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ABSTRACT

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Arrow's classic theorem shows that any collective choice function satisfying independence of irrelevant alternatives (IIA) and Pareto (P), where the range is a subset of weak orders, is based on a dictator. This thesis focuses on Arrovian collective choice functions in which the range is generalized to include acyclic, indifference-transitive (ACIT) relations on the set of alternatives. We show that Arrovian ACIT collective choice functions with domains satisfying the free-quadruple property are based on a unique weakly decisive voter; however, this is not necessarily true for ACIT collective choice functions where Arrow's independence condition is weakened. For ACIT collective choice functions with linear order domains, we present a complete characterization, as well as a recursive formula for counting the number of Arrovian ACIT collective choice functions with two voters.

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CHAPTER 1
INTRODUCTION

In social choice theory, "...we ask if it is formally possible to construct a procedure for passing from a set of known individual tastes to a pattern of social decision-making, the procedure in question being required to satisfy certain natural conditions" (Kenneth Arrow, [1]). Essentially, researchers in social choice theory study theoretical methods for compiling the individual preferences of an entire population into a single representation of the population as a whole. Given a set of candidates, each voter submits his or her preferences amongst those candidates, and the chosen procedure is applied to this information. Choosing how to "count" such votes is not as straight forward as it may seem. In fact, there are many widely-known "voting methods," each of which has both positive and negative characteristics. These methods are employed in various environments, both in and out of the political realm, and can be observed in any situation where two or more individuals must come together to make a decision between two or more alternatives. For a nice reference to Arrow's Theorem and its consequences in political theory, see [2].

In this dissertation, we will primarily focus on acyclic and indifference-transitive collective choice functions satisfying the classic Arrovian conditions of independence of irrelevant alternatives and the Pareto property. (These terms will be defined in the next chapter.) The domain of Arrovian-type collective choice functions is based on transitive relations such as linear orders and weak orders, and so capturing some transitivity in the output of those functions seems to be a reasonable requirement. See [5] and [6] for a version of Arrow's Theorem where the social output is only required to be a semiorder. The case where it is assumed that only the asymmetric part of the output is transitive has been studied [10], [11], [13]. In these works, the indifferent part of the output is allowed to be nontransitive. Recently, however, the opposite approach has been studied [12], and we continue this study in this thesis. Namely, we require the symmetric part of the output to be transitive, while simultaneously only requiring the asymmetric part to be acyclic. Finally, for the cases where the social output is just required to be acyclic, or reflexive and complete, we refer the reader to [4] and [14].

Throughout the next several chapters, we will introduce and, using an ultrafilter approach, prove a new generalization of Arrow's classic theorem. We will also provide a complete characterization of all such functions whose domain is restricted to the linear orders, which we investigate through both a mathematical analysis and an innovative software approach. Finally, we will investigate the effect of further weakening the independence of irrelevant alternatives condition.

CHAPTER 2
BACKGROUND AND NOTATION

In this chapter, we will introduce some of the notation and definitions that are central to the mathematical study of social choice. We will also discuss a cornerstone of social choice theory, Arrow's Impossibility Theorem, and briefly summarize a selection of the work which has been done to extend this landmark theorem. In what follows, we will introduce the basic model for a clear statement of Arrow's Impossibility Theorem.

We will first let $N = \{1, 2, 3, ..., n\}$ represent the set of n voters. In order for the "elections" that our functions represent to actually be interesting, we will assume that $n \geq 2$. We let the set X represent the set of alternatives or candidates that our voters are choosing from, and we let $m = |X|$.

As the nature of social choice theory is in representing how a society prefers some alternatives as compared to others, we will draw heavily from the study of ordered sets. By "ordered set", we mean a set of elements and a binary relation which determines the relationship between the elements in the set.

Definition 2.1. *A* ***binary relation*** *R on a set X is a collection of ordered pairs of elements in $X \times X$. We will write aRb instead of $(a, b) \in R$.*

- *A relation R on X is* ***reflexive*** *if xRx for all $x \in X$.*
- *A relation R on X is* ***complete*** *if at least one of xRy or yRx for all $x \neq y \in X$.*

- *A relation R on X is* ***transitive*** *if for all $x, y, z \in X$, xRy and yRz implies that xRz.*

- *A relation R on X is* ***antisymmetric*** *if for any $x, y \in X$, if xRy and yRx, then $x = y$.*

- *A relation R on X is* ***asymmetric*** *if for any $x \neq y \in X$, if xRy, then it is not the case that yRx.*

Two common types of ordered sets which we will use often are weak orders and linear orders.

Definition 2.2. *A* ***weak order*** *is a reflexive, complete, and transitive binary relation.*

Definition 2.3. *A* ***linear order*** *is an antisymmetric weak order.*

For a relation $\rho \in X \times X$, for any $a, b \in X$, if $(a, b) \in \rho$, then we say that "a is at least as good as b." We will define the following notation:

- $a \geq_\rho b$ if $(a, b) \in \rho$. (According to ρ, a is at least as good as b.)

- $a >_\rho b$ if $(a, b) \in \rho$ but $(b, a) \notin \rho$. (According to ρ, a is strictly better than b.)

- $a \sim_\rho b$ if $(a, b) \in \rho$ and $(b, a) \in \rho$. (According to ρ, a and b are indifferent or "tied.")

On a set X, we will let $L(X)$ represent the set of all linear orders on X, and we will let $W(X)$ represent the set of all weak orders on X. Note that $L(X) \subset W(X)$. Both linear orders and weak orders are transitive. The only difference between weak orders and linear orders is that, while weak orders allow two distinct elements to be tied, linear orders do not allow for this indifference.

We will be primarily concerned with *collective choice functions*. These functions attempt to map the preferences of a set of voters to a single relation representing the preferences of the group as a whole. Collective choice functions are formally defined as follows:

Definition 2.4. *Any mapping of the form $f\colon D(X) \to C(X)$, where $D(X) \subseteq W(X)^n$ and $C(X)$ is the set of all complete, reflexive, binary relations on X, is called a* ***collective choice function*** *with domain $D(X)$.*

The domain of a collective choice function is a set of preference profiles $D(X)$.

Definition 2.5. *A* ***preference profile*** *is an n-tuple $\pi = (\pi(1), \pi(2), ..., \pi(n))$ where each $\pi(i)$ is a relation representing voter i's preferences.*

In order to discuss a particular profile's preferences on a specific pair or subset of our alternatives, we will employ the following notation:

Definition 2.6. *For any $X' \subset X$ and any profile $\pi \in D(X)$, let $\pi|_{X'}$ represent the profile π restricted to X'. For all $i \in N$ and any $a \neq b \in X'$, $(a, b) \in \pi|_{X'(i)}$ if and only if $(a, b) \in \pi(i)$.*

One example of a collective choice function is known as *Absolute Majority Rule*, which is the function $f : W(X)^n \to C(X)$ where for all $\pi \in W(X)^n$, for any $a \neq b \in X$

- $a >_{f(\pi)} b$ if more than $n/2$ of voters prefer candidate a to b;
- $b >_{f(\pi)} a$ if more than $n/2$ of voters prefer candidate b to a;
- $a \sim_{f(\pi)} b$ otherwise.

This is one example, but a collective choice function could be defined in a number of ways. It is quite easy to define a function where the outcome is that all alternatives are indifferent; however, this is not a very interesting or useful function. Just as easily, we could define a function g such that $g(\pi) = \pi(1)$ for all π in our domain. In this particular function g, voter 1 would be a *dictator*.

DEFINITION 2.7. *A function f is based on a* dictator *if there exists an $i \in N$ such that for every $\pi \in D(X)$ and any $a, b \in X$, if $a >_{\pi(i)} b$, then $a >_{f(\pi)} b$.*

To ensure that our functions represent the actual preferences of the voters as accurately as possible, we choose to limit our functions according to certain fairness criteria. Two classic fairness criteria are the *Pareto Property* (P) and *Independence of Irrelevant Alternatives* (IIA).

DEFINITION 2.8. *A collective choice function f satisfies the* ***Pareto property (P)*** *if for all $\pi \in D(X)$ and any $a, b \in X$,*

$$\textit{if } a >_{\pi(i)} b \textit{ for all } i \in N, \textit{ then } a >_{f(\pi)} b.$$

DEFINITION 2.9. *A collective choice function f satisfies* ***independence of irrelevant alternatives (IIA)*** *if for all $\pi, \theta \in D(X)$ and any $a, b \in X$,*

$$\textit{if } \pi|_{\{a,b\}} = \theta|_{\{a,b\}}, \textit{ then } f(\pi)|_{\{a,b\}} = f(\theta)|_{\{a,b\}}.$$

The Absolute Majority Rule defined above satisfies both (P) and (IIA). For a given pair $(a, b) \in X \times X$, if $a >_{\pi(i)} b$ for all $i \in N$, then more than $n/2$ voters prefer a to b, so $a >_{f(\pi)} b$. If two profiles π and π' agree on a pair $(a, b) \in X \times X$, then the same number of voters prefer a to b or vice versa, so the function f must agree on the two profiles. Logically, this function appears to be as "fair" as possible. However, there is one potential problem with Absolute Majority Rule. Suppose there are exactly 3 voters and 3 alternatives with the following preference profile:

π:	1	2	3
	a	c	b
	b	a	c
	c	b	a

(Note that we will often use tables such as this to represent a preference profile. We will denote an indifferent or tied pair using brackets as so: $[ab]$.) According to Absolute Majority Rule, since two of the voters prefer a to b, $a >_{f(\pi)} b$. Similarly, $b >_{f(\pi)} c$ and $c >_{f(\pi)} a$. Putting these together, we have that $a >_{f(\pi)} b >_{f(\pi)} c >_{f(\pi)} a$, a cycle. Which candidate would win this election? There is no obvious answer.

In order to avoid this scenario, we could place further restriction on the range of our functions. If we limit the range to, for example, the set of weak orders only, this would require that our final results were transitive relations, and thus there can be no cycles. However, adding this requirement causes a whole new problem.

In 1951, Kenneth Arrow [1] published the following landmark result, one of the most foundational results in the research area of mathematical social choice theory.

Theorem 2.1 (Arrow's Theorem)**.** *Assume that a collective choice function $f\colon W(X)^n \to C(X)$ satisfies (IIA) and (P), and the range of f is a subset of $W(X)$. If $|X| = m \geq 3$, then f is based on a unique dictator.*

This theorem says that if a collective choice function whose range is a subset of weak orders satisfies (IIA) and (P), and there are at least 3 voters, then there is a single voter known as a dictator, whose strict preferences are represented in the final outcome. Note that Arrow's Theorem does not hold for exactly two voters, as Absolute Majority Rule satisfies (IIA) and (P) and is not based on a dictator.

Arrow's Theorem is often paraphrased to say that "no truly fair voting system exists," and in its classic form, it is sometimes referred to as Arrow's Impossibility Theorem. The conditions of (P) and (IIA) are sometimes called the "Arrovian conditions," in honor of Kenneth Arrow, who in part won his Nobel Prize in Economics due to this result. Much of the work in social choice has been to alter or omit Arrow's conditions in an attempt to produce a different, "more fair," non-dictatorial outcome. One such early attempt was the idea of replacing full transitivity of the outcome with just *quasi-transitivity* on the strict preference relation.

Definition 2.10. *A complete binary relation $\geq$ on a set X is* ***quasitransitive*** *if for all $x, y, z \in X$, $x > y$ and $y > z$ implies that $x > z$.*

Notice that quasitransitivity only requires transitivity on the strict preference.

A less-restrictive version of Arrow's dictator is the oligarchy, which is defined below.

Definition 2.11. *A collective choice function $f : W(X)^n \to C(X)$ is based on an* ***oligarchy*** *if*

- *there exists a nonempty $L \subseteq N$ such that for every $\pi \in W(X)^n$ and any $a, b \in X$, if $a >_{\pi(i)} b$ for all $i \in L$, then $a >_{f(\pi)} b$, and*
- *for any $i \in L$, if $x >_{\pi(i)} y$, then it cannot be the case that $y >_{f(\pi)} x$.*

Throughout the 1970s, several researchers independently established this result [10], [11], [13].

Theorem 2.2. *Assume that a collective choice function $f\colon W(X)^n \to C(X)$ satisfies (IIA) and (P), and the range of f is a subset of the quasitransitive relations on X. If $|X| = m \geq 3$, then f is based on a unique oligarchy.*

This result was further generalized by Weymark where the assumption of social completeness is dropped [16].

Thus, instead of one voter independently determining the outcome of the election, we have a sort of ruling body of voters such that, if all members agree, can determine an outcome on a pair, and any single member of the oligarchy can veto a strict preference. This outcome is an improvement on Arrow's dictator, but it is still not ideal.

Our work seeks to further the discussion of Arrovian-based collective choice functions by studying a subset known as *Acyclic, Indifference-Transitive Collective Choice Functions*, or ACIT Collective Choice Functions. This concept was introduced in early 2013 in a paper published by Iritani, et. al [12]. We will construct collective choice functions whose range is a subset of $ACIT(X)$, the set of all acyclic, indifference-transitive relations on our set of alternatives X. First, we will define what these terms mean.

Definition 2.12. *An asymmetric binary relation ρ on X is* ***cyclic*** *if there exist elements $a_1, ..., a_m$ such that $a_1 >_\rho a_2 >_\rho a_3 >_\rho ...... >_\rho a_m >_\rho a_1$. Otherwise, we say that ρ is* ***acyclic****. For any $a, b \in X$, we have a indifferent to b, denoted by $a \sim_\rho b$, if $a \not>_\rho b$ and $b \not>_\rho a$.*

To avoid the paradoxical cycle we saw earlier, we will require the range of our functions to be acyclic. Collective choice functions whose range consists of any acyclic order on X have been studied [4]. However, unlike Arrow's requirement that the outcomes be fully-transitive weak orders, and unlike the quasitransitive requirement of previous theorems, we will only assume that we do not have cycles, and that we have transitivity of indifference.

Definition 2.13. *An asymmetric relation ρ on X is* ***indifference-transitive*** *if, for any $a, b, c \in X$, if $a \sim_\rho b$ and $b \sim_\rho c$, then $a \sim_\rho c$.*

If ρ is indifference-transitive, then the corresponding relation $\sim_\rho$ is an equivalence relation, which is a familiar concept.

Combining these two concepts, we define the following:

Definition 2.14. *The set $ACIT(X)$ is the set of all complete, acyclic, indifference-transitive relations on X.*

It is worth noting that

$$L(X) \subset W(X) \subset ACIT(X) \subset C(X).$$

While $ACIT$ relations are acyclic, they are not necessarily quasi-transitive, so there could potentially be a scenario where $a > b > c$, but $a \sim c$. It cannot be the case that $c > a$. All linear and weak orders meet the requirements to be ACIT relations, but there are some interesting relations which can appear. For example:

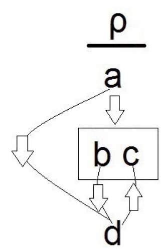

Figure 2.1: Example of $ACIT$ relation.

In this example, note that $b >_\rho d >_\rho c$, but we have $b \sim_\rho c$.

In this way, there is an interesting interplay between acyclicity and indifference-transitivity. These rules allow more flexiblity than rules restricted to weak orders, but the two conditions also interact in a unique way, which will be illustrated in some of the proofs presented in later chapters. We believe, as do Iritani, Kamo, and Nagahisa [12], that because of this trade-off between the two conditions, "AC-IT rules greatly merit investigation." Our goal for the remainder of this paper is to further that investigation.

