was thought to be valid. Impressed with his “proof,” Cayley and other mathematicians proposed that Kempe be elected a Fellow of the Royal Society, and Kempe later held the positions of vice president and treasurer of the Society. In 1890, the thoughts concerning Kempe’s “proof” shifted when Percy John Heawood, a lecturer at Durham College, revealed a fallacy in Kempe’s chain argument. Kempe had incorrectly analyzed a particular case where five regions surrounded an uncolored region. One may consider the part of the map shown in Figure 17, and suppose that a blue-yellow chain of regions connects region B to region E. Further, suppose that a green-blue chain of regions connects region C to region E. Then, the existence of these two chains implies that regions A and C belong to different red-green regions and regions B and D belong to different red-yellow regions. Using Kempe’s chain method, one would interchange the colors of region A’s red-green region and interchange region D’s red-yellow region. Thus, there would be no region colored red that is adjacent to the uncolored region. However, the possibility exists that a green region of A’s red-green region and a yellow region of D’s red-yellow region are adjacent to one another. Thus by interchanging both regions’ colors, two adjacent regions would be colored red,

contradicting the requirement that all adjacent regions be colored differently. Heawood stated that one of the region's colors could always be interchanged, but the first transposition prevented the second transposition from being effective. Kempe himself reported Heawood's findings to the London Mathematical Society, and stated that his own efforts to rectify the original proof had failed (Biggs, Lloyd, and Wilson 1976, 105-108).

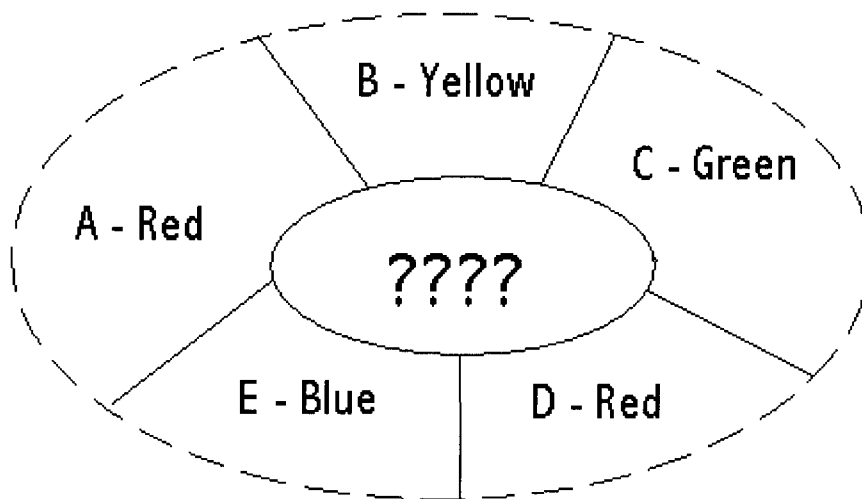


Figure 17. Refutation of Kempe's Chain Argument

It might seem that Heawood's study of the four-color conjecture was more destructive than constructive, but in addition to finding Kempe's flaw, Heawood also made a number of important contributions. One positive outcome demonstrated by Heawood was that five colors are sufficient for any planar map. Furthermore, he investigated the minimum number of colors needed to color maps on other 3-dimensional surfaces, such as the sphere, torus, and surfaces of genus 3 (Fritsch and Fritsch 1998, 22-24). Another notable discovery of Heawood involves the number of edges of each face in a planar graph. Using Euler's polyhedral formula (which Heawood mistakenly credited to

Cauchy), Heawood showed that the average number of edges of a face is less than six. Although Kempe had previously used this result, a simplification of his proof appeared in Heawood's paper (Coxeter 1959, 287). Although Heawood did give a counterexample of Kempe's supposed "proof," he was unable to provide a proof that four colors suffice to color a map. However, Heawood was "hooked" on the problem, and strove to find a solution until almost the time of his death in 1955.

The Four-Color Conjecture in the Early Twentieth Century

Following Heawood's counterexample to Kempe's argument, researchers typically employed one of four strategies when trying to prove the four-color conjecture. The first strategy, which was soon abandoned, was to use the key elements of Kempe's argument, but devise a new way of dealing with the case of the region with five neighbors. Another approach was to try to reformulate Kempe's strategy, but create a more complicated, yet successful, way of using it. Thirdly, some mathematicians believed that Kempe's ideas should be abandoned altogether; they believed that a new method should be sought for proving that four colors suffice. A final approach was to assume that the four-color theorem was false, and to focus research efforts on looking for maps that could not be four-colored (MacKenzie 1999, 22). One researcher on the subject, Harvard professor George Birkhoff, listed the possible alternative strategies that could be undertaken to solve the four-color problem in 1913. Although he did not list the first of the strategies noted above, he did seriously consider the last. Birkhoff was not the only person to believe that the four-color conjecture might be false. Edward R. Moore of the University of Wisconsin made several attempts to describe maps that were not four-

colorable, and some of his maps played a role in the eventual proof. By far, the second and third options were the two approaches on which the majority of researchers based their endeavors (MacKenzie 1999, 22).

Before continuing, it is important to define some terminology that is important to the development of the four-color problem. A **normal map** is a map in which no more than three regions meet at any given point and in which no region completely surrounds another region. If a map has either of these qualities, simple steps may be taken to transform the map into a normal map that requires the same number of colors as the original map. Thus, the four-color problem can be thought of as trying to prove that a normal map that requires five or more colors does not exist. One essential idea in the eventual proof of the four-color conjecture is the concept of an **unavoidable set of configurations**, or a set of possibilities that must occur in every normal map. For example, Kempe showed that in every planar map, there is at least one face that has less than six neighbors. Thus, a set of unavoidable configurations is a region with two neighbors, a region with three neighbors, a region with four neighbors, and a region with five neighbors. A configuration is said to be **reducible** if there is a way to show that by examining the configuration and the possible ways in which chains of countries can be aligned, then the configuration cannot occur in a minimal five-colored map. While Kempe essentially found an unavoidable set of configurations, it was in the reduction step that Kempe made his blunder, because he could not reduce the region with five neighbors. Soon after Heawood disproved Kempe's argument, it became apparent that a reducible, unavoidable set of configurations would be extremely complicated and quite large (MacKenzie 1999, 23-25).

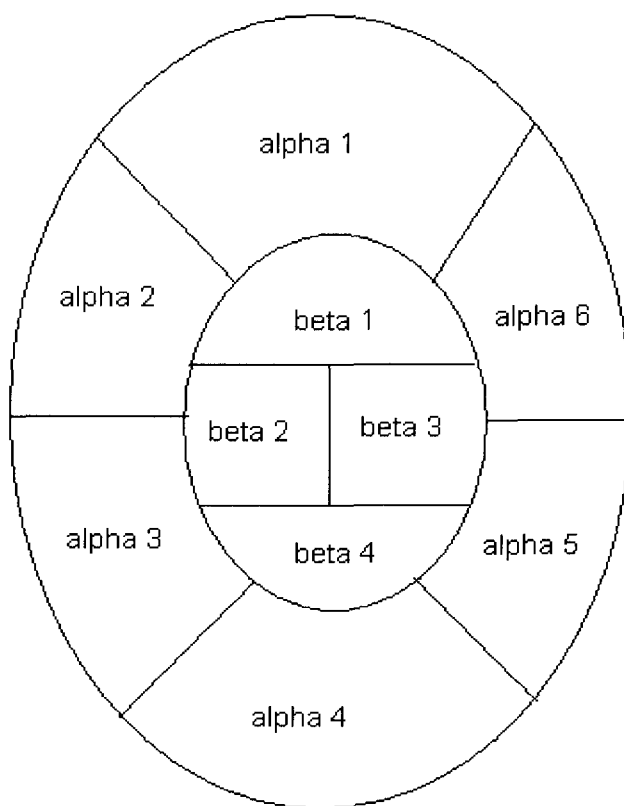


Figure 18. Birkhoff's Diamond

In the early twentieth century, it was quite a substantial task to prove that even one configuration was reducible. In 1913, Birkhoff reexamined Kempe's flawed arguments and constructed a basis for much of the future research on the four-color problem. One of Birkhoff's most important results was that by "systematizing the notion of 'reducibility,'" the configuration shown in Figure 18, known today as Birkhoff's diamond, was shown to be reducible (MacKenzie 1999, 26-27). In dual form, a **ring** is a simple closed path of vertices, and a **configuration** consists of the vertices surrounded by the ring. One can measure the size of a configuration by its **ring size**, which is the number of countries in the ring that surrounds a configuration. In the case of Birkhoff's diamond, the ring size is obviously six, because the six regions labeled "alpha" form the outer ring of the configuration. Birkhoff's goal was to try to determine the smallest

irreducible map (Holton and Purcell 1979, 13). Birkhoff immediately concluded that every map with 13 regions is four-colorable. Throughout the years, a number of proofs involving reducibility increased what became known as the **Birkhoff number**, or the lower bound on the number of regions in a minimal five-colorable map. For example, Philip Franklin increased the number to 26 in 1922.