CHAPTER 3
ACIT RULES WITH FREE-QUADRUPLE DOMAINS

In this chapter, we will discuss what happens when we generalize Arrow's theorem by enlarging the range of Arrovian functions to include the possibility of acylic, indifference-transitive relations on the set of alternatives. In addition, we will enlarge the class of domains of our functions. In Arrow's Theorem [1], the set of decisive subsets is an ultrafilter. In this chapter, we will show that for Arrovian ACIT functions, the set of weakly decisive subsets is an ultrafilter.

As before, we will let N be the set of voters, and let X be the set of alternatives. Recall that $L(X)$ represents the set of linear orders on X, $W(X)$ represents the set of weak orders on X, and $C(X)$ represents the set of all complete relations on X.

In early 2013 [12], the following result was proven:

Theorem 3.1 (IKN Generalization). *Assume that a collective choice function $f\colon W(X)^n \to C(X)$ satisfies (IIA) and (P), and the range of f is a subset of $ACIT(X)$. If $|X| = m \geq 4$, then f is based on a unique weakly decisive voter.*

(Note that we will refer to this theorem as the IKN Generalization.) The authors of [12], Iritani, Kamo, and Nagahisa, proved that weakening the requirement on the range of f implies that we no longer are guaranteed a dictator; instead, we are only guaranteed to have a unique weakly decisive voter.

Definition 3.1. *For $D(X) \subseteq W(X)^n$, and any function $f\colon D(X) \to C(X)$, an individual $i \in N$ is a* ***weakly decisive voter for*** *(x, y) if for every $\pi \in D(X)$,*

*$x >_{\pi_i} y$ implies $x \geq_{f(\pi)} y$. A set $I \subset N$ is **weakly decisive for** (x, y), if for every $\pi \in D(X)$, $x >_{\pi_i} y$ for all $i \in I$ implies $x \geq_{f(\pi)} y$.*

Depending on the context, we will either discuss weakly decisive voters or sets of voters. A weakly decisive voter is sometimes called a voter with "veto power." This is because if a voter is weakly decisive and she specifies a strict preference on a pair, she cannot be directly contradicted on this pair. If a weakly decisive voter is indifferent on a pair, then there is no restriction on the social outcome with respect to that pair.

Note that the IKN Generalization requires at least 4 alternatives, unlike Arrow's Theorem, which only requires 3 alternatives. In fact, this theorem does not hold for 3 alternatives, as shown in the following example from Iritani, et. al [12].

Example 3.1. *Let $X = \{a, b, c\}$. Define $f\colon W(X)^2 \to ACIT(X)$ as follows: For any $\pi \in W(X)^2$,*

- *If $b >_{\pi(i)} a$ for all $i \in N$, then $b >_{f(\pi)} a$; otherwise $a >_{f(\pi)} b$.*
- *If $c >_{\pi(i)} a$ for all $i \in N$, then $c >_{f(\pi)} a$; otherwise $a >_{f(\pi)} c$.*
- *If $b >_{\pi(i)} c$ (or $c >_{\pi(i)} b$) for all $i \in N$, then $b >_{f(\pi)} c$ (or $c >_{\pi(i)} b$); otherwise $b \sim_{f(\pi)} c$.*

Proof. There can be no indifference-transitive violation because there is at most one indifferent pair, $b \sim c$. On three elements, there are only two possible cycles, $a > b > c > a$ or $a > c > b > a$, neither of which are possible given the rules of the function.

- To demonstrate this, suppose that for $\pi \in W(X)^2$, $a >_{f(\pi)} b >_{f(\pi)} c >_{f(\pi)} a$. In order for there to be a strict preference on $\{b, c\}$, (P) must apply, so $b >_{\pi(i)} c$ for $i = 1, 2$. Since $c >_{f(\pi)} a$, it must be the case that $c >_{\pi(i)} a$ for $i = 1, 2$.

This implies that for $i = 1, 2$, by transitivity of weak orders, $b >_{\pi(i)} a$, which implies that $b >_{f(\pi)} a$, a contradiction.

- Similarly, if $a >_{f(\pi)} c >_{f(\pi)} b >_{f(\pi)} a$, then $c >_{\pi(i)} b >_{\pi(i)} a$ for all $i \in N$, and by transitivity, $c >_{\pi(i)} a$ for all $i \in N$. This implies that $c >_{f(\pi)} a$, a contradiction.

Thus, f is well-defined. It is easy to see that this example satisfies (P). To see why f satisfies (IIA), note that there are only two options for each pair: the Pareto case or the non-Pareto case; hence, if two profiles π and π' agree on a pair, then $f(\pi)$ and $f(\pi')$ agree on that pair. This implies that f satisfies (IIA). However, there is no unique weakly decisive voter because if the requirements for (P) are not met, it does not matter which of the voters disagrees with the outcome. $\square$

In this chapter, we will seek to further generalize this version of Arrow's Theorem. We have thus far discussed Arrow's Theorem in terms of functions whose domains are the set $W(X)^n$. In fact, Arrow's Theorem holds for functions $f\colon D(X) \to W(X)$ satisfying (IIA) and (P) where $D(X)$ satisfies the "free-triple property" [7].

Definition 3.2. *A set $D(X)$ such that $D(X) \subseteq W(X)^n$ satisfies the **free-triple property** if, for every 3-element subset X' of X, either*

$$\{\pi|_{X'} : \pi \in D(X)\} = L(X')^n, \text{ or}$$
$$\{\pi|_{X'} : \pi \in D(X)\} = W(X')^n.$$

In other words, $D(X)$ satisfies the free-triple property if, when restricted to any 3-element subset, the restriction is either the set of all linear orders or the set of all weak orders on that particular subset. Consider the following example [8]:

Example 3.2. *Let $X = \{a, b, c, d\}$, and let $L_d(X)$ represent the set of all linear orders on X such that d is neither at the top nor the bottom of the order. Let*

$D(X) = L_d(X)^n$. *For any* $\{x, y, z\} \subset X$, $\{\pi|_{\{x,y,z\}} : \pi \in D(X)\} = L(\{x, y, z\})^n$. *Hence,* $D(X)$ *satisfies the free-triple property.*

As it turns out, Arrow's Theorem can be generalized to include domains satisfying the free-triple property [7], such as the one described in Example 3.2.

Theorem 3.2 (Arrow's Theorem with Free-Triple Property)**.** *Assume that a collective choice function* $f\colon D(X) \to C(X)$ *satisfies (IIA) and (P). Also assume that* $D(X) \subseteq W(X)^n$ *satisfies the free-triple property, and the range of* f *is a subset of* $W(X)$*. If* $|X| = m \geq 3$*, then* f *is based on a unique dictator.*

Next, we will describe a new extension of the IKN Generalization. Since the ACIT theorem does not hold for only 3 alternatives, it does not necessarily hold for domains satisfying the free-triple property. However, since we know that it does hold for domains $W(X)^n$, where $|X| \geq 4$, we will make a similar attempt at generalizing the theorem. Consider the following property:

Definition 3.3. *A set* $D(X)$ *such that* $D(X) \subseteq W(X)^n$ *satisfies the* ***free-quadruple property*** *if, for every four-element subset* X' *of* X*, either*

$$\{\pi|_{X'} : \pi \in D(X)\} = L(X')^n, \text{ or}$$
$$\{\pi|_{X'} : \pi \in D(X)\} = W(X')^n.$$

Example 3.2 satisfies the free-triple property, but it does not satisfy the free-quadruple property because it does not contain linear or weak orders with d at the top or bottom. However, for a set $\{a, b, c, d, e\}$, if we let $L_d(X)$ be the set of all linear orders on X such that d is neither at the top nor the bottom of the order, then $L_d(X)^n$ satisfies the free-quadruple property. (Note that for $k \geq 5$, one could similarly define the "free k-tuple property.")

Because we must have at least 4 alternatives in order for ACIT rules with weak order domains to necessarily be based on a unique weakly decisive voter, we might naturally assume that generalizing our domain to include $D(X) \subseteq W(X)^n$ satisfying the free-triple property is not a strong enough condition to ensure a unique weakly decisive voter. As it happens, this assumption is correct. Consider the following example:

Example 3.3. *Let $X = \{a, b, c, d\}$, and let $L_d(X)$ represent the set of all linear orders on X such that d is neither at the top nor the bottom of the order. Let $D(X) = L_d(X)^n$. Define $f\colon D(X) \to ACIT(X)$ as follows:*

For any $\pi \in D(X)$,

- *If $b >_{\pi(i)} a$ for all $i \in N$, then $b >_{f(\pi)} a$; otherwise $a >_{f(\pi)} b$.*
- *If $c >_{\pi(i)} a$ for all $i \in N$, then $c >_{f(\pi)} a$; otherwise $a >_{f(\pi)} c$.*
- *If $b >_{\pi(i)} c$ (or $c >_{\pi(i)} b$) for all $i \in N$, then $b >_{f(\pi)} c$ (or $c >_{\pi(i)} b$); otherwise $b \sim_{f(\pi)} c$.*
- *If $b >_{\pi(i)} d$ for all $i \in N$, then $b >_{f(\pi)} d$; otherwise $d >_{f(\pi)} b$.*
- *If $c >_{\pi(i)} d$ for all $i \in N$, then $c >_{f(\pi)} d$; otherwise $d >_{f(\pi)} c$.*
- *If $a >_{\pi(i)} d$ (or $d >_{\pi(i)} a$) for all $i \in N$, then $a >_{f(\pi)} d$ (or $d >_{\pi(i)} a$); otherwise $a \sim_{f(\pi)} d$.*

Proof. The function f is defined such that the Pareto property must always hold. It is defined pairwise, so it must satisfy Independence of Irrelevant Alternatives. Notice that because the only possibilities for indifference are two disjoint pairs, an indifference-transitive violation is impossible.

We will demonstrate that a cycle is impossible. Let $\pi \in D(X)$. There are 4 possible 3-element subsets upon which a cycle could exist:

1. $\{a, b, c\}$

2. $\{d, b, c\}$

3. $\{a, b, d\}$

4. $\{a, c, d\}$

To demonstrate that a 3-cycle cannot exist on cases 1 and 2, let $x \in \{a, d\}$.

- If $x >_{f(\pi)} b >_{f(\pi)} c >_{f(\pi)} x$, then $b >_{\pi(i)} c >_{\pi(i)} x$ for $i = 1, 2$, and π must satisfy the following:

  $$\pi\big|_{\{x,b,c\}}: \quad \begin{array}{cc} 1 & 2 \\ \hline b & b \\ c & c \\ x & x \end{array}$$

  This implies that $b >_{\pi(i)} x$ for all $i \in N$, and since $x \in \{a, d\}$ (since b and c are already labelled), the function dictates that $b > f(\pi)x$. Since we assumed that $x >_{f(\pi)} b$, we have a contradiction.

- Similarly, if $x >_{f(\pi)} c >_{f(\pi)} b >_{f(\pi)} x$, then $c >_{\pi(i)} b >_{\pi(i)} x$ for $i = 1, 2$, and π must satisfy the following:

  $$\pi\big|_{\{x,b,c\}}: \quad \begin{array}{cc} 1 & 2 \\ \hline c & c \\ b & b \\ x & x \end{array}$$

  Since this implies that $c >_{\pi(i)} x$ for all $i \in N$, and $x \in \{a, d\}$, the function dictates that $c > f(\pi)x$. Since we assumed that $x >_{f(\pi)} c$, we have a contradiction.

Hence, there can be no cycles on either $\{a, b, c\}$ or $\{b, c, d\}$. There are two more possible 3- element subsets: $\{a, c, d\}$ and $\{a, b, d\}$. Let $x \in \{b, c\}$.

- Suppose that there is a cycle $a >_{f(\pi)} x >_{f(\pi)} d >_{f(\pi)} a$. Because $x >_{f(\pi)} d$, it is necessary that both $x >_{\pi(1)} d$ and $x >_{\pi(2)} d$. Since $d >_{f(\pi)} a$, it must be that both $d >_{\pi(1)} a$ and $d >_{\pi(2)} a$. As all of our voters submit linear order preferences, we have $x >_{\pi(i)} d >_{\pi(i)} a$, and thus $x >_{\pi(i)} a$ for all $i \in N$. In other words, π satisfies the folllowing:

$$\pi|_{\{a,d,x\}}: \quad \begin{array}{cc} 1 & 2 \\ \hline x & x \\ d & d \\ a & a \end{array}$$

  Thus it must be that $x >_{f(\pi)} a$, a contradiction. Thus this cycle cannot exist.

- Suppose that $a >_{f(\pi)} d >_{f(\pi)} x >_{f(\pi)} a$. Then it must be the case that $x >_{\pi(1)} a$ and $x >_{\pi(2)} a$, and $a >_{\pi(1)} d$ and $a >_{\pi(2)} d$. So π satisfies the following:

$$\pi|_{\{a,d,x\}}: \quad \begin{array}{cc} 1 & 2 \\ \hline d & d \\ x & x \\ a & a \end{array}$$

  This implies that $x >_{\pi(1)} d$ and $x >_{\pi(2)} d$, as both $\pi(1)$ and $\pi(2)$ are linear orders, and thus by (P) $x >_{f(\pi)} d$, a contradiction.

Therefore, there cannot be a cycle on either $\{a, b, d\}$ or $\{a, c, d\}$. This encompasses all possible 3-cycles; thus, there can be no 3-cycles.

Next, suppose there exists a 4-cycle, $d >_{f(\pi)} x >_{f(\pi)} y >_{f(\pi)} z >_{f(\pi)} d$, where $\{x, y, z\} = \{a, b, c\}$. There are three cases:

- Suppose $x = a$. Then $d >_{f(\pi)} a >_{f(\pi)} y >_{f(\pi)} z >_{f(\pi)} d$. This implies that π must satisfy the following when restricted to $\{a, d, z\}$:

$$\pi|_{\{a,d,z\}}: \quad \begin{array}{cc} 1 & 2 \\ \hline z & z \\ d & d \\ a & a \end{array}$$

Since these are linear orders, $z >_{\pi(i)} a$ for $i = 1$ and $i = 2$, which implies that $a >_{f(\pi)} y >_{f(\pi)} z >_{f(\pi)} a$, a 3-cycle, which we know cannot exist. Thus we have a contradiction.

- Suppose $y = a$. Then $d >_{f(\pi)} x >_{f(\pi)} a >_{f(\pi)} z >_{f(\pi)} d$. Then for both $i = 1, 2$, $z >_{\pi(i)} d$ and $x >_{\pi(i)} a$. Since $a >_{f(\pi)} z$, at least one of the voters must have $a > z$. Hence d must be at the bottom of one order, which is not allowed by our domain.

- Suppose $z = a$. Then $d >_{f(\pi)} x >_{f(\pi)} y >_{f(\pi)} a >_{f(\pi)} d$. This implies that π must satisfy the following when restricted to $\{a, d, z\}$:

$$\pi|_{\{a,d,y\}}: \quad \begin{array}{cc} 1 & 2 \\ \hline y & y \\ a & a \\ d & d \end{array}$$

Since $x >_{f(\pi)} y$, both voters cannot specify the opposite, so at least one must have $x > y$. This means that d must be at the bottom of either $\pi(1)$ or $\pi(2)$, which is not allowed by our domain.