The concept of reducibility was further investigated by a number of other mathematicians. Alfred Errera of Belgium proved in 1925, that every irreducible map must contain at least 13 pentagons (Biggs, Lloyd, and Wilson 1976, 180). Reynolds increased the Birkhoff number to 27 in 1926. Every few years, the estimate for the Birkhoff number increased, as Franklin extended it to 32 in 1938, Winn to 36 in 1940, Ore and Stemple (1970, 65-66) to 40 in 1968, Stromquist to 52 in 1975, and Jean Mayer to 96 in 1975.

One of the most significant participants in the investigation of the four-color problem in the twentieth century, especially in the study of reducible configurations, was the German mathematician Heinrich Heesch. While studying at Göttingen in the early 1930s, Heesch solved the very challenging regular parquet problem, which had been proposed by David Hilbert in 1900. Heesch's friend at Göttingen, Ernst Witt, believed that he had proved the four-color conjecture, and Heesch accompanied him when he went to share his solution with Richard Courant during a train ride from Göttingen to Berlin. Courant was not completely convinced of Witt's argument, and during their journey back to Göttingen, Heesch discovered an error in Witt's proof (MacKenzie 1999, 25). This experience involving the four-color problem initiated a long search for a proof that would dominate much of Heesch's future research. Heesch was a strong advocate that the four-

color conjecture could be proven by finding a large set of unavoidable, reducible configurations, and he was the first mathematician since Kempe to publicly state such a belief (Appel and Haken 1989, 5-6). He presented some of his findings at lectures at the University of Hamburg and the University of Kiel in the late 1940s. Present at the Kiel seminars was the young Wolfgang Haken, who later recalled that Heesch estimated “that an unavoidable set of reducible configurations might have 10,000 members” (MacKenzie 1999, 25-26).

The arrival of computer technology transformed the difficult task of producing an unavoidable, reducible set of configurations into one that could be “technically possible” (Appel and Haken, 1989, 6). By considering the dual of a map, Heesch determined that at least one of the methods of reduction, which he called D-reduction, could be realized through the use of computers. Recall that Heesch’s goal was to try to show that a configuration of a triangulation is reducible, that is it cannot be contained in any minimum counterexample to the four-color conjecture. Take for example, a triangulated graph T containing the Birkhoff Diamond shown in part in Figure 19. A “naive” method of determining whether a configuration is reducible can be executed by attempting to four-color a configuration by assigning one of four colors to each vertex. For example, suppose the vertices of the outer ring of the Birkhoff diamond, labeled u_1, u_2, u_3, u_4, u_5 , and u_6 in Figure 19, were assigned the colors c_1, c_2, c_3, c_4, c_3 , and c_2 , respectively. Notice that this is only one possible coloring of the outer ring. One is unaware of what the triangulation looks like outside of the ring, so one must consider all possible colorings of the ring. If every coloring of the ring can be extended in order to four-color T , then the configuration is reducible. In the example described above, the vertices of the

configuration, denoted v_1, v_2, v_3 , and v_4 , may be colored with c_3, c_4, c_2 , and c_1 respectively. Thus, in this case, one is able naively four-color T .

The naive method may seem rather easy, but it cannot always produce a four-coloring of T . Sometimes, one must utilize the method of Kempe chains that was discussed earlier in this chapter. Suppose the vertices of the outer ring in Figure 19, were assigned the colors c_2, c_1, c_3, c_1, c_3 , and c_1 , respectively. The vertices v_2 and v_4 of the configuration must be colored either c_2 or c_4 , because they are both adjacent to vertices colored c_1 and c_3 . In addition, v_2 and v_4 are also adjacent to one another, so they cannot be colored the same color either. Without loss of generality, let v_2 be colored c_2 and v_4 be colored c_4 . Now, v_3 is adjacent to vertices colored with all four colors. Thus, the triangulation cannot be naively colored, so one must apply the method of Kempe chains to show reducibility. Let T' be the triangulation T with the inner configuration (the Birkhoff Diamond) removed. Consider $H_{c_1c_4}$, which is the subgraph of T' that is induced by the colors c_1 and c_4 . One of the following four cases must occur: (1) u_4 is in different components of $H_{c_1c_4}$ than both u_2 and u_6 ; (2) u_4 is in the same component of $H_{c_1c_4}$ as u_2 , but in a different component than u_6 ; (3) u_4 is in the same component of $H_{c_1c_4}$ as u_6 , but in a different component than u_2 ; or (4) u_4 is in a same component of $H_{c_1c_4}$ as u_2 and u_6 . In the first case, the colors of the component containing u_4 can be interchanged, and the four-coloring of T can be extended by respectively coloring the configuration's vertices c_3, c_2, c_1 , and c_4 . The second case may be handled similarly by interchanging the colors of the component of $H_{c_1c_4}$ containing u_6 , and the third case can be demonstrated by interchanging the colors of the component of $H_{c_1c_4}$ containing u_2 . In the final case, u_2, u_4 , and u_6 are in the same component of $H_{c_1c_4}$, and thus u_3 cannot be in the same component

of $H_{c_2c_3}$ as u_1 or u_5 . Therefore, the colors of the component of $H_{c_2c_3}$ containing u_3 can be interchanged, and the coloring can be extended to the vertices of the configuration by coloring them c_4 , c_3 , c_4 , and c_2 respectively. A configuration is said to be **D-reducible** if it can be reduced by either the naive method or by utilizing Kempe's chain method (Holton and Sheehan 1993, 62-63). One can see that this process of D-reducibility becomes more difficult as the ring size of a configuration increases, thus requiring the need for computers in order to analyze the many cases involved.

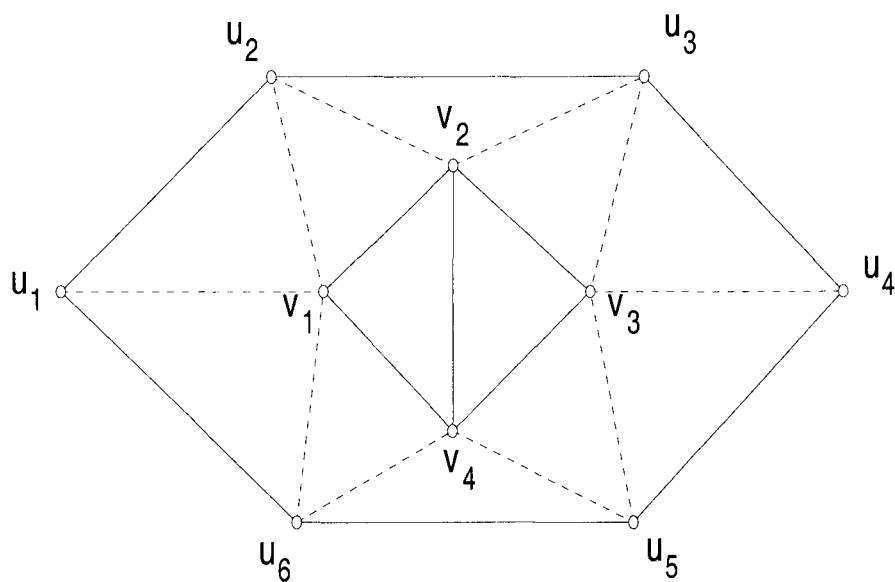


Figure 19. A Triangulation of the Birkhoff Diamond

Karl Dürre, one of Heesch's students, developed a computer program to decide if configurations were D-reducible. While Dürre's program was successful in determining which configurations were D-reducible, it did not show that a specific configuration is reducible in general. For configurations that failed to be D-reducible, Heesch could often combine information from the program that could be augmented by additional

calculations to prove reducibility through a technique called C-reduction (Appel and Haken 1989, 6). **C-reduction** is begun by first removing from T the configuration C that one is attempting to show reducible. Then, one can identify (see Chapter V for a full discussion of identifying vertices) certain non-adjacent vertices of the ring and add new edges to produce a new triangulation, T' . Any coloring of T' will also produce a coloring of $T \setminus C$. If one can extend this coloring to the configuration C , then the configuration is reducible. The Birkhoff Diamond shown in Figures 18 and 19 can be shown to be C-reducible. C-reduction can be performed much faster than D-reduction, because the identification of the vertices reduces the number of possible colorings (Holton and Sheehan 1993, 63-65).

Dürre's D-reduction program was written in Algol 60 for the CDC 1604A computer at Hanover. In November 1965, the program established the reducibility of the Birkhoff Diamond, and then it was used to examine evermore-complex configurations. While the program's results led to powerful conclusions, the 1604A machine had a few significant limitations. For example, computing time rose approximately four-fold as an additional vertex was added to the outer ring. Thus, a 12-ring configuration could be expected to take about six hours to analyze, but a 13-ring configuration could take anywhere from 16 to 61 hours (MacKenzie 1999, 27-28).

Heesch attempted to circumvent the CDC 1604's computing time limitations by locating a more powerful computer. Originally, Heesch sought to use the supercomputer ILLIAC IV, which was being constructed at the University of Illinois, but the machine was not yet functional. John Pasta at the University of Illinois referred Heesch to the United States Atomic Energy Commission's Brookhaven Laboratory, where Yoshio

Shimamoto was serving as chair of the applied mathematics department. The Brookhaven Laboratory possessed a Control Data 6600, recognized as the fastest computer of its day. Shimamoto was interested with the four-color problem himself and made arrangements for Heesch and Dürre to have access to the computer during the years 1968 and 1969. Dürre had to adapt his Angol D-reducibility program to be implemented in the Fortran computer language. The Brookhaven computer offered a considerable increase in computing power. 14-ring configurations could not be analyzed using the 1604A at Hanover, but the enhanced power of the Brookhaven 6600 allowed Dürre and Heesch the opportunity to study such configurations (MacKenzie 1999, 25-29).