Thus a 4-cycle cannot exist.

Since our function cannot contain cycles or indifference transitive violations (ITVs), and it satisfies (P) and (IIA), we may note that, similar to Example 3.1, there is no weakly decisive voter. $\square$

We will now investigate all functions $f\colon D(X) \to ACIT(X)$ satisfying (IIA) and (P), where $D(X)$ satisfies the free-quadruple property, and show that they are based on a unique weakly decisive voter. We use a somewhat different approach from that used in the proof of the IKN Generalization.

In [12], there seems to be a gap in the proof of the IKN Generalization. The problem occurs in footnote 7. Specifically, given an Arrovian ACIT rule f and individual $i \in N$, the authors consider the possibility that $N \setminus \{i\}$ is semidecisive for (b, a). They go on to say that, by their Lemma 4, it must be that $N \setminus \{i\}$ is semidecisive for (b, a), then this implies that $\{i\}$ is semidecisive for (b, a). The hypothesis of Lemma 4, however, requires that $N \setminus \{i\}$ is not a vetoer group. In other words, $N \setminus \{i\}$ is not a weakly decisive set. It appears that another argument is needed to address the case where $N \setminus \{i\}$ is weakly decisive. This gap has inspired us to use a different approach which completely describes the set of weakly decisive subsets using the concept of ultrafilters.

Before beginning, we will need a few definitions.

Definition 3.4. *A set $I \subseteq N$ is* ***semidecisive for*** (x, y) *(also written semidecisive for x against y), if for every $\pi \in D(X)$, if $x >_{\pi_i} y$ for all $i \in I$ and $y >_{\pi_j} x$ for all $j \notin I$, then $x >_{f(\pi)} y$.*

If we know that a set $I \subseteq N$ is semidecisive for (a, b), and we have a profile π such that all voters in I have $a > b$, and all other voters have $b > a$, the outcome $f(\pi)$ must have $a > b$. If all that we know is that I is semidecisive for a pair, and even just one voter outside of I agrees with I on that pair, we could not necessarily say that that is the outcome.

Definition 3.5. *Let C be a nonempty subset of N and $a \neq b \in X$. A profile $\pi \in D(X)$ is C-**competitive for*** (a, b) *if $a >_{\pi_i} b$ for all $i \in C$ and $b >_{\pi_j} a$ for all $j \notin C$.*

We will often use the concept of a C-competitive profile along with (IIA) in order to prove that the set C is semidecisive on a pair. For example, the following profile π is C-competitive for (a, b).

$$\pi\big|_{\{a,b\}}: \quad \begin{array}{c} C \\ \hline a \\ b \end{array} \quad \begin{array}{c} N \setminus C \\ \hline b \\ a \end{array}$$

Suppose we knew that for a collective choice function f satisfying (IIA), $a >_{f(\pi)} b$. Then, since any profile π' agreeing with π on the (a, b) pair would also require that $a >_{\pi'(i)} b$ for all $i \in I$, and $b >_{\pi'(j)} a$ for all $j \notin J$, (IIA) requires that $a >_{f(\pi')} b$. Thus I is semidecisive for (a, b).

The remainder of this chapter will be devoted to proving the next theorem, which is one of the main results of this thesis. To do so, we will employ a series of lemmas and one proposition.

Theorem 3.3. *Assume that a collective choice function $f\colon D(X) \to C(X)$ satisfies (IIA) and (P). Also assume that $D(X) \subseteq W(X)^n$ satisfies the free-quadruple property, and the range of f is a subset of $ACIT(X)$. If $n \geq 2$ and $|X| = m \geq 4$, then f is based on a unique weakly decisive voter.*

Assume f satisfies Independence of Irrelevant Alternatives (IIA), and Pareto (P) with $n \geq 2$ and $|X| = m \geq 4$.

For a diagram representing preference profile π, we will use the notation $\{x, y\}$ on a set $L \subseteq N$ to mean that for any $i \in L$, for$x \neq y \in X$, any one of the following could be true:

1. $x >_{\pi(i)} y$

2. $y >_{\pi(i)} x$

3. $x \sim_{\pi(i)} y$

Lemma 3.1. *If $C \subset N$ is semidecisive for a against b, then C is weakly decisive for (a, x), for all $x \neq a \in X$, and C is weakly decisive for (y, b) for all $y \neq b \in X$.*

Proof. Suppose that $C \subset N$ is semidecisive for (a, b), for $a \neq b \in X$. Let $x \in X \setminus \{a, b\}$. Consider any profile $\pi \in D(X)$ as shown below.

$$\pi: \quad \begin{array}{c|c} C & N \setminus C \\ \hline a & \{a, x\} \\ x & \end{array}$$

Construct any profile $\pi^1 \in D(X)$ such that $\pi|_{\{a,x\}} = \pi^1|_{\{a,x\}}$ which satisfies the following:

$$\pi^1: \quad \begin{array}{c|c} C & N \setminus C \\ \hline a & b \\ b & \{a, x\} \\ x & \end{array}$$

Because C is semidecisive for (a, b), $a >_{f(\pi^1)} b$. By (P), $b >_{f(\pi^1)} x$. So we have that $a >_{f(\pi^1)} b >_{f(\pi^1)} x$, and by acyclicity, $a \geq_{f(\pi^1)} x$. By (IIA), $a \geq_{f(\pi)} x$. Since the profile π above was arbitrary, it now follows that C is weakly decisive for (a, x).

Let $y \in X \setminus \{a, b\}$. Construct $\pi^2 \in D(X)$ which agrees with some arbitrary profile π' on the pair $\{y, b\}$ and satisfies the following:

$$\pi^2: \quad \begin{array}{c|c} C & N \setminus C \\ \hline y & \{b, y\} \\ a & a \\ b & \end{array}$$

Since C is semidecisive for (a, b), $a >_{f(\pi^2)} b$, and by (P), $y >_{f(\pi^2)} a$. Thus $y >_{f(\pi^2)} a >_{f(\pi^2)} b$, and by acyclicity, $y \geq_{f(\pi^2)} b$. Therefore, by (IIA), C is weakly decisive for (y, b).

We will now show that C is weakly decisive for the pair (a, b). Again, let $x \in X \setminus \{a, b\}$. Construct π^3 satisfying the following:

π^3:

C	$N \setminus C$
a	b
x	x
b	a

Because C is semidecisive for (a, b), $a >_{f(\pi^3)} b$. As shown above, C is weakly decisive for (a, x) and (x, b), so $a \geq_{f(\pi^3)} x \geq_{f(\pi^3)} b$. It cannot be the case that both $a \sim_{f(\pi^3)} x$ and $x \sim_{f(\pi^3)} b$, since that would imply that $a \sim_{f(\pi^3)} b$. Thus either $a >_{f(\pi^3)} x$ or $x >_{f(\pi^3)} b$. If $a >_{f(\pi^3)} x$, then C is semidecisive for (a, x). Since $b \in X \setminus \{a, x\}$, as shown above, C is weakly decisive for (a, b). If $x >_{f(\pi^3)} b$, then C is semidecisive for (x, b). Since $a \in X \setminus \{b, x\}$, C is weakly decisive for (a, b). In either case, C is weakly decisive for (a, b). Thus, C is weakly decisive for all (a, x), $x \neq a \in X$, and for all (y, b), $y \neq b \in X$.

□

In Example 3.3, notice that voter 1 is semidecisive for (a, b) and voter 2 is semidecisive for (d, c). As the next lemma shows, this cannot happen under the free-quadruple property.

Lemma 3.2. *There cannot exist two disjoint subsets C and I of N such that C is semidecisive for (a, b) and I is semidecisive for (c, d) with $\{a, b\} \cap \{c, d\} = \emptyset$.*

Proof. Suppose that there exist two disjoint subsets C and I of N such that C is semidecisive for (a, b), and I is semidecisive for (c, d), with $\{a, b\} \cap \{c, d\} = \emptyset$. Consider a profile $\pi \in D(X)$, satisfying the following:

π:

C	I	$N \setminus (C \cup I)$
d	b	b
a	c	d
b	d	a
c	a	c

Since C is semidecisive for (a, b), $a >_{f(\pi)} b$. Since I is semidecisive for (c, d), $c >_{f(\pi)} d$. For all $k \in N$, $b >_{\pi_k} c$, and $d >_{\pi_k} a$; thus, (P) implies that $b >_{f(\pi)} c$ and $d >_{f(\pi)} a$. Therefore we have a cycle: $a >_{f(\pi)} b >_{f(\pi)} c >_{f(\pi)} d >_{f(\pi)} a$, which is a contradiction.

$\square$

The proof of the following lemma is essentially identical to that found in the proof of Lemma 3.1 in [12]

Lemma 3.3. *Let C be any nonempty subset of N, and suppose that $a \sim b$ at a C-competitive profile for a against b. Let $x \in X \setminus \{a, b\}$. Then*

1. *$a \not\sim_{f(\pi)} x$ at any C-competitive profile $\pi \in D(X)$ for (a, x).*

2. *$b \not\sim_{f(\pi)} x$ at any C-competitive profile $\pi \in D(X)$ for (x, b).*

3. *$a \not\sim_{f(\pi)} x$ at any C-competitive profile $\pi \in D(X)$ for (x, a).*

4. *$b \not\sim_{f(\pi)} x$ at any C-competitive profile $\pi \in D(X)$ for (b, x).*

It follows from (P) that N is semidecisive for (a, b) for all $a \neq b \in X$. The next lemma shows how this observation can be extended to any proper nonempty subset of N.

Lemma 3.4. *For any nonempty proper subset C of N, there exists $a \neq b \in X$ such that either C or $N \setminus C$ is semidecisive for (a, b).*

Proof. Suppose there exists some $C \subset N$ such that neither C nor $N \setminus C$ is semidecisive for (x, y), for all $x \neq y \in X$. Then there exists a profile $\pi \in D(X)$ as follows,

π:	C	$N \setminus C$
	x	y
	y	x

where $y \geq_{f(\pi)} x$, because C is not semidecisive for (x, y). Similarly, there exists a $\pi' \in D(X)$ as follows,

$$\pi': \quad \begin{array}{c} C \\ \hline x \\ y \end{array} \quad \begin{array}{c} N \setminus C \\ \hline y \\ x \end{array}$$

where $x \geq_{f(\pi')} y$, because $N \setminus C$ is not semidecisive for (y, x). By (IIA), $f(\pi)|_{\{x,y\}} = f(\pi')|_{\{x,y\}}$. Thus, if π is C-competitive for (x, y), then $x \sim_{f(\pi)} y$.

Consider π'' satisfying:

$$\pi'': \quad \begin{array}{c} C \\ \hline a \\ c \\ b \end{array} \quad \begin{array}{c} N \setminus C \\ \hline c \\ b \\ a \end{array}$$

π'' is C-competitive for (a, b) and (a, c), which implies that $a \sim_{f(\pi'')} b$ and $a \sim_{f(\pi'')} c$, and by indifference-transitivity, $b \sim_{f(\pi'')} c$. However, (P) implies that $c >_{f(\pi'')} b$; therefore, this is a contradiction. $\square$

This next lemma will be helpful when proving our main result.

Lemma 3.5. *Assume C is semidecisive for (a, b). If $x \neq y \in X \setminus \{a, b\}$, then C is semidecisive for at least one of (a, x) or (a, y).*

Proof. Suppose C is semidecisive for (a, b), but C is not semidecisive for (a, x) or (a, y). Construct a profile π as follows, and note that π is C-competitive for (a, x) and (a, y).

$$\pi: \quad \begin{array}{c} C \\ \hline a \\ x \\ y \end{array} \quad \begin{array}{c} N \setminus C \\ \hline x \\ y \\ a \end{array}$$

Lemma 3.1 implies that C is weakly decisive for (a, x) and (a, y); therefore $a \geq_{f(\pi)} x$ and $a \geq_{f(\pi)} y$. However, if either $a >_{f(\pi)} x$ or $a >_{f(\pi)} y$, C would be semidecisive for that pair. Hence $a \sim_{f(\pi)} x$ and $a \sim_{f(\pi)} y$. By indifference transitivity, $x \sim_{f(\pi)} y$. However, by (P), $x >_{f(\pi)} y$, so this is a contradiction. $\square$

The preceding lemmas are combined in the proof of the following proposition. This proposition is essential in proving our theorem.

Proposition 3.1. *If $C \subset N$ is semidecisive for some (a, b), then C is weakly decisive. Conversely, if C is weakly decisive, then C is semidecisive for at least one pair (a, b).*

Proof. Suppose $C \subset N$ is semidecisive for (a, b), for some $a \neq b \in X$. By Lemma 3.1, C is weakly decisive for (a, x) for all $x \neq a \in X$, and (y, b) for all $y \neq b \in X$.

We still need to show that C is weakly decisive for the following pairs: (x, a) for all $x \neq a \in X$, (b, y) for all $y \neq b \in X$, and (x, y) for all $x \neq y \in X \setminus \{a, b\}$.

Our first step is to show that C is weakly decisive for (x, a). Let $x \neq y \in X \setminus \{a, b\}$. Since $D(X)$ satisfies the free-quadruple property, there exists a profile $\pi \in D(X)$ satisfying the following:

$$\pi: \quad \begin{array}{cc} C & N \setminus C \\ \hline x & b \\ a & y \\ b & a \\ y & x \end{array}$$

By Lemma 3.2 we know that $N \setminus C$ cannot be semidecisive for (y, x). This implies that $x \geq_{f(\pi)} y$. By Lemma 3.1, we have that C is weakly decisive for (x, b), so $x \geq_{f(\pi)} b$. By (P), $b >_{f(\pi)} y$. If $x >_{f(\pi)} b$, then by (IIA), C is semidecisive for (x, b), and by Lemma 3.1, C is weakly decisive for (x, a). If $x \sim_{f(\pi)} b$, then $x \not\sim_{f(\pi)} y$, because $b >_{f(\pi)} y$. This implies that $x >_{f(\pi)} y$, which implies that C is semidecisive for (x, y), and by Lemma 3.1, C is semidecisive for (x, a). In either case, C is weakly decisive for (x, a).

Consider π^2 satisfying:

$$\pi^2: \quad \begin{array}{cc} C & N \setminus C \\ \hline a & x \\ b & y \\ x & b \\ y & a \end{array}$$

Because C is semidecisive for (a, b), $a >_{f(\pi^2)} b$. By (P), $x >_{f(\pi^2)} y$, and by Lemma 3.1, $a \geq_{f(\pi^2)} x$, and $a \geq_{f(\pi^2)} y$. Since we know that $x >_{f(\pi^2)} y$, it must be the case that either $a >_{f(\pi^2)} x$, or $a >_{f(\pi^2)} y$. If $a >_{f(\pi^2)} y$, then C is semidecisive for (a, y), and by Lemma 3.1, C is weakly decisive for (b, y). If $a >_{f(\pi^2)} x$, then C is semidecisive for (a, x), and is thus weakly decisive for (b, x), which implies that $b \geq_{f(\pi^2)} x$. Since $N \setminus C$ cannot be semidecisive for (y, b), $b \geq_{f(\pi^2)} y$. Since we know that $x >_{f(\pi^2)} y$, then, in order to avoid an (ITV), it must be the case that either $b >_{f(\pi^2)} y$ or $b >_{f(\pi^2)} x$. In either case, C is weakly decisive for (b, y).

Consider π^3:

$$\pi^3: \quad \begin{array}{cc} C & N \setminus C \\ \hline x & a \\ y & x \\ a & y \end{array}$$

By (P), $x >_{f(\pi^3)} y$. Since C is weakly decisive for (x, a) and (y, a), $x \geq_{f(\pi^3)} a$ and $y \geq_{f(\pi^3)} a$. Since $x >_{f(\pi^3)} y$, both (x, a) and (y, a) cannot be indifferent. Thus either $x >_{f(\pi^3)} a$ or $y >_{f(\pi^3)} a$. Hence, C is either semidecisive for (x, a) or (y, a). In either case, Lemma 3.1 gives that C is weakly decisive for (b, a).

Finally, we will show that C is weakly decisive for the pair (x, y), for $x \neq y \in X \setminus \{a, b\}$. Consider profile π^4:

$$
\pi^4: \quad \begin{array}{c} C \\ \hline x \\ a \\ b \\ y \end{array} \quad \begin{array}{c} N \setminus C \\ \hline b \\ y \\ x \\ a \end{array}
$$

By (P), $x >_{f(\pi^4)} a$ and $b >_{f(\pi^4)} y$. Since C is semidecisive for (a, b), $a >_{f(\pi^4)} b$. Combining this information, we have that $x >_{f(\pi^4)} a >_{f(\pi^4)} b >_{f(\pi^4)} y$. Acyclicity gives that both $x \geq_{f(\pi^4)} b$ and $x \geq_{f(\pi^4)} y$. Because $b >_{f(\pi^4)} y$, one of these pairs must be a strict preference. If $x >_{f(\pi^4)} b$, then C is semidecisive for (x, b), which implies that C is weakly decisive for (x, y). If $x >_{f(\pi^4)} y$, then C is semidecisive for (x, y), which implies that C is weakly decisive for (x, y).

Thus, C is weakly decisive.

Suppose there exists a $C \subset N$ that is weakly decisive for all pairs, but semidecisive for no pairs. Then by Lemma 3.4, there exists $a \neq b \in X$ such that $N \setminus C$ is semidecisive for (a, b). Then for any profile $\pi \in D(X)$ satisfying the following:

$$
\pi: \quad \begin{array}{c} C \\ \hline b \\ a \end{array} \quad \begin{array}{c} N \setminus C \\ \hline a \\ b \end{array}
$$

we have that $a >_{f(\pi)} b$. However, C is weakly decisive for (b, a), which implies that $a \not>_{f(\pi)} b$. Thus, this is a contradiction, so C must be semidecisive for at least one pair.

□

We will now prove Theorem 3.3. To do so, we will need the following definition [2].

Definition 3.6 (Ultrafilter)**.** *For any set T, let $\mathcal{T}$ be a family of subsets of T. $\mathcal{T}$ is an **ultrafilter** if and only if the following all hold*

- *$T \in \mathcal{T}$ and $\emptyset \notin \mathcal{T}$.*
- *If $T_1 \in \mathcal{T}$, and $T_1 \subseteq T_2$, then $T_2 \in \mathcal{T}$.*
- *If $T_1 \in \mathcal{T}$ and $T_2 \in \mathcal{T}$, then $T_1 \cap T_2 \in \mathcal{T}$.*
- *For any $T_1 \subseteq T$, either $T_1 \in \mathcal{T}$ or $T \setminus T_1 \in \mathcal{T}$.*

It is known that a finite ultrafilter has a smallest element, which will be very useful in this proof. Recall that in the case of Arrow's Theorem, the collection of decisive sets is an ultrafilter. We will prove that for our ACIT functions, the set of weakly decisive sets is an ultrafilter.

Proof. (Theorem 3.3) We will first show that the set of weakly decisive subsets of N is an ultrafilter. Let S be that set. Since the empty set is not weakly decisive, the empty set is not an element of S. If $I \subset J$ and I is weakly decisive, then if for some $\pi \in D(X)$, for all $j \in J$, $a >_{\pi_j} b$, then it must be that for all $i \in I$, $a >_{\pi_i} b$. Since I is weakly decisive, $a \geq_{f(\pi)} b$. Thus J is weakly decisive.

By Lemma 3.4, for every subset I of N, either I or $N \setminus I$ is semidecisive for some $(a, b) \in X \times X$, and by Proposition 3.1, one of those sets must be weakly decisive. (This also tells us that there must be at least one weakly decisive subset of N.)

We will next show that if I and J are weakly decisive, then $I \cap J$ must also be weakly decisive. If I and J are both weakly decisive, then by Proposition 3.1 we know that for each set I and J, there exist $x \neq y \in X$ such that the set is semidecisive for (x, y). Then there are 4 cases:

1. *I is semidecisive for (a, b) and J is semidecisive for (b, c).*

   Consider $\pi \in D(X)$:

π:	$I \setminus J$	$I \cap J$	$J \setminus I$	$N \setminus (I \cap J)$
	c	a	b	c
	d	b	c	b
	a	c	d	d
	b	d	a	a

Because I is semidecisive for (a, b), $a >_{f(\pi)} b$. Because J is semidecisive for (b, c), $b >_{f(\pi)} c$. Acyclicity implies that $a \geq_{f(\pi)} c$. By (P), $c >_{f(\pi)} d$, and by acyclicity, since $a >_{f(\pi)} b >_{f(\pi)} c >_{f(\pi)} d$, it must be that $a \geq_{f(\pi)} d$. Since $c >_{f(\pi)} d$, it cannot be the case that both $a \sim_{f(\pi)} c$ and $a \sim_{f(\pi)} d$. Thus $a >_{f(\pi)} c$ or $a >_{f(\pi)} d$. In either case, $I \cap J$ is semidecisive for some pair, and by Proposition 3.1, $I \cap J$ is weakly decisive.

2. *I is semidecisive for (a, b) and J is semidecisive for (c, d).*

   Consider $\pi \in D(X)$:

π:	$I \setminus J$	$I \cap J$	$J \setminus I$	$N \setminus (I \cap J)$
	d	a	b	b
	a	b	c	d
	b	c	d	c
	c	d	a	a

   Since I is semidecisive for (a, b), $a >_{f(\pi)} b$, and since J is semidecisive for (c, d), $c >_{f(\pi)} d$. By (P), $b >_{f(\pi)} c$. Thus $a >_{f(\pi)} b >_{f(\pi)} c >_{f(\pi)} d$. This implies that $a \geq_{f(\pi)} c$ and $a \geq_{f(\pi)} d$. Since $c >_{f(\pi)} d$, it cannot be the case that both $a \sim_{f(\pi)} c$ and $a \sim_{f(\pi)} d$. If $a >_{f(\pi)} c$, then I is semidecisive for (a, c), and this is equivalent to case 1. If $a >_{f(\pi)} d$, then $I \cap J$ is semidecisive for (a, d), and by Proposition 3.1, $I \cap J$ is weakly decisive.

3. *I is semidecisive for (a, b) and J is semidecisive for (a, c).*

   Consider $\pi \in D(X)$:

π:	$I \setminus J$	$I \cap J$	$J \setminus I$	$N \setminus (I \cap J)$
	c	b	b	c
	d	a	a	d
	b	c	c	b
	a	d	d	a

Because J is semidecisive for (a, c), $a >_{f(\pi)} c$. By (P), $b >_{f(\pi)} a$ and $c >_{f(\pi)} d$. This implies that $b >_{f(\pi)} a >_{f(\pi)} c >_{f(\pi)} d$. Acyclicity implies that $b \geq_{f(\pi)} c$ and $b \geq_{f(\pi)} d$. Since $c >_{f(\pi)} d$, both pairs cannot be indifferent. If $b >_{f(\pi)} c$, then J is semidecisive for (b, c). If $b >_{f(\pi)} d$, then J is semidecisive for (b, d). Either way, this is equivalent to case 1.

4. *I is semidecisive for (a, b) and J is semidecisive for (c, b).*

   Since I is semidecisive for (a, b) and $c \neq d \in X \setminus \{a, b\}$, by Lemma 3.5, I is semidecisive for either (a, c) or (a, d). If I is semidecisive for (a, c), this is equivalent to case 1. If I is semidecisive for (a, d), this is equivalent to case 2.

5. *I is semidecisive for (a, b) and J is semidecisive for either (a, b) or (b, a).*

   If J is semidecisive for (a, b), then by Lemma 3.5, J is semidecisive for (a, c), which is equivalent to case 3. If J is semidecisive for (b, a), then by Lemma 3.5, J is semidecisive for (b, c), which is equivalent to case 1.

Thus, if I and J are both weakly decisive, then $I \cap J$ is also weakly decisive.

Therefore by definition, S, the set of weakly decisive subsets of N is an ultrafilter. Since N is finite, S is a finite ultrafilter, and therefore S has a least element, the intersection of all sets in the ultrafilter. This least element must be a set $I \subset N$ such that $|I| = 1$. If $|I| > 1$, then there exist $i \neq j \in I$. As a result of Lemma 3.4 and Proposition 3.1, either $\{i\}$ or $N \setminus \{i\}$ are weakly decisive, and either $\{j\}$ or $N \setminus \{j\}$ are weakly decisive. Both $\{i\}$ and $\{j\}$ cannot be weakly decisive, since both are subsets of I, and I is the smallest element of S. This implies that

$N \setminus \{i\}$ and $N \setminus \{j\}$ are weakly decisive, which implies that one of the singleton sets cannot be contained in the least element subset. Thus, this is a contradiction, so $|I| = 1$. Hence, f is based on a unique weakly decisive voter. $\square$

Therefore, we have shown that ACIT collective choice functions satisfying (P) and (IIA) whose domain satisfies the free-quadruple property are based on a unique weakly decisive voter. While we know that Arrovian ACIT functions with free-triple domains are not necessarily based on a weakly decisive voter, describing all such functions remains an open question.

CHAPTER 4
A SOFTWARE APPROACH TO ACIT RULES

In an effort to generate some examples of functions satisfying the conditions of Theorem 3.3, we have designed an algorithm, implemented in C++, to generate a list of all such functions with two voters and four alternatives, where the domain is the set $L(X)^n$. In doing so, we determined the following theorem:

THEOREM 4.1. *The number of collective choice functions $f : L(X)^2 \to ACIT(X)$ satisfying (IIA) and (P), with $|X| = m = 4$, is 92.*

In the first section, we discuss the case where $m = 4$. In the second section, we show why this algorithm can be extended to $m \geq 5$.

4.1 The Algorithm for $m = 4$

We will demonstrate the above claim using a combination of mathematical proof and through a computer algorithm designed to represent such functions.

For now, we will limit our discussion to, as shown in the theorem, functions whose domain consists of preference profiles containing two voters and four alternatives, with only linear orders, and whose preference profiles map into $C(X)$. With four alternatives $\{a, b, c, d\}$ there are six different relationships which determine an ordering in $C(X)$. Thus, for any $\rho \in C(X)$, we will let σ_ρ be the following six-tuple:

$$(ab, ac, ad, bc, bd, cd),$$

where each position in the six-tuple represents the relationship between the two elements named by that pair, according to ρ. If $a > b$, then we set $ab = 1$. If

$b > a$, then $ab = -1$, and if $a \sim b$, then $ab = 0$. The same follows for all pairs, with the particular position containing a 1 if the element who comes first alphabetically in the pair is preferred to the other element, and so forth. We will reference the elements of σ_ρ as follows:

- $\sigma_\rho(1) = ab$;
- $\sigma_\rho(2) = ac$;
- $\sigma_\rho(3) = ad$;
- $\sigma_\rho(4) = bc$;
- $\sigma_\rho(5) = bd$;
- $\sigma_\rho(6) = cd$.

For example, if ρ is the linear ordering $a >_\rho b >_\rho c >_\rho d$, then ρ is represented by the 6-tuple

$$\sigma_\rho = (1, 1, 1, 1, 1, 1).$$

Similarly, if ρ^2 is the linear ordering $b >_{\rho^2} c >_{\rho^2} a >_{\rho^2} d$, then ρ^2 is

$$\sigma_{\rho^2} = (-1, -1, 1, 1, 1, 1).$$

If the order $\rho^3 \in W(X)$ satisfies $a >_{\rho^3} [bc] >_{\rho^3} d$, then

$$\sigma_{\rho^3} = (1, 1, 1, 0, 1, 1).$$

For the next result, we need to introduce nine functions. For $i = 1, 2, ..., 9$, we define $g_i : \{-1, 1\}^2 \to \{-1, 0, 1\}$ as follows:

- $g_i(1, 1) = 1$ and $g_i(-1, -1) = -1$ for all i;
- $g_1(r, s) = 1$ for all $(r, s) \in \{-1, 1\}^2$ where $r \neq s$;
- $g_2(r, s) = -1$ for all $(r, s) \in \{-1, 1\}^2$ where $r \neq s$;
- $g_3(r, s) = 0$ for all $(r, s) \in \{-1, 1\}^2$ where $r \neq s$;

- $g_4(r,s) = r$ for all $(r,s) \in \{-1,1\}^2$ where $r \neq s$;
- $g_5(r,s) = s$ for all $(r,s) \in \{-1,1\}^2$ where $r \neq s$;
- $g_6(r,s) = sign\{|r| + s\}$ for all $(r,s) \in \{-1,1\}^2$ where $r \neq s$;
- $g_7(r,s) = -sign\{|r| + s\}$ for all $(r,s) \in \{-1,1\}^2$ where $r \neq s$;
- $g_8(r,s) = sign\{r + |s|\}$ for all $(r,s) \in \{-1,1\}^2$ where $r \neq s$;
- $g_9(r,s) = -sign\{r + |s|\}$ for all $(r,s) \in \{-1,1\}^2$ where $r \neq s$.

(Note that since r and s are derived from linear orders, we do not need to allow for either to be 0.)

Recall that a profile $\pi \in L(X)^2$ contains two linear orders, and we denote $\pi = (\pi(1), \pi(2))$. For brevity, we may refer to a g_i function evaluated at a profile π restricted to a certain pair, as in $g_i(\pi|_{\{a,b\}})$. By convention, what we really mean by this notation is $g_i(\sigma_{\pi(1)}(ab), \sigma_{\pi(2)}(ab))$.

Lemma 4.1. *A function $f : L(X)^2 \to C(X)$ where $|X| = 4$ satisfies (IIA) and (P) if and only if for each $x \neq y \in X$, there exists $i \in \{1, 2, ..., 9\}$ such that $\sigma_{f(\pi)}(xy) = g_i(\pi|_{\{x,y\}})$ for all $\pi \in L(X)^2$.*

Proof. Suppose $f : L(X)^2 \to C(X)$ satisfies the following condition:

For each $x \neq y \in X$, there exists $i \in \{1, 2, ..., 9\}$ such that

$$\sigma_{f(\pi)}(xy) = g_i(\pi|_{\{x,y\}}) \text{ for all } \pi \in L(X)^2. \textbf{ (*)}$$

We will demonstrate that f satisfies (IIA) and (P).

Let $\pi \in L(X)^2$ be a profile such that $x >_{\pi(1)} y$ and $x >_{\pi(2)} y$. Then $\pi|_{\{x,y\}}$ is represented by either $(1,1)$ or $(-1,1)$. Since $g_i(1,1) = 1$ and $g_i(-1,-1) = -1$, it follows from property **(*)** that we have $x >_{f(\pi)} y$. Thus f satisfies (P).