Heesch returned to Brookhaven in August 1971, to study the C-reducibility of some of the configurations which Dürre's program had determined not to be D-reducible. Wolfgang Haken also visited the laboratory the following month. While attending a meeting of Brookhaven department chairs on the morning of September 30, Shimamoto began to "play" with some of the essential configurations and eventually created Figure 20, which has come to be known as Shimamoto's horseshoe. Shimamoto's construction and supplemental work showed that if this horseshoe configuration was D-reducible, then the four-color conjecture would be true. Shimamoto shared his findings with Heesch and Haken, who told Shimamoto that the configuration had already been checked, and that it had been found to be D-reducible.

Excitement rose among the researchers, and it was decided that the horseshoe should be rechecked for D-reducibility. Dürre, himself, returned to Brookhaven, and after 26 hours of computing time, the researchers discovered that the results of the original program had been incorrect. Shimamoto's horseshoe had not yet been shown to be

reducible. These findings were disappointing to a number of interested graph theorists, as well as to Shimamoto who had developed the horseshoe. Even though the same program had run the two tests, it is believed that the installation of a new computer system during the first test may have produced instability in the system thereby causing the inaccurate result. Another notable outcome of this episode is the origin of widespread skepticism concerning computer generated mathematical results. The skepticism would eventually cause many to doubt the logical validity of the eventual results of Appel and Haken during the mid-1970s (MacKenzie 1999 25-31).

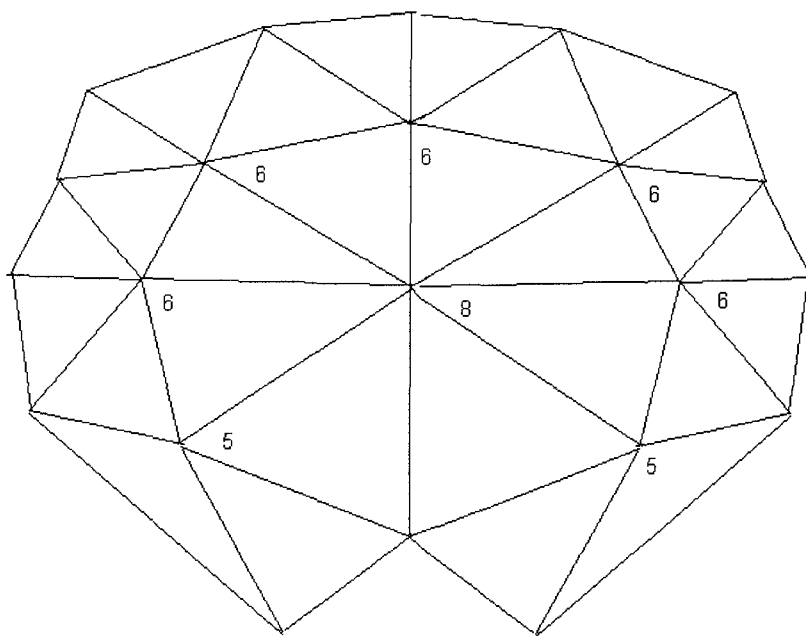


Figure 20. Shimamoto's Horseshoe

Appel and Haken “Prove” the Four-Color Theorem

In the early 1970s, a number of mathematicians, including Heesch, Frank Allaire, Edward Swart, and Frank Berhardt, were independently attempting to determine an

unavoidable set of reducible configurations. Haken had originally collaborated with Heesch in an effort to find such a set of configurations. Haken had invited Heesch to give a lecture at the University of Illinois, and then worked with him in the fall of 1971 at Brookhaven to find discharging procedures. When the work of Shimamoto seemed to settle the four-color question, their collaboration was suspended. Haken seemed to lose hope that such a set could be found using the computers that were then available. Lack of computing time and money to acquire such time also affected Heesch's efforts in Germany. One of Haken's students, Thomas W. Osgood, was working on a thesis about the four-color problem. Kenneth Appel, a mathematical logician, was also serving on Osgood's thesis committee and asked Haken to make a presentation on the four-color problem to the logic seminar at the University of Illinois in order to better understand Osgood's work. Appel, who was experienced in computer programming, was interested in Haken's presentation. Although Haken had proclaimed that he was ready to quit his work on the four-color problem for the present time, Appel encouraged him by saying, "I don't know anything involving computers that can't be done; some things just take longer than others. Why don't we take a shot at it?" (MacKenzie 1999, 34). In 1972, the collaboration of Appel and Haken began with an attempt to study discharging procedures to determine which configurations were reducible and which were not. After almost three years of work, the duo determined that no configurations were necessary with ring size greater than 14, and the computer power they needed to carry out discharging procedures on such a set of configurations was finally available. A University of Illinois computer science graduate student, John Koch, joined Appel and Haken in their work in 1974. Within a year, Koch had developed a program to check D-reducibility through

configurations with ring size 11. Koch later modified his work to allow for checking ring sizes of 12, 13, and 14 (Appel and Haken 1989, 8-9).

Finally, by June of 1976, Appel and Haken had had compiled an unavoidable set consisting of 1,936 reducible configurations, thus “proving” the four-color theorem. The number of configurations was later reduced to 1,482 and later to 1,405. The construction of such a set involved 1,200 hours of time on 3 separate computers, and required the analysis of 487 discharging rules by hand, without the use of technology. Appel and Haken’s wives and children were also directly involved in the reducing process, checking one another’s work and pointing out errors. Appel’s daughter, Laurel, located approximately 800 mistakes, of which she was able to correct 650 by herself. In early July, about 50 errors remained, and Appel spent the weekend of the Fourth of July reducing that number to 12. Haken replaced those configurations with about 20 others, and two of them failed, but he later reworked them as well. As Haken would later admit, it took one month to find approximately 800 mistakes, and only about five days to repair them (MacKenzie 1999, 39). The Appel and Haken paper was first published in the *Illinois Journal of Mathematics* in two parts that when combined total almost 140 pages. In addition, 400 pages of microfiche contained diagrams and verifications of claims made by 24 lemmas in the main text (Appel and Haken 1986, 10). The articles and microfiche were printed by the American Mathematical Society in the form of a 741-page book in the 1980s (See Appel and Haken 1989).

In their proof, Appel and Haken considered planar triangulations, and used many ideas that Kempe had introduced, such as the notion that vertices of less than degree 6 must be contained in planar triangulations. They also used Kempe’s mathematical

induction argument by assuming that “every planar triangulation with fewer than N vertices can be properly four-colored” (Appel and Haken 1986, 11) and then attempting to show that every planar triangulation with N vertices can be four-colored. Using the work of Birkhoff and Heesch, Appel and Haken also utilized the process of reducing configurations. In drawing their configurations, Appel and Haken used the notation based on the degree of each vertex that Heesch had first introduced in 1969. The majority of Appel and Haken’s case analysis was performed by computer and cannot be checked manually. Using a method originally stated by Heesch, Appel and Haken tested configurations for reducibility obstacles. Heesch’s rule states,

“Given a configuration ... proceed as follows:

- (1) Whenever (either initially or after a previous step of the procedure) a vertex of any degree d is connected to fewer than $d - 3$ other vertices of the configuration it may be removed (along with all incident edges) to form a smaller configuration. Such a vertex is connected to at least four other vertices of the ring around the configuration and thus is called a ‘ ≥ 4 legger vertex.’
- (2) Whenever a pair of vertices of degree 5 both of which are connected to a third vertex and to one another but to no further vertices (a ‘hanging pair’) appears then both vertices may be removed to form a smaller configuration
- (3) Whenever a cut-vertex (i.e. a vertex whose removal disconnects the configuration) of degree d is connected to fewer than $d - 2$ other vertices of the configuration (a ‘ ≥ 3 -legger cut-vertex’) then it may be removed to form a pair of smaller configurations” (Appel and Haken 1986, 12).

A configuration fails the above test if after repeating these steps, the resulting configuration is empty or is already known to be irreducible. A configuration that fails is almost certainly not reducible, but configurations that pass this test may or may not be able to be reduced. Another important strategy used by Appel and Haken is called the m -and- n rule:

“for given ring-size n the likelihood of reducibility increases rapidly with the number m of vertices inside the ring In particular, if any configuration satisfies

$$m > \frac{3n}{2} - 6 \quad (\#)$$

then it contains an obstacle-free sub-configuration that also satisfies (#), ... and is almost certainly reducible” (Appel and Haken 1986, 13).

The researchers claimed that they spent about ninety percent of their time on the proof developing methods to produce unavoidable sets of likely-to-be-reducible configurations. A **likely-to-be-reducible configuration** was defined by Appel and Haken to be one that is not known to be irreducible, obstacle free, and satisfies the inequality $m \geq n - 5$ where n is the ring-size and m is the number of vertices in the configuration (Appel and Haken 1986, 13). In his quest to find an unavoidable set of reducible configurations, Heesch had devised a technique that Haken later called a “discharging procedure,” due to its relationship to electrical networks. Each vertex was given a “charge.” During the process of discharging, charges are distributed from degree 5 vertices that are positively charged to vertices of other degrees. After the discharging of the vertices occurs, some positively charged vertices remain, and it is these vertices that “infer the existence of an unavoidable set of configurations” (Fritsch and Fritsch 1998, 223-224). Appel and Haken utilized a number of different discharging procedures to arrive at their unavoidable set. However, they admit that thousands of different proofs may exist of the four-color theorem, since “any particular proof is only selected by a series of choices among the many proofs extant” (Appel and Haken 1986, 13).

Appel and Haken’s work was greeted by a number of diverse reactions. The University of Illinois postage meter proclaimed their findings by printing “Four Colors Suffice” in the postmark on metered mail. Many in the mathematical community championed their significant and difficult work. However, the research also received much criticism from those filled with disdain for the heavy reliance of Appel and Haken

on computer analysis. One early announcement of the proof was made at a summer meeting of the American Mathematical Society and Mathematical Association of America at the University of Toronto. Haken presented the lecture, and at the end of his presentation, rather than the room erupting into cheers, he was met with “polite applause.” At the time, Armin Haken was a graduate student at the University of California at Berkley and later remarked

“[A]t the end of his talk the audience split into two groups, roughly at the age 40. The people over 40 could not be convinced that a proof by computer could be correct, and the people under 40 could not be convinced that a proof that took 700 pages of hand calculations could be correct” (MacKenzie 1999, 41).

The controversy over the proof by computer continues today, although most individuals in the mathematical community now accept Appel and Haken’s computer analysis proof as acceptable. Furthermore, since the time of their publication, several others have provided additional independent proofs of the four-color theorem. In 1977, Frank Allaire described another proof using different discharging procedures. A third proof by Neil Robertson and Daniel P. Sanders of Ohio State University, Paul Seymour of Princeton, and Robin Thomas of the Georgia Institute of Technology was published in 1993. Their efforts, using a discharging procedure developed by Jean Mayer, found a much smaller unavoidable set of 633 reducible configurations. However, their proof was still heavily reliant upon computer analysis.

The beginning and ending of the four-color problem introduced many important influences on the field of mathematics. Its origins were a driving force in establishing graph theory as an independent branch of mathematical study. Its final proof initiated an era during which mathematics employs not only the human mind, but also the use of technological devices. As the first significant proof to be established using a computer,

Appel and Haken's proof of the four-color theorem will likely forever be remembered as being a groundbreaking event in the history of mathematics.

Generalizations of the Four-Color Theorem

As Kempe first suggested in his 1879 paper on the four-color theorem, more than four colors are needed in order to color maps on certain surfaces besides the plane or sphere. In fact, Kempe gave an example of a map on a torus in which six colors are necessary. In the context of this chapter, a **surface** is considered to be a closed orientable 2-manifold, which one could think of as a sphere with a certain number of handles attached as described in Chapter II. S_k is commonly used to denote a sphere of with k handles, and we say that this surface has **genus** k . Heawood and later writers hoped that by studying the more general problem, they could be enlightened on the coloring problem for the sphere or plane (Biggs, Lloyd, and Wilson 1976, 109-110). Ironically, the general version was settled by Ringel and Youngs in 1968 – eight years prior to Appel and Haken's proof of the four-color theorem (White 1980, 20).

The **chromatic number** of a map M , denoted $\chi(M)$, is the smallest number of colors which suffices for coloring the faces of M . The chromatic number of a specific surface, S_k , is the maximum $\chi(M)$ of all maps drawn on S_k . In his 1890 paper on map coloring, Heawood attempted to prove some properties regarding maps on surfaces besides the sphere or plane. One very important feature of Heawood's work was the following equation that describes the chromatic number of S_k :

$$\chi(S_k) = \left\lfloor \frac{7 + \sqrt{1 + 48k}}{2} \right\rfloor,$$

for some positive integer k . The right-hand side of the equation is sometimes denoted $H(S_k)$, and the equation itself is often referred to as the Heawood conjecture. By including the four-color theorem, one can see that Heawood's map coloring theorem is true for all nonnegative integers k . Heawood was able to successfully show that $\chi(S_k) \leq H(S_k)$ for all positive integers k , and although he claimed that $\chi(S_k) \geq H(S_k)$ was also true, his proof was insufficient. A map on a torus requiring seven colors was given by Heawood, which supports his map-coloring theorem, since $\chi(S_1) = H(S_1) = 7$ (White 1980, 20). One year later, Lothar Heffter noticed the missing part of Heawood's proof and showed further that equality is true for $k \leq 6$. Heffter's method involved the use of neighboring regions to find the minimum value of k that allows n regions to be drawn on the surface of genus k . The genus of a specific surface S can be denoted $g(S)$. By considering the neighboring regions in their dual form, Heffter considered the equivalent problem of finding the minimum value k that allows the complete graph K_n to be embedded on the surface of S_k (Biggs, Lloyd, and Wilson 1990, 96-97). Using this dual form and applying Euler's polyhedral formula, Heffter was able to establish that

$$g(K_n) \geq \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil \text{ for } n \geq 3.$$

Further, he conjectured the equality of the statement for all natural numbers, and this equation later became known as the **complete graph conjecture**.

The map coloring problem for non-orientable surfaces was discussed by Heinrich Tietze in 1910. For a non-orientable surface, N_k , with genus k , Tietze proved that for $k \geq 1$,

$$\chi(N_k) \leq H(N_k) = \left\lfloor \frac{7 + \sqrt{1 + 24k}}{2} \right\rfloor.$$

He was able to show equality for $k = 1$, but only that $6 \leq \chi(N_2) \leq H(N_2) \leq 7$. Franklin settled the case of a non-orientable surface of genus 2 in 1934 when he showed $\chi(N_2) = 6$, but this is the single exception to the general rule. Equality in Tietze's statement was later proved for $k = 3, 4$, and 6 by Kango in 1935, for $k = 7$ by Bose in 1939, and for $k = 5$ by Coxeter in 1943 (Biggs, Lloyd, and Wilson 1990, 97).

In 1952, Gerhard Ringel settled the complete graph conjecture for orientable surfaces of genus 13. This was a "prelude" to research by many mathematicians during the 1950s and 1960s (Biggs, Lloyd, and Wilson 1990, 98). It was determined that the complete graph theorem could essentially be separated into 12 different cases, numbered 0 through 11, where case k corresponds to $n \equiv k \pmod{12}$. Ringel settled case 5 as a by-product to a result that he had found concerning non-orientable surfaces. By 1961, Ringel had resolved cases 7, 10, and 3. Cases 3, 4, and 7 were independently settled by Gustin, although his original treatment of case 4 contained a misprint. In 1963, Terry, Welsh, and Youngs proved case 0, and soon cases 1 and 9 were settled by Gustin and Youngs during the period 1963-1965. Youngs completed case 6 in 1966 and cases 2, 8, and 11 were finally finished by Ringel and Youngs during 1967. Although all the key cases had been analyzed, the work was not yet finished, as the proofs did not hold for a certain number of small values. Jean Mayer worked on all odd values up to 23, and by early 1968, only four values remained. Guy, Mayer, and Ringel and Youngs produced proofs for these four values during February 1968, thus proving Heawood's original conjecture. The orientable case may be combined with a statement of the non-orientable case yielding

what is known today as the Map Color Theorem: Let S be the orientable surface S_k of genus k ($k \geq 1$) or the non-orientable surface N_k of genus k ($k \neq 2$). Then the chromatic number $\chi(S)$ is given by

$$\chi(S) = \left\lceil \frac{7 + \sqrt{49 - 24\eta}}{2} \right\rceil,$$

where $\eta = 2 - 2k$ or $2 - k$, is the Euler characteristic of S . The one exception to the non-orientable case, found by Franklin, is the Klein bottle, N_2 , for which $\chi(N_2) = 6$, not 7 (Biggs, Lloyd, and Wilson 1990, 98-99). Heawood's conjecture, the Map Color Theorem, and many other properties of graphs can be proved using a modern tool of graph theory known as rotation schemes. Rotation schemes, introduced by Heffter, Edmonds, and Ringel, is a common method to think of embeddings combinatorially. For more information on rotation schemes and their uses, consult Hartsfield and Ringel (1994, 208-240).

One of the most important open questions in graph theory today is another generalization of the four-color theorem, known as Hadwiger's Conjecture. In 1943, H. Hadwiger hypothesized the following: Every connected t -colorable graph is contractible to K_t . An equivalent statement is that for every $t \geq 0$, every loopless graph with no K_t minor is $(t - 1)$ -colorable. Hadwiger's conjecture is obviously true when $t \leq 3$, and Hadwiger showed that it holds true when $t = 4$. About a decade later, Dirac, oblivious of Hadwiger's results, independently showed this case was true. Wagner had shown in 1937, prior to the formulation of Hadwiger's conjecture, that in the case when $t = 5$, the conjecture is equivalent to the four-color theorem. Thus, Appel and Haken showed the proof of this case in their 1976 proof of the four-color theorem (Kotlov 2002, 241-242).