Suppose $\pi|_{\{x,y\}} = \pi'|_{\{x,y\}}$. By property (*), there exists $i \in \{1, 2, ..., 9\}$ such that $\sigma_{f(\pi)}(xy) = g_i(\pi|_{\{x,y\}}) = g_i(\pi'|_{\{x,y\}}) = \sigma_{f(\pi')}(xy)$. Now $\sigma_{f(\pi)}(xy) = \sigma_{f(\pi')}(xy)$ which implies that $f(\pi)|_{\{x,y\}} = f(\pi')|_{\{x,y\}}$. Thus f satisfies (IIA).

Next, suppose that $f : L(X)^2 \to C(X)$ satisfies (IIA) and (P). For each $x \neq y \in X$, we define a function $g_{xy} : \{-1, 1\}^2 \to \{-1, 0, 1\}$ in the following way. For any ordered pair $(r, s) \in \{-1, 1\}^2$, choose $\pi \in L(X)^2$ such that $\sigma_{\pi(1)}(xy) = r$ and $\sigma_{\pi(2)}(xy) = s$, and let $g_{xy}(r, s) = \sigma_{f(\pi)}(xy)$.

Since f satisfies (IIA), $\pi|_{\{x,y\}} = \pi'|_{\{x,y\}}$ implies that $\sigma_{f(\pi)}(xy) = \sigma_{f(\pi')}(xy)$, and so the definition of g_{xy} does not depend on the choice of profile π.

By (P), we get $g_{xy}(1, 1) = 1$ and $g_{xy}(-1, -1) = -1$.

Next, note that $g_{xy}(1, -1)$ and $g_{xy}(-1, 1)$ are both elements of $\{-1, 0, 1\}$. Thus there are three possibilities for $g_{xy}(1, -1)$ and three possibilities for $g_{xy}(-1, 1)$, for a total of nine possible mappings g_{xy}. These nine possibilities are the functions $g_1, g_2, ...g_9$ defined above.

For example, the function g_1 maps all pairs in $\{-1, 1\}^2$ to 1 except for the pair $(-1, -1)$, which (P) determines must be mapped to -1. In this way, we see that f satisfies (*).

$\square$

Thus we can represent every function $f : L(X)^2 \to C(X)$ with $|X| = 4$ satisfying (IIA) and (P) by a 6-tuple of functions from the list of g_i functions. Because there are nine possible outcomes for each of the six pairs, there are 9^6 or $53, 1441$ possible functions from $L(X)^2$ to $C(X)$ such that $|X| = 4$ and each function satisfies (IIA) and (P). The next step is to narrow down these functions to those whose range only consists of elements of $ACIT(X)$.

For each possible function, our algorithm is designed to cycle through all of the possible domain elements, checking the outcome determined by the function for each input, after which it determines whether or not the outcome is in fact an acyclic, indifference-transitive relation. We first need to verify that our outcome contains no cycles. We also need to ensure that our outcome does not contain an indifference-transitive violation (ITV). An (ITV) occurs whenever there exists a set of three elements $\{x, y, z\} \subseteq X$ such that

$$x{\sim}y{\sim}z, \text{ but } x{\nsim}z.$$

In order to eliminate all functions whose range contains either a cycle or an (ITV), we employ the following lemma:

Lemma 4.2. *Let ρ be a binary relation on X. $\rho|_{\{x,y,z\}}$ satisfies $(xy+yz+xz = \pm 1)$ and $(xy)(yz)(xy-yz) = 0$ if and only if $\rho|_{\{x,y,z\}}$ contains a 3-cycle or an indifference-transitive violation.*

Proof. Note that $(xy)(yz)(xy - yz) = 0$ if and only if either $xy = 0$, $yz = 0$, or $xy = yz$.

To prove the forward direction, consider first the case where $xy = 0$ and $xy + yz + xz = \pm 1$. In this case, since $xy = 0$, $yz + xz = \pm 1$. This implies that either $yz = 0$ and $xz = \pm 1$, or $xz = 0$ and $yz = \pm 1$. If the former is true, then $x{\sim}y$ and $y{\sim}z$, but $x{\nsim}z$, which is an (ITV). If the latter is true, then $x{\sim}z$ and $x{\sim}y$, but $y{\nsim}z$, which is also an (ITV). Similarly, if $yz = 0$ and $xy + yz + xz = \pm 1$, there will also be an (ITV).

If $xy = yz$ and neither are 0 and $xy + yz + xz = 1$, then it must be the case that $xy = yz = 1$, because otherwise, $xz = 3$, which is not possible. Therefore $xz = -1$, and so $x > y > z > x$, which is a 3-cycle. Similarly, if $xy = yz$ and neither are 0 and $xy + yz + xz = -1$, then $xy = yz = -1$ and $xz = 1$, which implies that $x > z > y > x$, another 3-cycle.

To verify the converse, first suppose that $\rho_{\{x,y,z\}}$ contains a 3-cycle. If that cycle is $x > y > z > x$, then $xy = 1$, $xz = -1$, and $yz = 1$, so $xy = yz$, $xy + yz + xz = 1$, and $(xy)(yz)(xy - yz) = 0$. If that cycle is $x > z > y > x$, then $xy = -1$, $xz = 1$, and $yz = -1$, so $xy = yz$, and $xy + yz + xz = -1$, and $(xy)(yz)(xy - yz) = 0$.

Next, suppose that $\rho_{\{x,y,z\}}$ contains an (ITV).

- If $x \sim y$, $y \sim z$, and $x \nsim z$, then $xy = 0$,$yz = 0$, and $xz = \pm 1$. Thus $xy + yz + xz = \pm 1$ and $xy = 0$.

- If $x \sim y$, $x \sim z$, and $y \nsim z$, then $xy = 0$,$xz = 0$, and $yz = \pm 1$. Thus $xy + yz + xz = \pm 1$ and $xy = 0$.

- If $x \sim z$, $y \sim z$, and $x \nsim y$, then $xz = 0$,$yz = 0$, and $xy = \pm 1$. Thus $xy + yz + xz = \pm 1$ and $yz = 0$.

On the three-element subset, these cases comprise all possible 3-cycles and (ITV)s. Thus, the converse is true. □

This test eliminates the possibility of a 3-cycle or an (ITV). Because $|X| = 4$, we must also consider possible 4-cycles. In this case, there are 6 possible 4-cycles, so we can check for each individually. They are

- $a > b > c > d > a$
- $a > b > d > c > a$
- $a > c > b > d > a$
- $a > c > d > b > a$
- $a > d > b > c > a$
- $a > d > c > b > a$.

Putting all of this together, our algorithm cycles through every possible function satisfying our constraints, testing the outcome of the function on each of the possible profiles in $L(X)^2$ for 3-cycles, 4-cycles, or (ITV)s. If any are found, that function is thrown out. If the function passes our tests for all possible inputs, we update our counter and add the function to the list of qualifying functions. The following flowchart illustrates this algorithm:

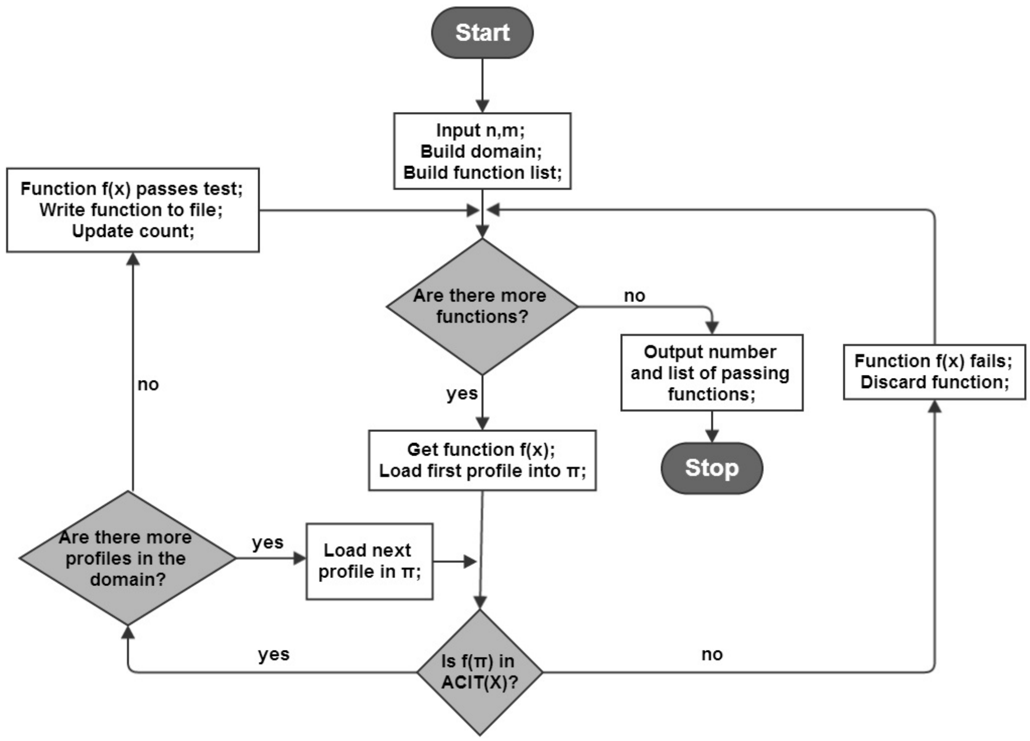

Figure 4.1: Function Search Algorithm Flowchart

For the particular case where $n = 2$ and $|X| = 4$, exactly 92 functions survived our tests, all, of course, based on a unique weakly decisive voter.

4.2 The case where $m \geq 5$

To consider functions with more alternatives, we are still able to use the approach described above. This is a result of the following lemma:

Lemma 4.3. *Let $\rho \in C(X)$ such that ρ is indifference-transitive. For $i \geq 5$, $|X| \geq i$, if there exist unique elements of X, x_1, x_2, x_3, $....x_i$, such that $x_1 > x_2 >$*

$x_3 >\ldots.> x_i > x_1$, *then there is either an underlying* 3*-cycle or* 4*-cycle contained in* ρ.

Proof. We will prove Lemma 4.3 using induction. First consider the case where there is a cycle of length 5. Thus the following cycle exists: $x_1 > x_2 > x_3 > x_4 > x_5 > x_1$. Suppose that there are no 4-cycles comprised of elements from this cycle. Then $x_1 \geq x_4$ and $x_3 \geq x_1$ because otherwise, there would be a 4-cycle. It cannot be the case that both $x_1 \sim x_4$ and $x_3 \sim x_1$, because $x_3 > x_4$. Thus either $x_1 > x_4$ or $x_3 > x_1$. If $x_1 > x_4$, then we have the 3-cycle $x_1 > x_4 > x_5 > x_1$. If $x_3 > x_1$, then we have the 3-cycle $x_1 > x_2 > x_3 > x_1$. Therefore a cycle of length 5 must either have an underlying 3-cycle or 4-cycle.

Secondly, consider the case where we have a 6-cycle, $x_1 > x_2 > x_3 > x_4 > x_5 > x_6 > x_1$. Suppose the cycle does not contain any underlying 4-cycles on those elements. Then we have $x_1 \geq x_4$ and $x_4 \geq x_1$, or $x_1 \sim x_4$. Since $x_4 > x_5$ and there are no (ITV)s, $x_1 \nsim x_5$, so either $x_1 > x_5$ or $x_5 > x_1$. If $x_1 > x_5$, then we have the 3-cycle $x_1 > x_5 > x_6 > x_1$. If $x_5 > x_1$, we have the 5-cycle $x_1 > x_2 > x_3 > x_4 > x_5 > x_1$. As shown above, this 5-cycle must contain an underlying 3 or 4-cycle. Thus any 6-cycle must contain an underlying 3 or 4-cycle.

Next, suppose that any indifference-transitive cycle of length k where $k \geq 5$ contains either a 3 or 4-cycle. Consider an indifference-transitive cycle of length $k+1$, $x_1 > x_2 > x_3 > \ldots > x_{k-1} > x_k > x_{k+1} > x_1$. It cannot be the case that both $x_1 \sim x_{k-1}$ and $x_1 \sim x_k$ because $x_{k-1} > x_k$. Thus we have four possible cases:

1. Suppose $x_1 > x_k$. Then we have $x_1 > x_k > x_{k+1} > x_1$, a 3-cycle.

2. Suppose $x_1 > x_{k-1}$. Then we have $x_1 > x_{k-1} > x_k > x_{k+1} > x_1$, a 4-cycle.

3. Suppose $x_{k-1} > x_1$. This creates a cycle of length $k-1$. By the induction assumption, this cycle contains an underlying 3 or 4-cycle.

4. Suppose $x_k > x_1$. This creates a cycle of length k. By the induction assumption, this cycle contains an underlying 3 or 4-cycle.

Thus any cycle of length $k + 1$ must contain either a 3-cycle or a 4-cycle.

Therefore, any indifference-transitive profile ρ containing a cycle of size greater than or equal to 5 must also contain a 3-cycle or 4-cycle. □

This theorem is very useful to us because we now know that if we test for (ITV)s, 3-cycles, and 4-cycles, our algorithm will also detect any cycles of size greater than 4, should there be enough alternatives to create one. Using a similar approach to the one outlined above, we have determined that the number of collective choice functions satisfying $f : L(X)^2 \to ACIT(X)$ satisfying (IIA) and (P), with $|X| = 5$, is 332. This approach could be implemented to determine the number of collective choice functions on any number of alternatives, given enough computing time.

Our overall algorithm could be adapted to collective choice functions meeting a wide variety of criteria. Computer-aided proofs of Arrow's Theorem have been studied [15] using a different approach from the one presented here.

CHAPTER 5
A CHARACTERIZATION RESULT

In this chapter we present a complete characterization of all functions f: $L(X)^n \rightarrow ACIT(X)$ satisfying (IIA) and (P), where $n \geq 2$. Note that for this chapter, we are again restricting our domain to $L(X)^n$. We will introduce a property called the Principal Ultrafilter Property and show that there is a 1-1 correspondence between our functions and families of sets satisfying this property. We will then use this characterization to verify the counting results presented in the previous chapter.

5.1 The Partial Principal Ultrafilter Property (PPUP)

To begin, let N be the set of voters, and let X be the set of alternatives. Let $|N| = n \geq 2$ and $|X| = m \geq 4$. Let f: $L(X)^n \rightarrow C(X)$ be an arbitrary function, where $C(X)$ represents the set of complete, binary relations on X.

For any $j \in N$, let $\mathcal{F}_j = \{I \subseteq N\colon\ j \in I\}$. $\mathcal{F}_j$ is known as the **principal ultrafilter generated by** j, and the singleton set $\{j\}$ is the smallest element of $\mathcal{F}_j$, or the intersection of all of the sets in $\mathcal{F}_j$. We will use the set $\mathcal{F}_j$ to define the following property, which will prove essential to characterizing our class of functions.