In 1993, Robertson, Seymour, and Thomas (1993, 279-361) showed that the conjecture is true when $t = 6$. Further values of the conjecture remain unverified, although proving the conjecture is one of the most significant unanswered questions in graph theory.

Map coloring problems have formed an essential part of graph theory almost since its genesis. The four-color conjecture, Heawood's conjecture, and similar problems have been a driving force in the study of planar graphs and their properties. As the next chapter will demonstrate, the coloring problems of this chapter can be generalized and extended to describe coloring problems involving graphs that are not planar, but where a graph's closeness to planarity can be measured.

CHAPTER V

MEASURING CLOSENESS TO PLANARITY

As the past chapters have discussed, many valuable results concerning planar graphs have been discovered. There are also many properties of nonplanar graphs that involve the notion of planarity, such as the minimum number of edge crossings in a graph or the minimum number of planar graphs into which a specific graph can be decomposed. Because many of these concepts were not given attention until the mid-twentieth century, their history is shorter than that of the characterizations of planar graphs or map coloring problems.

Crossing Number

The **crossing number** of a graph G , denoted $\nu(G)$, is the smallest number k such that G can be drawn in the plane with no less than k edge crossings (Liebers 2001, 39). It is obvious that the crossing number of a planar graph is zero, because such graphs can be drawn in the plane with no edge crossings. Furthermore, $\nu(K_{3,3}) = 1$ and $\nu(K_5) = 1$, because these two graphs cannot be drawn in the plane without edge crossings, but the minimum number of edge crossings necessary is only one. For these basic examples, determining the crossing number of the graph seems intuitive, but determining the crossing number of a nonplanar graph is generally a difficult task (Beineke 1989, 210).

Much of the significant research on crossing numbers has focused on determining a general crossing number for the graphs of K_n and $K_{m,n}$.

One of the earliest problems concerning crossing numbers was suggested by Paul Turán and was originally thought to have been solved by Zarankiewicz in 1954. Turán was a member of a labor combattation in 1944, and as Turán later wrote, “had the extreme luck...to work in a brick factory in Budapest” (Guy 1969, 63). At the brick factory, Turán and his fellow workers had to transport bricks from the ovens where they were required to empty storage facilities via small vehicles that ran on rails. At certain times, any of a number of storerooms might be available, so it was necessary that each oven be connected to each storage facility by rail. The relevant problem, known today as Turán’s brick factory problem, was described by Zarankiewicz (1954, 137) as follows:

“In a brickworks the bricks are made in burning-ovens. When they are burnt out, they are carried away to storerooms by workers on small trucks rolling on rails. The trucks move easily and fast except when they pass a crossing of the rails. Here the trucks are usually derailed a great loss of time and bricks occurs and the traffic is hindered on all rails crossing that point. This loss will be reduced to minimum when the number of intersections of the rails is as small as possible and no three rails intersect each other at an inner point.” [sic]

It is clear that Turán’s brick factory problem is equivalent to finding the crossing number for a complete bipartite graph, $K_{m,n}$ where m represents the number of ovens and n represents the number of storage facilities.

Turán originally mentioned the problem in lectures at Warsaw and Wroclaw in October of 1952. Zarankiewicz was at the former lecture, and Urbaník was at the latter. They each independently submitted proposed solutions to Turán’s brick factory problem in 1953 (Guy 1969, 64). Zarankiewicz’s (1954, 137-145) solution was the first to appear

in print the next year and concluded the following three statements about the crossing numbers of complete bipartite graphs:

$$\nu(K_{2k,2p}) = (k^2 - k)(p^2 - p)$$

$$\nu(K_{2k,2p+1}) = (k^2 - k)p^2$$

$$\nu(K_{2k+1,2p+1}) = k^2 p^2.$$

At the end of his paper, Zarankiewicz admitted that another formula had been noticed by Rényi and Turán and proved by Urbaník. The following formula, similar to Urbaník's, is equivalent to the Zarankiewicz's three equations shown above:

$$\nu(K_{m,n}) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor.$$

The result was originally known as Zarankiewicz's theorem, but in 1965 and 1966, Kainen and Ringel noticed an error in Zarankiewicz's work. Thus, today, this equation is often known as Zarankiewicz's conjecture. An attempt to repair the proof was made by Kainen, but his efforts did not resolve the issue (Guy 1969, 64). It was established, however, that the crossing number of $K_{m,n}$ is less than or equal to the right hand side of the equation, and Zarankiewicz managed to prove that equality did hold for $\min(m,n) = 3$.

In 1969, Kleitman (1971, 315-323) proved that Zarankiewicz's conjecture holds for the graph $K_{m,n}$ when $\min(m,n) = 6$. In 1993, Woodall extended this work by proving Zarankiewicz's Theorem for $m \leq 8$, $n \leq 10$. Furthermore, Kleitman gave the following lower bound for the crossing number of a complete bipartite graph:

$$\nu(K_{m,n}) \geq \frac{1}{5}m(m-1) \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor.$$

Only slight improvements have been made to this lower bound (See Nahas 2003, 1-6).

Like the graph of $K_{m,n}$, the investigation of the crossing number of K_n , the complete graph on n vertices, has also yielded significant results. In 1960, Richard Guy published the following upper bound for the crossing number of K_n :

$$\nu(K_n) \leq \begin{cases} \frac{1}{64}(n-1)^2(n-3)^2 & \text{if } n \text{ is odd} \\ \frac{1}{64}n(n-4)(n-2)^2 & \text{if } n \text{ is even} \end{cases}$$

or equivalently,

$$\nu(K_n) \leq \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor.$$

Harary and Hill (1963, 335) claim that this result had been discovered independently several times prior to Guy's publication in 1960. Equality of these statements was conjectured by Guy, but his hypothesis is still not proved. Although Guy (1972, 111-118) was able to prove equality for $n \leq 10$, in the words of Hartsfield and Ringel (1994, 185), Guy's "proofs for $7 \leq n \leq 10$ are very uncomfortable."

A general lower bound has been determined for the crossing number of a general graph G with n vertices and m edges whenever $m \geq 7.5n$. The result that follows was originally found without the second term on the right side by Ajtai, Chvátal, Newborn, and Szemerédi in 1982 and discovered independently by Leighton one year later. The second term is an improvement made by Pach and Tóth in 1997:

$$\nu(G) \geq \frac{m^3}{33.75 \cdot n^2} - 0.9n$$

(Liebers 2001, 40).

A concept similar to the crossing number is the **rectilinear crossing number** of a graph G , denoted $\bar{\nu}(G)$, or the minimum number of crossings when G is drawn in the plane in which every edge is represented by a straight line segment. The concept was originally introduced in 1963 by Harary and Hill (1963, 333-338). Recall from Chapter III that Fáry and Wagner independently showed that every planar graph could be drawn in the plane so that every edge is a straight line segment. The rectilinear crossing number is a natural extension of this notion. It is obvious that $\bar{\nu}(G) \geq \nu(G)$ because the minimum number of crossings in general must be smaller than the minimum number of crossings in a straight edge graph. Guy (1972, 112-118) showed that $\bar{\nu}(K_n) = \nu(K_n)$ holds for $n \leq 7$ and $n = 9$, however in the case of $n = 8$, $\nu(K_8) = 18$, but $\bar{\nu}(K_8) = 19$.

Thickness

The crossing number of a graph offers one way to determine “how close” a graph is to being planar, but several other alternative measurements exist. The **thickness** of a graph G , denoted $\theta(G)$, is the minimum number of planar subgraphs of G whose union is G . Determining the thickness of an arbitrary graph is also notoriously difficult (White and Beineke 1978, 43). In fact, in 1983, Mansfield found that determining a graph’s thickness is an NP-incomplete problem (Liebers 2001, 34-35). The study of thickness originated in a conjecture proposed by John L. Selfridge while working with networks to be used as printed circuits (Harary 1962, 301). In 1961, Frank Harary (1961, 542) submitted the following research problem to the *Bulletin of the American Mathematical Society*:

“Prove or disprove the following conjecture suggested by J. Selfridge.... For any graph G with 9 points, G or its complementary graph \overline{G} is nonplanar. Experimental evidence appears to support this conjecture, which in turn would imply the validity of the conclusion for any graph with at least 9 points. A simple argument using Euler’s polyhedron formula serves to prove that if G is a graph with p points and q lines for which $q > 3p - 6$, then G is nonplanar. This proves the conclusion of the conjecture for all graphs with at least 11 points. For graphs G with 9 or 10 points, it is still open.”

In terms of thickness, the problem of Selfridge asks for which values of n is $\theta(K_n) > 2$ when $n \geq 9$. The following year, Harary (1962, 301-303) published a proof of his statement about graphs with at least 11 points. The more general question posed by Selfridge was proved independently three times during 1962. Battle, Harary, and Komada (1962, 569-571) utilized Kuratowski’s theorem and graph partitioning to prove Selfridge’s conjecture. A similar proof was given by John R. Ball of the Carnegie Institute of Technology. A third, independent proof of Selfridge’s problem was given by W.T. Tutte, who chose to employ a “brutal method” of constructing every triangulation of the sphere having 9 vertices. After only two days of work, Tutte verified that each of the triangulations’ complements was nonplanar (Harary 1962, 303).

In another paper in 1963, Tutte introduced the word “thickness” in the sense of the current mathematical usage of that term, and established several basic results concerning the thickness of a graph (Beineke 1988,128). Among his observations, Tutte found that if a graph G has thickness $\theta(G) = t$, then every subgraph of G has a thickness that is less than or equal to t . Furthermore, if a subgraph G' of a graph G is created by removing only one edge or only one vertex from G , then either $\theta(G') = t$ or $\theta(G') = t - 1$ (Liebers 2001, 36).

As in the study of the crossing number of a graph, in tackling the question of thickness, researchers have given heavy consideration to specific classes of graphs, such as the complete graphs and complete bipartite graphs. Beineke and Harary laid the foundations for the search for the determination of the thickness of a complete graph, K_n . In 1965, they proved that the following result holds for $n \not\equiv 4 \pmod 6$:

$$\theta(K_n) \leq \begin{cases} \left\lfloor \frac{n+7}{6} \right\rfloor & \text{if } n \geq 1, n \neq 9, n \neq 10 \\ 3 & \text{if } n = 9, n = 10 \end{cases}.$$

Beineke and Harary employed a method that required decomposing a graph into n triangular regions. A number of results for the case $n \equiv 4 \pmod 6$ were proven on a case-by-case basis, such as for $n = 16, 22, 28, 34, 40$, and 46 . The upper bound was finally established for all $n \geq 1$ in 1976 by Alekseev and Gonchakov and independently by Vasak. All of the proofs use Beineke and Harary's decomposition method, except when $n = 16$. Many mathematicians had attempted to decompose K_{16} into three planar graphs, but all were unsuccessful. Many had concluded that perhaps $\theta(K_{16}) = 4$, until Jean Mayer proved that $\theta(K_{16}) = 3$ in 1972 (White and Beineke 1978, 43-44).

Unfortunately, the issue of the thickness of complete bipartite graphs, $K_{m,n}$, has not been completely resolved, although several results have been found on the subject. In 1964, Beineke, Harary, and Moon determined hypotheses under which the following result holds:

$$\theta(K_{m,n}) = \left\lceil \frac{m \cdot n}{2(m+n-2)} \right\rceil.$$

While it does seem to work for most cases, it does not always hold when m and n are both odd, $m \leq n$, and there exists an integer k such that $n = \left\lfloor \frac{2k(m-2)}{m-2k} \right\rfloor$. The issue of thickness has also been resolved for the family of graphs known as the hypercubes, and was shown by Kleinert in 1967 to be:

$$\theta(Q_n) = \left\lceil \frac{n+1}{4} \right\rceil$$

(Liebers 2001, 37).

Splitting Number

Consider the planar graph on the left in Figure 21 with two vertices labeled “5.” The graph on the right demonstrates what the graph would look like if one were to “glue together” the two vertices labeled “5” to create the graph K_5 . To generalize, suppose G is a graph and u and v are two vertices of G . If a new graph, G' can be constructed by replacing vertices u and v with a new vertex, w , in such a way that any vertex that was adjacent to either u or v in G is also adjacent to w in G' , then this process is known as **identifying** two vertices. Conversely, the reverse procedure to identifying two vertices is the process of splitting a vertex (Hartsfield and Ringel 1994, 193-194). One may **split** a vertex by replacing a single vertex w of G with two separate vertices, u and v , such that whenever u is adjacent to some of the vertices to which w was originally adjacent, it follows that v is adjacent to the remaining vertices originally adjacent to w . The processes of identifying two vertices and splitting a vertex allow for another measurement of a graph’s closeness to planarity, called the splitting number. The **splitting number** of a graph G , denoted $\sigma(G)$, is the smallest number k such that G can be obtained from a

planar graph by performing k vertex identifications of two vertices. Equivalently, the splitting number can be thought of as the minimum number of vertex splittings that are required to be performed on a graph G in order to produce a planar graph (Hartsfield and Ringel 1994, 194). Clearly, $\sigma(G) = 0$ if and only if G is a planar graph.

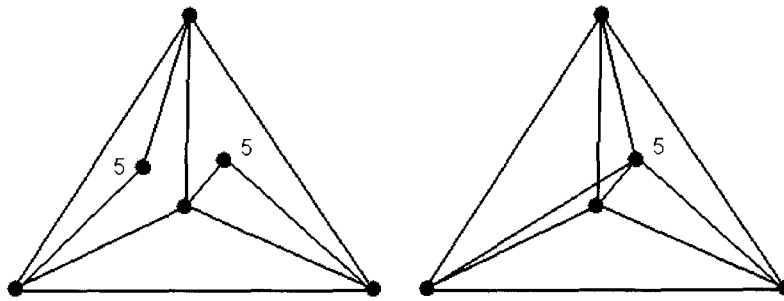


Figure 21. Identifying Two Vertices

The origins of the splitting number can be traced to the 1980s work of Nora Hartsfield, Brad Jackson, and Gerhard Ringel on determining lower bounds for the splitting number and the procedure of splitting vertices of complete graphs and complete bipartite graphs in order to embed such graphs on a particular surface (Liebers 2001, 25). It is extremely difficult to determine the splitting number of a graph. During the late 1990s and early 2000s, Luérbio Faria, Celina Miraglia Herrera de Figueiredo and Candido Ferreira Xavier de Mendonça Neto showed that finding the splitting number of a given graph is NP-incomplete (Liebers 2001, 27).

As is the case with other measurements of closeness to planarity, results have been found for specific families of graphs. The first class of graphs for which the splitting

number was determined is the family of complete bipartite graphs. In 1984, Jackson and Ringel gave the following equation for this splitting number:

$$\sigma(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{2} \right\rceil.$$

One year later, Hartsfield, Jackson, and Ringel found the following result for the family of complete graphs:

$$\sigma(K_n) = \left\lceil \frac{(n-3)(n-4)}{6} \right\rceil.$$

(Hartsfield and Ringel 1994, 193-196).

Coarseness

One might say that the discovery of the concept of the coarseness of a graph was a propitious mistake of Paul Erdős, one of the greatest mathematical minds of the twentieth century. Erdős attempted to define the thickness of a graph by speaking of the maximum number of edge-disjoint nonplanar subgraphs contained in a given graph. This of course, is not the definition of thickness, but of the **coarseness** of a graph G , denoted $\xi(G)$ (Harary 1969, 121). Thickness and coarseness are similar concepts, since both involve the decomposition of graphs, but the former is the minimum number of planar graphs, while the latter is the maximum number of nonplanar graphs.

Equations involving the coarseness of a graph are not as compact as those of the other measurements in this chapter. Erdős originally conjectured that $\binom{n}{2}$ was a lower bound for the coarseness of K_n whenever n is a multiple of 3. Beineke and Chartrand

improved Erdős' lower bounds for $n \geq 30$ (Guy 1967, 38). Subsequently, in 1968, Guy and Beineke (1968, 888-894) proved the following result:

$$\xi(K_{3p}) = \begin{cases} \binom{n}{2} & \text{if } n = 3p \leq 15 \\ \binom{n}{2} + \left\lfloor \frac{n}{5} \right\rfloor & \text{if } n = 3p \geq 30 \end{cases}$$

$$\xi(K_{3p+1}) = \binom{n}{2} + 2 \left\lfloor \frac{n}{3} \right\rfloor \quad \text{if } n = 3p + 1 \geq 19 \text{ and } n \neq 9p + 7$$

$$\xi(K_{3p+2}) = \binom{n}{2} + \left\lfloor \frac{14n+1}{15} \right\rfloor.$$

Since the statements above do not give the explicit coarseness of every complete graph, Table 1 gives a list of the conjectured coarseness for a complete graph on n vertices as hypothesized by Guy and Beineke (1968, 894).

Table 1

Conjectured Values of Coarseness

| | | | | | | |
|------------|----|----|----|----|----|----------------------------|
| N | 13 | 18 | 21 | 24 | 27 | $9n + 7$ |
| $\xi(K_n)$ | 7 | 15 | 21 | 28 | 36 | $\frac{9n^2 + 13n + 2}{2}$ |

Analagous to the statements above about the complete graph, the computations of coarseness of a complete bipartite graph are also complex, because they involve several cases and the cases are often incomplete. The following results concerning the coarseness of a complete bipartite graph are due to Beineke and Guy:

$$\xi(K_{3r+d, 3s+e}) = rs + \min\left(\left\lfloor \frac{er}{3} \right\rfloor, \left\lfloor \frac{ds}{3} \right\rfloor\right) \text{ for } d = 0 \text{ or } 1 \text{ and } e = 0 \text{ or } 1$$

$$\xi(K_{3r+2, 3s}) = rs + \left\lfloor \frac{s}{3} \right\rfloor \text{ when } r \geq 1$$

$$\xi(K_{3r+2, 3s+1}) \begin{cases} \leq rs + \min\left(\left\lfloor \frac{r+s}{3} \right\rfloor, \left\lfloor \frac{2s}{3} \right\rfloor, \left\lfloor \frac{8r+16s+2}{39} \right\rfloor\right) \\ \geq rs + \max\left(\left\lfloor \frac{s+2}{3} \right\rfloor, \min\left(\left\lfloor \frac{r}{3} \right\rfloor, \left\lfloor \frac{2s}{3} \right\rfloor\right)\right) \end{cases} \text{ for } r \geq 2, s \geq 7$$

$$\xi(K_{3r+2, 3s+2}) \begin{cases} \leq rs + \min\left(\left\lfloor \frac{r+2s}{3} \right\rfloor, \left\lfloor \frac{2r+s}{3} \right\rfloor, \left\lfloor \frac{16r+16s+4}{39} \right\rfloor\right) \\ \geq rs + \left\lfloor \frac{r}{3} \right\rfloor + \left\lfloor \frac{s}{3} \right\rfloor + \left\lfloor \frac{r}{9} \right\rfloor \end{cases} \text{ for } 1 \leq r \leq s$$

(Harary 1969, 121-122).