Definition 5.1. *For each $x \neq y \in X$, let $\mathcal{E}_{(x,y)}$ be a collection of nonempty subsets of N. We will say that the collection of all these sets $\mathcal{E} = \{\mathcal{E}_{(x,y)}\}$ satisfies the* ***Partial Principal Ultrafilter Property*** *(PPUP) if there exists a $j \in N$ such that*

1. $\{N\} \subseteq \mathcal{E}_{(x,y)} \subseteq \mathcal{F}_j$ *for all* $x \neq y \in X$*, and*

2. $\mathcal{E}_{(a,b)} \neq \mathcal{F}_j$ *implies that* $\mathcal{E}_{(x,y)} = \mathcal{F}_j$ *for any* $x \neq y \in X$ *such that* $|\{x,y\} \cap \{a,b\}| = 1$.

If $\{\mathcal{E}_{(x,y)}\}$ satisfies (PPUP), then we will say $\{\mathcal{E}_{(x,y)}\}$ is a (PPUP) family on N. For example, consider the following family of sets.

Example 5.1. *Let* $j \in N$*. For any* $x \neq y \in X$*, let* $\mathcal{E}_{(x,y)} = \mathcal{F}_j$*. Then* $\{N\} \subseteq \mathcal{E}_{(x,y)} = \mathcal{F}_j$ *for all* $x \neq y \in X$*, and since these sets are equal for all pairs, the second point is vacuously true. Thus, the family* $\{\mathcal{E}_{(x,y)}\}$ *satisfies (PPUP), and in this particular case, we actually have a principal ultrafilter on* N*.*

For any $\pi \in L(X)^n$ and for any $x \neq y \in X$, we will let

$$K_{(x,y)}(\pi) = \{i \in N \colon x >_{\pi_i} y\}.$$

So $K_{(x,y)}(\pi)$ is the set of all voters which, according to profile π, rank x above y. If we let $C = K_{(x,y)}(\pi)$, then π is a C-competitive profile for (x,y).

We will now show how to generate a collective choice function based on a given (PPUP) family $\mathcal{D} = \{\mathcal{D}_{(x,y)}\}$.

Define $f_{\mathcal{D}} \colon L(X)^n \to C(X)$ as follows: for any profile $\pi \in L(X)^n$, and for any $a \neq b \in X$,

$$a >_{f_{\mathcal{D}}(\pi)} b \text{ iff } K_{(a,b)}(\pi) \in \mathcal{D}_{(a,b)}.$$

Notice that, since $f_{\mathcal{D}}$ maps into $C(X)$, the result must be complete. Thus the following is true:

$$a \sim_{f_{\mathcal{D}}(\pi)} b \text{ iff } K_{(a,b)}(\pi) \notin \mathcal{D}_{(a,b)} \text{ and } K_{(b,a)}(\pi) \notin \mathcal{D}_{(b,a)}.$$

To verify that $f_{\mathcal{D}}$ is well-defined, note that since $\mathcal{D}$ satisfies (PPUP), if $I \in \mathcal{D}_{(x,y)}$ for some $x \neq y \in X$, then $N \setminus I \notin \mathcal{D}_{(a,b)}$ for all $a \neq b \in X$. This is because if $I \in \mathcal{D}_{(x,y)}$, then $j \in I$, which implies that $j \notin N \setminus I$, and thus $N \setminus I \notin \mathcal{F}_j$. Since

$N \setminus I \notin \mathcal{F}_j$, $N \setminus I$ cannot be contained in $\mathcal{D}_{(a,b)}$ for any $a \neq b \in X$, according to (PPUP). This means that for any $\pi \in L(X)^n$, there is exactly one possible strict outcome in $f_{\mathcal{D}}(\pi)$ for each $x \neq y$. Thus $f_{\mathcal{D}}$ is well-defined.

Also observe that if there exists $j \in N$ such that $\mathcal{D}_{(x,y)} = \mathcal{F}_j$ for all $x \neq y \in X$, then $\mathcal{D} = \{\mathcal{D}_{(x,y)}\}$ satisfies (PPUP) and $f_{\mathcal{D}}(\pi) = \pi_j$ for all $\pi \in L(X)^n$.

Next we will show that for such a family $\mathcal{D}$ satisfying (PPUP), the function based on it, $f_{\mathcal{D}}$, maps into $ACIT(X)$ and must satisfy independence of irrelevant alternatives and the Pareto property. This is the first step towards our characterization result.

THEOREM 5.1. *For each $x \neq y \in X$, let $\mathcal{D}_{(x,y)}$ be a collection of nonempty subsets of N, and let $\mathcal{D}$ denote the collection $\{\mathcal{D}_{\{x,y\}}\}$. If $\mathcal{D}$ satisfies (PPUP), then the function $f_{\mathcal{D}}$ satisfies (IIA), (P), and for all $\pi \in L(X)^n$, $f_{\mathcal{D}}(\pi) \in ACIT(X)$.*

Proof. For each $x \neq y \in X$, let $\mathcal{D}_{(x,y)}$ be a collection of subsets of N, and let $\mathcal{D}$ denote the collection $\{\mathcal{D}_{(x,y)}\}$. Suppose $\mathcal{D}$ satisfies (PPUP).

To show that $f_{\mathcal{D}}$ satisfies (IIA), suppose we have two profiles π_1 and π_2 in $L(X)^n$ such that $\pi_1|_{\{a,b\}} = \pi_2|_{\{a,b\}}$ for some $a \neq b \in X$. This means that $K_{(a,b)}(\pi_1) = K_{(a,b)}(\pi_2)$ and $K_{(b,a)}(\pi_1) = K_{(b,a)}(\pi_2)$. Then we have three cases:

1. $K_{(a,b)}(\pi_1) = K_{(a,b)}(\pi_2) \in \mathcal{D}_{(a,b)}$ and therefore $a >_{f_{\mathcal{D}}(\pi_1)} b$ and $a >_{f_{\mathcal{D}}(\pi_2)} b$, which implies that $f_{\mathcal{D}}(\pi_1)|_{\{a,b\}} = f_{\mathcal{D}}(\pi_2)|_{\{a,b\}}$.

2. $K_{(b,a)}(\pi_1) = K_{(b,a)}(\pi_2) \in \mathcal{D}_{(b,a)}$ and therefore $b >_{f_{\mathcal{D}}(\pi_1)} a$ and $b >_{f_{\mathcal{D}}(\pi_2)} a$, which implies that $f_{\mathcal{D}}(\pi_1)|_{\{a,b\}} = f_{\mathcal{D}}(\pi_2)|_{\{a,b\}}$.

3. $K_{(a,b)}(\pi_1) = K_{(a,b)}(\pi_2) \notin \mathcal{D}_{(a,b)}$ and $K_{(b,a)}(\pi_1) = K_{(b,a)}(\pi_2) \notin \mathcal{D}_{(b,a)}$, and therefore $a \sim_{f_{\mathcal{D}}(\pi_1)} b$ and $a \sim_{f_{\mathcal{D}}(\pi_2)} b$, which implies that $f_{\mathcal{D}}(\pi_1)|_{\{a,b\}} = f_{\mathcal{D}}(\pi_2)|_{\{a,b\}}$.

Hence, $f_{\mathcal{D}}$ satisfies (IIA).

To show that $f_{\mathcal{D}}(\pi)$ satisfies (P), recall from the definition of (PPUP) that $N \in \mathcal{D}_{(x,y)}$ for all $x \neq y \in X$. Thus for all $x \neq y \in X$, if $x >_{\pi_i} y$ for all $i \in N$, then $x >_{f_{\mathcal{D}}(\pi)} y$. Therefore $f_{\mathcal{D}}$ satisfies (P).

To show that $f_{\mathcal{D}}(\pi) \in ACIT(X)$ for all $\pi \in L(X)^n$, we need to show that $f_{\mathcal{D}}(\pi)$ cannot contain an indifference-transitive violation (ITV) or a cycle.

Note that, since $\mathcal{D}$ satisfies (PPUP), we have a voter $j \in N$ such that $\{N\} \subseteq \mathcal{D}_{(x,y)} \subseteq \mathcal{F}_j$ for all $x \neq y \in X$, and whenever $\mathcal{D}_{(a,b)} \neq \mathcal{F}_j$, $\mathcal{D}_{(x,y)} = \mathcal{F}_j$ for any $x \neq y \in X$ such that $|\{x,y\} \cap \{a,b\}| = 1$.

We will first show that an (ITV) cannot occur. Suppose that for $x \neq y \in X$, for some $\pi \in L(X)^n$, $x \sim_{f_{\mathcal{D}}(\pi)} y$. According to the definition of $f_{\mathcal{D}}$, this implies that $K_{(x,y)}(\pi) \notin \mathcal{D}_{(x,y)}$, and $K_{(y,x)}(\pi) \notin \mathcal{D}_{(y,x)}$.

Let $z \in X \setminus \{x,y\}$. Because $\mathcal{D}$ satisfies (PPUP), and either $\mathcal{D}_{(x,y)} \neq \mathcal{F}_j$ or $\mathcal{D}_{(y,x)} \neq \mathcal{F}_j$, $\mathcal{D}_{(y,z)} = \mathcal{D}_{(z,y)} = \mathcal{F}_j$, so any set containing j must be semidecisive for (y,z) and (z,y). (Similarly, any set containing j must be semidecisive for (x,z) and (z,x).) Either $j \in K_{(y,z)}(\pi)$ or $j \in K_{(z,y)}(\pi)$, but not both. If $j \in K_{(y,z)}(\pi)$, then $y >_{f_{\mathcal{D}}(\pi)} z$. If $j \in K_{(z,y)}(\pi)$, then $z >_{f_{\mathcal{D}}(\pi)} y$. (Similarly, it must be the case that either $x >_{f_{\mathcal{D}}(\pi)} z$ or $z >_{f_{\mathcal{D}}(\pi)} x$.) Thus if $x \sim_{f_{\mathcal{D}}(\pi)} y$, then neither $x \sim_{f_{\mathcal{D}}(\pi)} z$ nor $y \sim_{f_{\mathcal{D}}(\pi)} z$, so there can never be an (ITV) in $f_{\mathcal{D}}(\pi)$.

Finally, we will show that $f_{\mathcal{D}}(\pi)$ cannot contain a cycle. Because for all $x \neq y \in X$, we know that $\mathcal{D}_{(x,y)} \subseteq \mathcal{F}_j$, if $I \in \mathcal{D}_{(x,y)}$, then $j \in I$. Thus if $x >_{f_{\mathcal{D}}(\pi)} y$, then since $K_{(x,y)}(\pi) \in \mathcal{D}_{(x,y)}$, $j \in K_{(x,y)}(\pi)$, which implies that $x >_{\pi_j} y$. Because a cycle is a chain of strict preferences $x_1 >_{f_{\mathcal{D}}(\pi)} x_2 >_{f_{\mathcal{D}}(\pi)} \ldots.. >_{f_{\mathcal{D}}(\pi)} x_m >_{f_{\mathcal{D}}(\pi)} x_1$, then each of these strict preferences must be contained in π_j ($x_1 >_{\pi_j} x_2 >_{\pi_j} \ldots. >_{\pi_j} x_m >_{\pi_j} x_1$). However, $\pi_j \in L(X)$ and so π_j is acyclic. Hence, $f(\pi)$ is acyclic.

Since $f_{\mathcal{D}}(\pi)$ cannot contain an (ITV) or a cycle, and $f_{\mathcal{D}}(\pi)$ is complete, $f_{\mathcal{D}}(\pi) \in ACIT(X)$ for all $\pi \in L(X)^n$. $\square$

The next result is a converse of Theorem 5.1. In order to state this theorem, we need the following notation: for any $x \neq y \in X$, let

$$\mathcal{D}^f_{(x,y)} = \{L \subset N\colon\ L \text{ is semidecisive for } (x,y) \text{ under the function } f\}.$$

THEOREM 5.2. *Suppose $f : L(X)^n \to ACIT(X)$ satisfies (P) and (IIA). Then the collection $\mathcal{D}^f = \{\mathcal{D}^f_{\{x,y\}}\}$ for all $x \neq y \in X$ satisfies (PPUP).*

Proof. Suppose $f : L(X)^n \to ACIT(X)$ satisfies (P) and (IIA). Let $\mathcal{D}^f$ represent the collection of sets given above. We now show that $\mathcal{D}^f$ satisfies (PPUP).

Because f satisfies (P), for each $x \neq y \in X$, $N \in \mathcal{D}^f_{(x,y)}$. Thus $\{N\} \subseteq \mathcal{D}^f_{(x,y)}$ for all $x \neq y \in X$.

Due to our previous Theorem 3.3, we know that there exists $j \in N$ such that j is a unique weakly decisive voter with respect to f. For any $x \neq y$, if there exists some set $K \in \mathcal{D}^f_{(x,y)}$, then for any profile $\pi \in L(X)^n$ satisfying the following:

$$\pi|_{\{x,y\}}\colon \quad \begin{array}{cc} K & N \setminus K \\ \hline x & y \\ y & x \end{array}$$

Since K is semidecisive for (x,y), then $x >_{f(\pi)} y$. Since j is weakly decisive, it cannot be the case that $y >_{\pi_j} x$. Thus every set in $\mathcal{D}^f_{(x,y)}$ (for any $x \neq y \in X$) must contain j, and therefore $\mathcal{D}^f_{(x,y)} \subseteq \mathcal{F}_j$ for all $x \neq y \in X$.

Assume that there exists $a \neq b \in X$ such that $\mathcal{D}^f_{(a,b)} \neq \mathcal{F}_j$. Then there exists $I \subset N$ where $j \in I$ and I is not semidecisive for (a,b).

To show that $\mathcal{F}_j \subseteq \mathcal{D}^f_{(x,y)}$ for any $x \neq y \in X$ such that $|\{x,y\} \cap \{a,b\}| = 1$, we need to show that for any $c \in X \setminus \{a,b\}$, any set H such that $\{j\} \subseteq H$ is semidecisive for (a,c), (c,a), (b,c), and (c,b). Since $j \in I$, $\{j\} \subseteq H \cap I$ for all such H.

First, we will prove that any J containing I is in $\mathcal{D}^f_{(a,c)}$. Suppose to the contrary that J is not semidecisive for (a,c). Consider a profile $\pi^1 \in L(X)^n$ satisfying the following:

$$\pi^1|_{\{a,b,c\}}: \quad \begin{array}{c c c} I & J\setminus I & N\setminus J \\ \hline a & b & b \\ b & a & c \\ c & c & a \end{array}$$

Because $I \notin \mathcal{D}^f_{(a,b)}$ and $a >_{\pi^1_i} b$ for all $i \in I$, $b >_{\pi^1_k} a$ for all $k \in N\setminus I$, a cannot be strictly over b in $f(\pi^1)$. Thus $b \geq_{f(\pi^1)} a$. Since $j \in I$, $a >_{\pi^1_j} b$. This implies that b is not strictly over a in $f(\pi^1)$. Thus $a \sim_{f(\pi^1)} b$. Similarly, since $J \notin \mathcal{D}^f_{(a,c)}$, $a \sim_{f(\pi^1)} c$. Indifference-transitivity implies that $b \sim_{f(\pi^1)} c$; however, by (P), $b >_{f(\pi^1)} c$. Thus this is a contradiction, so for any J such that $I \subseteq J$, $J \in \mathcal{D}^f_{(a,c)}$.

Next we will prove that any $H \in \mathcal{F}_j$ is semidecisive for

(i) (b, c)

(ii) (c, b)

(iii) (c, a), and finally

(iv) (a, c)

(i) Assume $H \notin \mathcal{D}^f_{(b,c)}$ for some H such that $j \in H$. Consider $\pi^2 \in L(X)^n$ satisfying

$$\pi^2|_{\{a,b,c\}}: \quad \begin{array}{c c c c} I\setminus H & I\cap H & H\setminus I & N\setminus (I\cap H) \\ \hline a & a & b & c \\ c & b & c & b \\ b & c & a & a \end{array}$$

Since $I \notin \mathcal{D}^f_{(a,b)}$, but $j \in I$, $a \sim_{f(\pi^2)} b$. Since $H \notin \mathcal{D}^f_{(b,c)}$, but $j \in H$, $b \sim_{f(\pi^2)} c$. Indifference-transitivity implies that $a \sim_{f(\pi^2)} c$; however, since any set J containing I is in $\mathcal{D}^f_{(a,c)}$, we know that $I \in \mathcal{D}^f_{(a,c)}$. Thus since $a >_{\pi^2_i} c$ for all $i \in I$, and $c >_{\pi^2_k} a$ for all $k \notin I$, this implies that $a >_{f(\pi^2)} c$, a contradiction. Thus $H \in \mathcal{D}^f_{(b,c)}$.

(ii) Assume $H \notin \mathcal{D}^f_{(c,b)}$ for some H such that $j \in H$. Consider $\pi^3 \in L(X)^n$ satisfying

$\pi^3|_{\{a,b,c\}}$:

$I \setminus H$	$I \cap H$	$H \setminus I$	$N \setminus (I \cap H)$
a	a	c	b
b	c	b	c
c	b	a	a

Since $I \notin \mathcal{D}^f_{(a,b)}$, but $j \in I$, $a{\sim}_{f(\pi^3)}b$. Since $H \notin \mathcal{D}^f_{(c,b)}$, but $j \in H$, $c{\sim}_{f(\pi^3)}b$. Indifference-transitivity implies that $a{\sim}_{f(\pi^3)}c$; however, since $I \in \mathcal{D}^f_{(a,c)}$, we have that $a >_{f(\pi^3)} c$, a contradiction. Therefore $H \in \mathcal{D}^f_{(c,b)}$.

(iii) Assume $H \notin \mathcal{D}^f_{(c,a)}$ for some H such that $j \in H$. Consider $\pi^4 \in L(X)^n$ satisfying

$\pi^4|_{\{a,b,c\}}$:

$I \setminus H$	$I \cap H$	$H \setminus I$	$N \setminus (I \cap H)$
a	c	c	b
b	a	b	a
c	b	a	c

Since $I \notin \mathcal{D}^f_{(a,b)}$, but $j \in I$, $a{\sim}_{f(\pi^4)}b$. Since $H \notin \mathcal{D}^f_{(c,a)}$, but $j \in H$, $c{\sim}_{f(\pi^4)}a$. Indifference-transitivity implies that $c{\sim}_{f(\pi^4)}b$; however, since $H \in \mathcal{D}^f_{(c,b)}$, we have that $c >_{f(\pi^4)} b$, a contradiction. Therefore $H \in \mathcal{D}^f_{(c,a)}$.

(iv) Finally, we complete the (a,c) case by proving it true for any H such that $j \in H$.

Assume $H \notin \mathcal{D}^f_{(a,c)}$ for some H such that $j \in H$. Consider $\pi^5 \in L(X)^n$ satisfying

$\pi^5|_{\{a,b,c\}}$:

$I \setminus H$	$I \cap H$	$H \setminus I$	$N \setminus (I \cap H)$
c	a	b	c
a	b	a	b
b	c	c	a

Since $I \notin \mathcal{D}^f_{(a,b)}$, but $j \in I$, $a \sim_{f(\pi^5)} b$. Since $H \notin \mathcal{D}^f_{(a,c)}$, but $j \in H$, $a \sim_{f(\pi^5)} c$. Indifference-transitivity implies that $b \sim_{f(\pi^5)} c$; however, since $H \in \mathcal{D}^f_{(b,c)}$, we have that $b >_{f(\pi^5)} c$, a contradiction. Therefore $H \in \mathcal{D}^f_{(a,c)}$.

Thus, $\mathcal{F}_j \subseteq \mathcal{D}^f_{(x,y)}$ for all $|\{x, y\} \cap \{a, b\}| = 1$.

To summarize, we have show that there exists $j \in N$ such that

1. $\{N\} \subseteq \mathcal{D}^f_{(x,y)} \subseteq \mathcal{F}_j$ for all $x \neq y \in X$, and
2. $\mathcal{D}^f_{(a,b)} \neq \mathcal{F}_j$ implies that $\mathcal{D}^f_{(x,y)} = \mathcal{F}_j$ for any $x \neq y \in X$ such that $|\{x, y\} \cap \{a, b\}| = 1$.

In other words, $\{\mathcal{D}^f_{(x,y)}\}$ satisfies (PPUP). □

We can now combine the previous two theorems to prove our main characterization result.

Theorem 5.3. *The function $f : L(X)^n \to ACIT(X)$ satisfies (P) and (IIA) if and only if the collection $\mathcal{D}^f = \{\mathcal{D}^f_{\{x,y\}}\}$ for all $x \neq y \in X$ satisfies (PPUP).*

In order to prove our main result, we need the following lemma.

Lemma 5.1. *If $f : L(X)^n \to C(X)$ satisfies (IIA), then $f_{\mathcal{D}^f} = f$.*

Proof. To begin, note that f and $f_{\mathcal{D}^f}$ are functions with the same domain, $L(X)^n$. To show that the two functions are equal, we need to show that for any $\pi \in L(X)^n$, $f(\pi) = f_{\mathcal{D}^f}(\pi)$. Let $\pi \in L(X)^n$, and let $a \neq b \in X$. Because $f_{\mathcal{D}^f}$ is complete, there are three possibilities for $f_{\mathcal{D}^f}(\pi)|_{\{a,b\}}$:

1. $a >_{f_{\mathcal{D}^f}(\pi)} b$, which is true if and only if $K_{(a,b)}(\pi) \in \mathcal{D}^f_{(a,b)}$, if and only if $a >_{f(\pi)} b$ (since $\mathcal{D}^f_{(a,b)}$ is the set of semidecisive sets for (a, b), if $K_{(a,b)}(\pi) \in \mathcal{D}^f_{(a,b)}$, then $a >_{f(\pi)} b$).
2. $b >_{f_{\mathcal{D}^f}(\pi)} a$, which, similarly, is true if and only if $b >_{f(\pi)} a$.

3. $a \sim_{f_{\mathcal{D}^f}(\pi)} b$, which is true if and only if neither $K_{(a,b)}(\pi) \notin \mathcal{D}^f_{(a,b)}$ nor $K_{(b,a)}(\pi) \notin \mathcal{D}^f_{(b,a)}$, which is true if and only if neither set is semidecisive for its respective pair. This is equivalent to saying that for our given π, $a \sim_{f(\pi)} b$.

Hence, for any $a \neq b \in X$, $f_{\mathcal{D}^f}(\pi)|_{\{a,b\}} = f(\pi)|_{\{a,b\}}$. Therefore, $f_{\mathcal{D}^f} = f$. □

We are now ready to prove Theorem 5.3.

Proof. (Theorem 5.3) The forward direction is shown in our proof of Theorem 5.2.

To show the converse, assume that the collection $\mathcal{D}^f = \{(D)^f_{\{x,y\}}\}$ for all $x \neq y \in X$ satisfies (PPUP). Then, as a direct consequence of Theorem 5.1, $f_{\mathcal{D}^f}$ satisfies (IIA), (P), and for all $\pi \in L(X)^n$, $f_{\mathcal{D}^f}(\pi) \in ACIT(X)$. Since we know from the previous lemma that $f_{\mathcal{D}^f} = f$, we can conclude that f satisfies (P), (IIA), and the range of f is a subset of $ACIT(X)$. □

5.2 The Number of (PPUP) Families With $n = 2$

Now that we have a complete characterization of all ACIT functions satisfying the constraints given in Theorem 5.3, we return to a discussion of the number of functions satisfying our conditions. As it turns out, combinatorially counting the number of unique families which satisfy (PPUP) is possible, and since every (PPUP) family corresponds to a unique function with the given conditions, we can use this as a tool for counting the number of functions. We will count the number of functions $f : L(X)^2 \to ACIT(X)$ satisfying (IIA) and (P). By Theorem 5.3, the collection $\mathcal{D}^f = \{\mathcal{D}^f_{(x,y)}\}$ for all $x \neq y \in X$ satisfies (PPUP). Therefore,

$$\{N\} \subseteq \mathcal{D}^f_{(x,y)} \subseteq \mathcal{F}_j.$$

For all $x \neq y \in X$, $N = \{1,2\}$, and $j \in N$. We will fix $j = 1$. There are two possibilities for $\mathcal{D}^f_{(x,y)}$: either $\mathcal{D}^f_{(x,y)} = \mathcal{F}_1$ or $\mathcal{D}^f_{(x,y)} = \{N\}$.

Consider a directed graph G where X is the set of vertices and $E(G)$ is the set of all directed edges. We get a directed edge $\overrightarrow{xy}$ as follows:

$$\overrightarrow{xy} \in E(G) \leftrightarrow \mathcal{D}^f_{(x,y)} = \{N\}.$$

For any $x \neq y \in X$, we have four possibilities:

(i) $E(G) \cap \{\overrightarrow{xy}, \overrightarrow{yx}\} = \emptyset$;

(ii) $E(G) \cap \{\overrightarrow{xy}, \overrightarrow{yx}\} = \{\overrightarrow{xy}\}$;

(iii) $E(G) \cap \{\overrightarrow{xy}, \overrightarrow{yx}\} = \{\overrightarrow{yx}\}$;

(iv) $E(G) \cap \{\overrightarrow{xy}, \overrightarrow{yx}\} = \{\overrightarrow{xy}, \overrightarrow{yx}\}$.

We will say that the vertices x and y are "connected" if $E(G) \cap \{\overrightarrow{xy}, \overrightarrow{yx}\} \neq \emptyset$.

By the definition of (PPUP), $\overrightarrow{xy} \in E(G)$ implies that if $\{a, b\} \cap \{x, y\} = 1$, then $\overrightarrow{ab}$ is not in $E(G)$. So any vertex of G can be connected to at most one other vertex of G. Conversely, any directed graph G with vertex set X such that any vertex of G is connected to at most one other vertex gives a (PPUP) family on N. Thus, counting (PPUP) families is equivalent to determining the number of directed graphs on X with the property that every vertex of G is connected to at most one other vertex of G.

We will demonstrate this relationship by discussing the cases where $|X| = 3$, 4, and 5. These will collectively serve as the base case for a recursive relationship.

5.2.1 The Case $|X| = 3$

Let $X = \{a_1, a_2, a_3\}$ and so $|X| = 3$. In this case, our graph G could consist of no edges or at most one connection. A connection between two vertices means that there is at least one directed edge between these two vertices. There are $\binom{3}{2}$

pairs of verticies $\{a_i, a_j\}$ and 3 possible connections between a_i and a_j as given in items (ii), (iii), and (iv) above. This gives us

$$1 + 3\binom{3}{2} = 10$$

possible (PPUP) families with $j = 1$. Similarly, $j = 2$ gives us 10 more possible (PPUP) families, for a total of 20 unique (PPUP) families when $|X| = 3$. Thus, there are 20 functions $f : L(X)^2 \to ACIT(X)$ satisfying (IIA) and (P) when $|X| = 3$.

5.2.2 The Case $|X| = 4$

Now let $X = \{a_1, a_2, a_3, a_4\}$, and so $|X| = 4$. Again, fix $j = 1$. There is 1 graph with no edges. We now calculate the number of graphs where there is exactly one connected pair. There are $\binom{4}{2}$ pairs of vertices, and each pair can be connected in 3 different ways (cases (ii), (iii), and (iv)). This gives us a total of $3\binom{4}{2}$ graphs with one connection. Because with four elements, we can have at most two disjoint pairs of vertices, we also need to count the number of graphs with exactly two connections. There are 3 possibilities for the two connected pairs:

- $\{a_1, a_2\}$ and $\{a_3, a_4\}$;
- $\{a_1, a_3\}$ and $\{a_2, a_4\}$;
- $\{a_1, a_4\}$ and $\{a_2, a_3\}$.

There are 3 possible connections for each pair, which gives us $(3)(3^2) = 27$ graphs with exactly two connections. Thus there are

$$1 + 3\binom{4}{2} + (3)(3^2) = 46$$

possible (PPUP) families with $j = 1$. If $j = 2$, there are 46 additional (PPUP) families, for a total of 92 (PPUP) families. This gives us 92 functions $f : L(X)^2 \to$

$ACIT(X)$ satisfying (IIA) and (P) when $|X| = 4$. Recall in Chapter 4, we stated this result as a Theorem 4.1.

5.2.3 The Case $|X| = 5$

Next, we will let $X = \{a_1, a_2, a_3, a_4, a_5\}$, and so $|X| = 5$. As with the previous two cases, we will fix $j = 1$. With 5 elements, there can be again at most two simultaneously disjoint pairs, so we must consider graphs with no connections, exactly one connection, or exactly two connections. There is 1 graph with no connections. To calculate the number of graphs with exactly one connection, consider that there are $\binom{5}{2}$ pairs of vertices, and each pair can be connected in 3 different ways, for a total of $3\binom{5}{2}$ graphs with one connection. We will next count the number of graphs with exactly two connections. There are 15 possibilities for the two connected pairs:

- $\{a_1, a_2\}$ and $\{a_3, a_4\}$;
- $\{a_1, a_2\}$ and $\{a_3, a_5\}$;
- $\{a_1, a_2\}$ and $\{a_4, a_5\}$;
- $\{a_1, a_3\}$ and $\{a_2, a_4\}$;
- $\{a_1, a_3\}$ and $\{a_2, a_5\}$;
- $\{a_1, a_3\}$ and $\{a_4, a_5\}$;
- $\{a_1, a_4\}$ and $\{a_2, a_3\}$;
- $\{a_1, a_4\}$ and $\{a_2, a_5\}$;
- $\{a_1, a_4\}$ and $\{a_3, a_5\}$;
- $\{a_1, a_5\}$ and $\{a_2, a_3\}$;
- $\{a_1, a_5\}$ and $\{a_2, a_4\}$;
- $\{a_1, a_5\}$ and $\{a_3, a_4\}$;
- $\{a_2, a_3\}$ and $\{a_4, a_5\}$;
- $\{a_2, a_4\}$ and $\{a_3, a_5\}$;
- $\{a_3, a_4\}$ and $\{a_2, a_5\}$.

There are 3 possible connections for each pair, so that gives us $(15)(3^2)$

possible graphs with exactly two connections. This gives us a total of

$$1 + 3\binom{5}{2} + (15)(3^2) = 166$$

(PPUP) families when $j = 1$, or 332 families for either j. Thus when $|X| = 5$, there are exactly 332 functions $f : L(X)^2 \to ACIT(X)$ satisfying (IIA) and (P).

These three scenarios will constitute the base case for the following theorem:

Theorem 5.4. *For $n = 2$ and $|X| = m \geq 3$, let $\delta(m)$ represent the number of families $\mathcal{D} = \{\mathcal{D}_{(x,y)}\}$ where $x \neq y \in X$ such that $\mathcal{D}$ satisfies (PPUP). Then, for $m \geq 5$,*

$$\delta(m) = \delta(m-1) + 3(m-1)\delta(m-2)$$

where $\delta(3) = 20$ and $\delta(4) = 92$.