Heawood's Empire Problem

Many of the methods of measuring a graph's closeness to planarity described in this chapter can be directly applied to specific problems. In Heawood's 1890 paper that refuted Kempe's argument and explored the notion of coloring maps on surfaces of higher genus, another problem was discussed concerning empires. The four-color problem, described in Chapter IV, required that each "country" be a connected region. However, in reality that is often not the case; for example Alaska is not connected to the mainland of the United States and the Kaliningrad Oblast (formerly Königsburg) of Euler's bridges problem is separated from mainland Russia. At the time Heawood proposed his problem, a number of European nations had colonies throughout the world, many of which continue to exist today. Heawood asked how many colors would be

needed to color maps of empires such that each colony was colored the same as the mother country, and empires sharing borders would receive different colors. Heawood was able to prove that if every empire consisted of M connected regions, then every map could be colored with $6M$ colors. Jackson and Ringel later gave the name **M -pire** to an empire of M connected regions. Obviously, Heawood's upper bound is not always the best, since a 1-pire can be colored with four colors due to the four-color theorem. However, Heawood gave an example of a 2-pire in which every set of 12 empires share a common border, so 12 colors are needed (Hutchinson 1993, 212-215). Heawood regretted not being able to present a symmetric map of his twelve mutually adjacent 2-pires, but almost 80 years later, Scott Kim provided such a figure, as shown in Figure 22 (Hartsfield and Ringel 1994, 198).

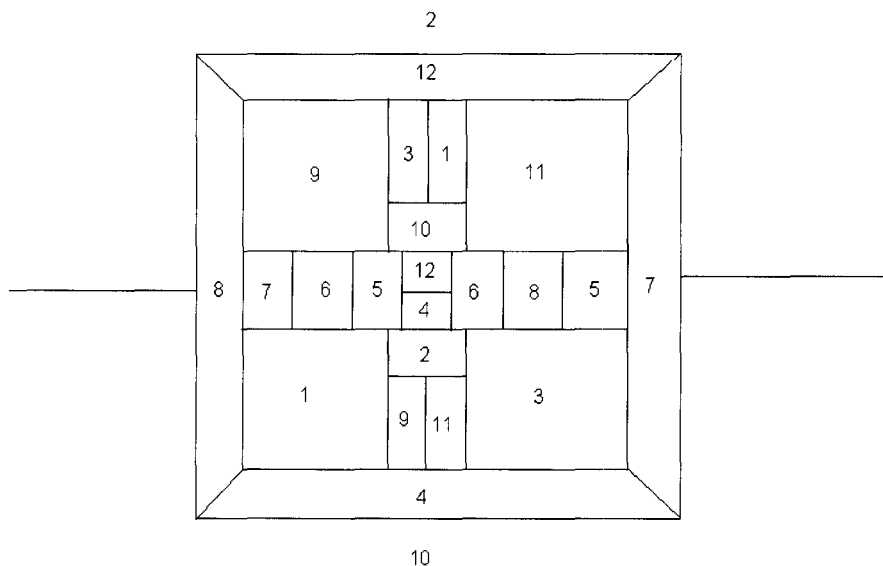


Figure 22. Kim's Symmetric Map of 12 Mutually Adjacent 2-pires

While Heawood was certain that every M -pire was $6M$ -colorable, he was not able to show that this was always the best coloring. By the early 1980s, Herbert Taylor had

created maps of 3-pires requiring 18 colors and 4-pires requiring 24 colors. In 1983, Jackson and Ringel used one of Taylor's maps in their proof of the M -pire theorem (Hartsfield and Ringel 1994, 200-201). The M -pire theorem can be stated as follows: "For every $M > 1$ there is an M -pire graph that requires $6M$ colors. In fact, the graph consisting of $6M$ mutually adjacent vertices is an M -pire graph" (Hutchinson 1993, 215). Jackson and Ringel proved this theorem using techniques that had been developed by Ringel and Youngs in their proof of Heawood's conjecture on coloring maps on surfaces of higher genus that was discussed in Chapter IV.

There is an interesting connection between the M -pire problem and the method of identifying and splitting vertices. For example, the map shown in Figure 22 can serve as a proof that the splitting number of K_{12} is 12. One may notice that the dual of the given map is merely a planar splitting of K_{12} , and by identifying vertices labeled the same, the map will result in the graph of K_{12} (Hartsfield and Ringel 1994, 202).

Ringel's Earth-Moon Problem

In 1949, Ringel introduced a variation to Heawood's problem about empires. The problem is often referred to as the earth-moon problem because Ringel presented a scenario in which each country on earth had a colony on the moon with similar stipulations as those in the empire problem concerning common borders. Ringel wanted to know what is the smallest number of colors, k_2 , necessary to color every earth-moon map. It is essentially a problem concerning graphs that have thickness of at most 2, sometimes called **biplanar graphs**. One may infer from the empire problem that $k_2 \leq 12$. Furthermore, the thickness of K_9 is 3, so there is no map of mutually exclusive 2-pires on

the earth and moon. Thus, immediately, one can conclude that $8 \leq k_2 \leq 12$. For several years that is all that was known on the subject. Finally, Sulanke created an earth-moon map consisting of 11 earth-moon 2-pires. After vertices are identified, the resulting graph is the complete graph on 11 vertices minus a single cycle of 5 vertices. Of course the six mutually connected vertices must each be colored a different color, and the five vertices of the missing cycle may be colored using three colors. Thus, Sulanke's map requires nine colors, so we may conclude $9 \leq k_2 \leq 12$; however, the general problem remains open (Hartsfield and Ringel 1994, 203-205).

The earth-moon problem may be generalized to a problem involving any number of spheres. It has been shown that for the problem involving three spheres, $k_3 = 16, 17$, or 18 , and for four spheres, $k_4 = 22, 23$, or 24 . Using properties of the thickness of a complete graph, the concepts can be generalized to show that for a problem involving m spheres, $k_m = 6m - 2, 6m - 1$, or $6m$ (Hartsfield and Ringel 1994, 203-204). Applications of the earth-moon problem and of thickness in general include devising procedures to test for errors in printed circuit boards. For example, it might be the goal of an engineer to print electronic circuits in layers in such a way that each layer does not have edge crossings (Beineke 1997, 4).

The study of planar graphs has had a significant impact on the field of graph theory and in the overall field of mathematics. In the history of mathematical ideas, planar graphs have played a relatively short, but rich, role. As this paper has discussed, many of the origins of graph theoretic ideas are based in the puzzles of recreational mathematics, such as the problems of the Königsburg bridges, the utilities, and the brick factory of Turán. A driving force in the development of many ideas concerning planar

graphs has been the four-color theorem and other coloring problems. One could argue that if the four-color theorem had never been conjectured, graph theory would not have the prominence that it has today. The effects of coloring problems extend into many other topics in graph theory. Even coloring problems involving the thickness and splitting number of a graph have been introduced, although these tools may not initially seem to be applicable to problems of map coloring. This thesis has not attempted to address every known result involving planar graphs, but has rather concentrated on the areas of research that have led to the most significant contributions to our knowledge of planar graphs. Many other conclusions, algorithms, and applications have been found regarding planar graphs; for example, in 1990 Schnyder (1990, 138-148) and de Fraysseix, Pach, and Pollack (1990, 41-51) independently proved results on drawing n -vertex triangulated planar graphs as crossing-free straight-line grid drawings. Problems, both solved and unsolved, regarding drawing planar graphs are numerous, and have many important applications. It is likely that the study of planar graphs and their generalizations will not end soon because many questions, such as those involving coarseness of a graph, Hadwiger's conjecture, and the earth-moon problem, remain unanswered. This history provides only a brief glimpse at the origins and early development of concepts relating to planar graphs, as many of the most significant results may appear in the future.

REFERENCES

- Aigner, Martin. 1986. *Graph Theory: A Development from the 4-Color Problem*. Moscow, ID: BCS Associates.
- Appel, Kenneth and Wolfgang Haken. 1986. The Four Color Proof Suffices. *The Mathematical Intelligencer* 8:10-20.
- Appel, Kenneth and Wolfgang Haken. 1989. *Every Planar Map is Four Colorable*. Providence, RI: American Mathematical Society.
- Battle, Joseph, Frank Harary, and Yukihiro Kodama. 1962. Every Planar Graph with Nine Points has a Nonplanar Complement. *Bulletin of the American Mathematical Society* 68:569-571.
- Beineke, Lowell W. 1988. Fruited Planes. *Congressus Numerantium* 63:127-138.
- Beineke, Lowell W. 1989. Are Graphs Finally Surfacing? *The College Mathematics Journal* 20:206-225.
- Beineke, Lowell W. 1997. Biplanar Graphs: A Survey. *Computers and Mathematics with Applications* 34:1-8.
- Biggs, Norman L., E. Keith Lloyd, and Robin J. Wilson. 1976. *Graph Theory 1736-1936*. Oxford: Clarendon Press.
- Biggs, Norman L., E. Keith Lloyd, and Robin J. Wilson. 1990. The Historical Background to Gerhard Ringel's Work. *Topics in Combinatorics and Graph Theory: Essays in Honour of Gerhard Ringel*. Heidelberg: Physica-Verlag. 93-99.
- Biggs, Norman L., E. Keith Lloyd, and Robin J. Wilson. 1995. The History of Combinatorics. *Handbook of Combinatorics: Volume II*. Cambridge, MA: The MIT Press. 2163-2198.
- Coxeter, H.S.M. 1959. The Four-Color Map Problem 1840-1890. *The Mathematics Teacher* 52:283-289.
- de Fraysseix, H., J. Pach, and R. Pollack. 1990. How to Draw a Planar Graph on a Grid. *Combinatorica* 10:41-51.

- Dudeney, Henry Ernest. 1958. *Amusements in Mathematics*. Paperback ed. New York: Dover Publications.
- Dudeney, Henry Ernest. 1967. *536 Curious Problems & Puzzles*. New York: Barnes & Noble, Inc.
- Eves, Howard. 1990. *An Introduction to the History of Mathematics*. 9th Ed. Orlando, FL: Saunders College Publishing.
- Fritsch, Rudolf and Gerta Fritsch. 1998. *The Four-Color Theorem: History, Topological Foundations, and Idea of Proof*. New York: Springer-Verlag.
- Guy, Richard K. 1967. A Coarseness Conjecture of Erdős. *Journal of Combinatorial Theory* 3:38-42.
- Guy, Richard K. 1969. The Decline and Fall of Zarankiewicz's Theorem. *Proof Techniques in Graph Theory: Proceedings of the Second Ann Arbor Graph Theory Conference, February 1968*. New York and London: Academic Press. 63-69.
- Guy, Richard K. 1972. Crossing Numbers of Graphs. *Graph Theory and Applications*. Proceedings of the Conference on Graph Theory, Western Michigan University, 1972. New York: Springer-Verlag.
- Guy, Richard K. and Lowell W. Beineke. 1968. The Coarseness of the Complete Graph. *Canadian Journal of Mathematics* 20:888-894.
- Harary, Frank. 1960. Some Historical and Intuitive Aspects of Graph Theory. *SIAM Review* 2:123-131.
- Harary, Frank. 1961. Research Problem. *Bulletin of the American Mathematical Society* 67:542.
- Harary, Frank. 1962. A Complementary Problem on Nonplanar Graphs. *Mathematics Magazine* 35:301-303.
- Harary, Frank. 1969. *Graph Theory*. Reading, MA: Addison Wesley Publishing.
- Harary, Frank. 1981. Homage to the Memory of Kazimierz Kuratowski. *Journal of Graph Theory* 5:217-219.
- Harary, Frank and Anthony Hill. 1963. On the Number of Crossings in a Complete Graph. *Proceedings of the Edinburgh Mathematical Society, Series II* 13:333-338.

- Harborth, Heiko. 1990. Gerhard Ringel – Seventy. *Topics in Combinatorics and Graph Theory: Essays in Honour of Gerhard Ringel*. Heidelberg: Physica-Verlag. XIII-XV.
- Hartsfield, Nora, and Gerhard Ringel. 1994. *Pearls in Graph Theory: A Comprehensive Introduction*. San Diego: Academic Press.
- Holton, D.A. and J. Sheehan. 1993. *The Petersen Graph*. Cambridge: Cambridge University Press.
- Holton, D.A. and S. Purcell. 1979. The Four-Color Theorem - A Short History. *Australian Mathematical Society Gazette* 6:11-14.
- Hutchinson, Joan P. 1993. Coloring Maps, Maps of Empires, and Maps of the Moon. *Mathematics Magazine* 66:211-226.
- Kennedy, John W., Louis V. Quintas, and Maciej M. Syslo. 1985. The Theorem on Planar Graphs. *Historia Mathematica* 12:356-368.
- Kleitman, Daniel J. 1970. The Crossing Number of $K_{5,n}$. *Journal of Combinatorial Theory* 9:315-323.
- König, Dénes. 1990. *Theory of Finite and Infinite Graphs*. Boston: Birkhauser.
- Kotlov, Andreï. 2002. Matchings and Hadwiger's Conjecture. *Discrete Mathematics* 244:241-252.
- Krasinkiewicz, J. 1981. A Note on the Work and Life of Kazimierz Kuratowski. *Journal of Graph Theory* 5:221-223.
- Kruja, Eriola, Joe Marks, Ann Blair, and Richard Waters. 2002. A Short Note on the History of Graph Drawing. *Graph Drawing: Selected Papers from the 9th International Symposium held in Vienna, September 23-26, 2001*. Lecture Notes in Computer Science, 2265. Berlin: Springer-Verlag. 272-286.
- Kullman, David E. 1979. The Utilities Problem. *Mathematics Magazine* 52:299-302.
- Lakatos, Imre. 1976. *Proofs and Refutations: The Logic of Mathematical Discovery*. Cambridge: Cambridge University Press.
- Liebers, Annegret. 2001. Planarizing Graphs – A Survey and Annotated Bibliography. *Journal of Graph Algorithms and Applications* 5:1-74.
- Lloyd, E. Keith. 1976. Facts and Myths in the History of Graph Theory. Proceedings of the Fifth British Combinatorial Conference (University of Aberdeen, 1975). *Congressus Numerantium* 411-416.

- Lyusternik, L.A. 1966. *Convex Figures and Polyhedra*. Boston: D.C. Heath and Company.
- MacKenzie, Donald. 1999. Slaying the Kracken: The Sociohistory of a Mathematical Proof. *Social Studies of Science* 29:7-60.
- May, Kenneth O. 1965. The Origin of the Four Color Conjecture. *Isis* 56:346-348.
- Mitchem, John. 1981. On the History and Solution of the Four-Color Problem. *The Two-Year College Mathematics Journal* 12:108-116.
- Mulder, Henry Martyn. 1988. To See the History for the Trees. 250th Anniversary Conference on Graph Theory. *Congressus Numerantium* 64:25-43.
- Nahas, Nagi H. 2003. On the Crossing Number of $K_{m,n}$. *The Electronic Journal of Combinatorics*. 10:1-6.
- Ore, Oystein and Joel Stemple. 1970. Numerical Calculations on the Four-Color Problem. *Journal of Combinatorial Theory* 8:65-78.
- Robertson, Neil, Paul Seymour and Robin Thomas. 1993. Hadwiger's Conjecture for K_6 -free Graphs. *Combinatoria* 13:279-361.
- Sachs, Horst, Michael Stiebitz, and Robin J. Wilson. 1988. An Historical Note: Euler's Königsburg Letters. *Journal of Graph Theory* 12:133-139.
- Schnyder, Walter. 1989. Planar Graphs and Poset Dimension. *Order* 5:323-343.
- Schnyder, Walter. 1990. Embedding Planar Graphs on the Grid. *Proceedings of 1st ACM-SIAM Symposium on Discrete Algorithms*. 138-148.
- Thomassen, Carsten. 1981. Kuratowski's Theorem. *Journal of Graph Theory* 5:225-241.
- White, Arthur T. 1978. The Proof of the Heawood Conjecture. *Selected Topics in Graph Theory*. New York: Academic Press. 51-81.
- White, Arthur T. 1980. Colouring Maps on Surfaces. *Bulletin of the Institute of Mathematics and its Applications* 16:20-23.
- White Arthur T. and Lowell W. Beineke. 1978. Topological Graph Theory. *Selected Topics in Graph Theory*. New York: Academic Press. 15-49.

- Wilson, Robin J. 1978. 200 Years of Graph Theory: A Guided Tour. *Theory and Applications of Graphs (Proceedings of the International Conference, Western Michigan University, Kalamazoo, Michigan, 1976)*. Lecture Notes in Mathematics, Vol. 642. Berlin: Springer-Verlag. 1-9.
- Wilson, Robin J. 1999. Graph Theory. *History of Topology*. Amsterdam: Elsevier Science.
- Wilson, Robin J. and John J. Watkins. 1990. *Graphs: An Introductory Approach*. New York: Wiley.
- Woodall, Douglass R. and Robin J. Wilson. 1978. The Appel-Haken Proof of the Four-Color Theorem. *Selected Topics in Graph Theory*. New York: Academic Press. 83-101.
- Zarankiewicz, K. 1954. On a Problem of P. Turan Concerning Graphs. *Fundamenta Mathematicae* 41:137-145.

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