Proof. We will prove this theorem by induction on m, with $m \geq 3$. We know from above that $\delta(3) = 20$, $\delta(4) = 92$, and $\delta(5) = 332$. Using this recursive relationship, it should be the case that $\delta(5) = \delta(4) + 3(4)\delta(3)$. In fact, $\delta(4) + 3(4)\delta(3) = 92 + 3(4)20 = 332 = \delta(5)$. Thus the base case is satisfied.

Next, assume that the recursion relationship $\delta(k) = \delta(k-1) + 3(k-1)\delta(k-2)$ holds for $k = |X| = 5, ..., m-1$. Let $X = \{a_1, a_2, ..., a_m\}$ and so $|X| = m \geq 5$.. Recall that (PPUP) requires that $\{N\} \subseteq \mathcal{D}_{(x,y)} \subseteq F_j$ for all $x \neq y \in X$. Then we have two cases:

Case 1: Suppose that $\mathcal{D}_{(a_m,x)} = \mathcal{D}_{(x,a_m)} = F_j$ for all $x \in \{a_1, a_2, ..., a_{m-1}\}$. In other words a_m is not connected to any other element in X. Since there are $\delta(m-1)$ possible families on $\{a_1, a_2, ..., a_{m-1}\}$, and only one possibility for each set involving a_m, there are $\delta(m-1)$ families satisfying this case.

Case 2: Suppose there exists $x \in \{a_1, a_2, ..., a_{m-1}\}$ such that $\mathcal{D}_{(a_m,x)} \neq \mathcal{F}_j$ or $\mathcal{D}_{(x,a_m)} \neq \mathcal{F}_j$, so either $\mathcal{D}_{(a_m,x)} = \{N\}$, $\mathcal{D}_{(x,a_m)} = \{N\}$, or both sets are $\{N\}$. This gives us $3(m-1)$ possibilities for the pair $\{a_m, x\}$, since there are $(m-1)$ choices for x and 3 ways to connect them.

For any $a_k \in X \setminus \{x, a_m\}$, (PPUP) requires

$$\mathcal{D}_{(a_k,a_m)} = \mathcal{D}_{(a_m,a_k)} = \mathcal{F}_j \text{ and } \mathcal{D}_{(a_k,x)} = \mathcal{D}_{(x,a_k)} = \mathcal{F}_j.$$

Now notice that the subset $\mathcal{E}$ of $\mathcal{D}$ consisting of all $\mathcal{D}_{(u,v)}$ such that $u \neq v$ with $u, v \in X \setminus \{x, a_m\}$ is a (PPUP) family on N with respect to the set $X \setminus \{x, a_m\}$. Moreover, the above display equation implies that $\mathcal{E}$ can be any (PPUP) family on N with respect to the set $X \setminus \{x, a_m\}$. Since $|X \setminus \{x, a_m\}| = m-2$, the number of possibilities for $\mathcal{E}$ is $\delta(m-2)$.

In sum, there are $(m-1)$ choices for x, there are 3 possible connections on each pair $\{x, a_m\}$, and there are $\delta(m-2)$ (PPUP) families on N with respect to $X \setminus \{x, a_m\}$. Therefore, the total number of (PPUP) families for Case 2 is

$$3(m-1)\delta(m-2).$$

Combining the two cases, we have $\delta(m) = \delta(m-1) + 3(m-1)\delta(m-2)$. □

From Theorem 5.3, we know that counting the number of families $\mathcal{D}^f = \{\mathcal{D}^f_{\{x,y\}}\}$ for all $x \neq y \in X$ satisfying (PPUP) is equivalent to counting the number of functions $f : L(X)^n \to ACIT(X)$ which satisfy (P) and (IIA). The following corollary is a direct consequence of this fact and Theorem 5.4.

Corollary 5.1. *The number of Arrovian ACIT rules with $n = 2$ and $m \geq 5$ satisfies the following recursive relation:*

$$\delta(m) = \delta(m-1) + 3(m-1)\delta(m-2) \tag{5.1}$$

where $\delta(3) = 20$ and $\delta(4) = 92$.

As far as we know, there is no easy way to solve for a closed form of Equation 5.1. It would be interesting to investigate the related problem of a recursive relationship on the number of voters ($n \geq 2$), and then to eventually generalize our Corollary 5.1 to $n \geq 2$.

CHAPTER 6
WEAK INDEPENDENCE AND ACIT RULES

Thus far, we have primarily discussed collective choice functions where the range is a subset of $ACIT(X)$, which satisfy the standard Arrovian assumptions of (IIA) and (P). We have established that with at least four voters, any such f is based on a unique weakly decisive voter.

Interestingly, this is not the only set of assumptions which results in a function based on a unique weakly decisive voter. Introduced by Nick Baigent in 1987 [3] and established in 2000 by Donald Campbell and Jerry Kelly [8], we know that replacing the (IIA) condition in Arrow's theorem with a condition called weak independence of Irrelevant Alternatives leads to the conclusion that such rules are determined by a unique weakly decisive voter. This notion of Weak Independence (WI) can be defined as follows:

Definition 6.1. *A collective choice function $f : D(X) \to C(X)$ satisfies* weak independence of irrelevant alternatives (WI) *if for all $\pi, \theta \in D(X)$ and for all $a \neq b \in X$, if $\pi|_{\{a,b\}} = \theta|_{\{a,b\}}$ and $a >_{f(\pi)} b$, then $a \geq_{f(\theta)} b$.*

Equivalently, we can say that a function f satisfies (WI) if for all $\pi, \theta \in D(X)$ and any $a, b \in X$,

$$\text{if } a >_{f(\pi)} b \text{ and } b >_{f(\theta)} a, \text{ then } \pi|_{\{a,b\}} \neq \theta|_{\{a,b\}}.$$

Notice that (IIA) implies (WI), but the converse is not necessarily true. Thus (WI) is a weaker assumption than (IIA).

The following example satisfies (WI), but it does not satisfy (IIA). (Note that it does not satisfy (P).)

EXAMPLE 6.1. *Let $X = \{a, b, c\}$ and $n = 2$. Define $f : L(X)^2 \to C(X)$ as follows: For all $\pi \in L(X)^2$,*

- $f(\pi)|_{\{x,y\}} = \pi(1)|_{\{x,y\}}$ *for all* $\{x, y\} \neq \{a, b\}$.
- *If* $a >_{f(\pi)} c$, *then* $a >_{f(\pi)} b$. *Otherwise,* $a \sim_{f(\pi)} b$.

Suppose that for $\pi, \theta \in L(X)^2$, $\pi|_{\{x,y\}} = \theta|_{\{x,y\}}$ and $x >_{f(\pi)} y$.

1. If $\{x, y\} = \{a, b\}$, then $a >_{f(\pi)} b$, as the function never allows for $b >_{f(\pi)} a$. According to the definition of f, either $a >_{f(\theta)} b$ or $a \sim_{f(\theta)} b$, so $a \geq_{f(\theta)} b$.
2. If $\{x, y\} \neq \{a, b\}$, then $x >_{\pi(1)} y$ and since $\pi|_{\{x,y\}} = \theta|_{\{x,y\}}$, $x >_{\theta(1)} y$. By the definition of f, $x >_{f(\theta)} y$.

Hence, f satisfies (WI). However, f does not satisfy (IIA).

Consider the following scenario:

π:	1	2
	a	b
	b	a
	c	c

By the definition of our function f, $a >_{f(\pi)} c$ (from voter 1), so we have that $a >_{f(\pi)} b$.

θ:	1	2
	c	b
	a	a
	b	c

In this profile, voter 1 determines that $c >_{f(\theta)} a$, so we have that $a \sim_{f(\theta)} b$. Notice that we have $\pi|_{\{a,b\}} = \theta|_{\{a,b\}}$, but it is not the case that $f(\pi)|_{\{a,b\}} = f(\theta)|_{\{a,b\}}$. Thus, this function does not satisfy (IIA).

By incorporating the concept of (WI) into otherwise Arrovian collective choice functions, researchers Baigent [3], Campbell and Kelly [8] proved the following result.

Theorem 6.1 (Baigent, Campbell/Kelly)**.** *Assume that a collective choice function $f\colon W(X)^n \to C(X)$ satisfies **(WI)** and (P), and the range of f is a subset of **W(X)**. If $|X| = m \geq 4$, then f is based on a unique weakly decisive voter.*

(Note that the researchers also proved the theorem to be true for domains satisfying the free-quadruple property, but we are using the more specific version for our purposes. In [9], Coban and Sanver investigate collective choice functions satisfying (WI) where (P) is no longer required.)

Theorem 6.1 is strikingly similar to the IKN generalization [12] discussed in Chapter 3.

Theorem 6.2 (IKN)**.** *Assume that a collective choice function $f\colon W(X)^n \to C(X)$ satisfies **(IIA)** and (P), and the range of f is a subset of **ACIT(X)**. If $|X| = m \geq 4$, then f is based on a unique weakly decisive voter.*

(Note that we present a slightly reformulated version of the previous two results in order to better compare the results. The theorems are equivalent to their original formulations.)

Both theorems are modified versions of Arrow's Theorem, with at least four alternatives and the constraints altered slightly. Theorem 6.1 presents a function satisfying a weaker version of independence, but it restricts the range to the set of weak orders, while Theorem 6.2 uses the traditional strong version of independence, but allows for the less restrictive ACIT range. Both theorems result in functions based on a unique weakly decisive voter. It is natural to wonder whether it is possible to give a single result that generalizes both theorems. In other words, can we describe all collective choice functions $f\colon W(X)^n \to C(X)$ satisfying **(WI)** and (P), where the range of f is a subset of **ACIT(X)**? Are these functions also necessarily based on a unique weakly decisive voter?

The following chart summarizes what has been established thus far:

Source	n	**Independence**	**Range**	**Result**
Arrow	≥ 3	(IIA)	$W(X)$	unique dictator
Campbell/Kelly	≥ 4	(WI)	$W(X)$	unique weakly decisive voter
Iritiani, et. al	≥ 4	(IIA)	$ACIT(X)$	unique weakly decisive voter
Open Question	≥ 4	(WI)	$ACIT(X)$	???

Table 6.1: Summary of Theorems

The following function is a counterexample proving that, for a function whose domain is $L(X)^n$, when the range is a subset of $ACIT(X)$ and the (IIA) condition is weakened to (WI), the function need not be based on a unique weakly decisive voter.

EXAMPLE 6.2. *Let $|X| \geq 4$. Consider $f\colon L(X)^2 \to ACIT(X)$ such that for any $\pi \in L(X)^2$, and for some $a \neq b \in X$:*

1. $f(\pi)|_{\{x,y\}} = \pi(1)|_{\{x,y\}}$ *for all* $\{x, y\} \neq \{a, b\}$.
2. $f(\pi)|_{\{a,b\}} = \pi(2)|_{\{a,b\}}$ *if this allows* $f(\pi) \in ACIT(X)$*; otherwise,* $a \sim_{f(\pi)} b$.

It is easy to show that f satisfies (P). We will demonstrate that the function satisfies (WI). Suppose that $x >_{f(\pi)} y$ and $y >_{f(\pi')} x$. Then we have one of two cases:

- If $\{x, y\} \neq \{a, b\}$, then $x >_{\pi(1)} y$ and $y >_{\pi'(1)} x$. This implies that $\pi|_{\{x,y\}} \neq \pi'|_{\{x,y\}}$.
- If $\{x, y\} = \{a, b\}$, then by the definition of f, $a >_{\pi(2)} b$ and $b >_{\pi'(2)} a$. This implies that $\pi|_{\{x,y\}} \neq \pi'|_{\{x,y\}}$.

Hence, f satisfies (WI). In fact, this function satisfies (IIA) on all pairs $\{x, y\}$ such that $\{x, y\} \neq \{a, b\}$, but it only satisfies (WI) on the pair $\{a, b\}$.

However, the function f is not based on a unique weakly decisive voter. Voter 2 is weakly decisive on the pair $\{a, b\}$, but voter 1 is weakly decisive for all other pairs in $X \times X$, and voter 1 is not weakly decisive on the pair $\{a, b\}$.

Therefore, we know that it is not necessary that functions $f : L(X)^n \to ACIT(X)$ satisfying (P) and (WI) are based on a unique weakly decisive voter. Unfortunately, if we extend the domain of this function to include all weak orders on X, then this function is no longer well-defined, because if voter 1 is indifferent between either a or b and any other element of X, this could potentially cause an indifference-transitive violation. Thus to prove that (WI) and a range of $ACIT(X)$ does not imply a unique weakly decisive voter, we need a counterexample that is well defined when the domain of the function is $W(X)^n$. To describe our counterexample, we will need to introduce some notation.

- For any $\rho \in W(X)$, we define ρ^* to be the asymmetric part of ρ. In other words, $(x, y) \in \rho^*$ if and only if $(x, y) \in \rho$ and $(y, x) \notin \rho$.

- Assume $X = \{a, b, c, d\}$. We will let l_a be the linear order on X such that $a >_{l_a} b >_{l_a} c >_{l_a} d$.

- We will define the following function: $L_a : W(X) \to L(X)$ as follows: for all $\rho \in W(X)$,

$$L_a(\rho) = \rho^* \cup \{\rho \cap l_a\}$$

This function L_a acts as a sort of "tie breaker" for weak orders, based on a given linear order l_a. (Note that l_a could be any linear order. We have chosen the alphabetical order for convenience.) Basically, for any $x \neq y \in X$, if ρ specifies a strict preference for the pair, that will be the outcome; otherwise, the preference specified by l_a will be the outcome. For example, consider a weak order ρ satisfying the following:

$$\begin{array}{c} \rho\text{:} \\ \hline [bc] \\ [ad] \end{array}$$

Then $b >_{L_a(\rho)} a$, $c >_{L_a(\rho)} a$, $b >_{L_a(\rho)} d$, and $c >_{L_a(\rho)} d$. Since $b \sim_\rho c$ and $a \sim_\rho d$, the tie is broken by l_a, so $b >_{L_a(\rho)} c$, and $a >_{L_a(\rho)} d$. Thus the result is $b >_{L_a(\rho)} c >_{L_a(\rho)} a >_{L_a(\rho)} d$.

We will use L_a as a building block for the following example:

EXAMPLE 6.3. *Define $f : W(X)^2 \to ACIT(X)$ as follows:*

- *For any 2-element subset $\{x, y\} \neq \{a, b\} \in X \times X$, $f(\pi)|_{\{x,y\}} = L_a(\pi(1))|_{\{x,y\}}$.*
- *For $\{a, b\}$, if $L_a(\pi(1)) \backslash L_a(\pi(1))|_{\{a,b\}} \cup \pi(2)|_{\{a,b\}} \in ACIT(X)$, then $f(\pi)|_{\{a,b\}} = \pi(2)|_{\{a,b\}}$.*
- *Otherwise, $a \sim_{f(\pi)} b$.*

We must first establish that our function is well-defined, and that it satisfies (P) and (WI). To show that f is well-defined, observe that for all pairs $\{x, y\} \neq \{a, b\}$, the function outputs a linear order. For $\{a, b\}$, the function either outputs voter 2's preference (if that would be well-defined), or a tie between a and b. If $a \sim_{f(\pi)} b$, this cannot cause an indifference-transitive violation because for all other pairs, there is a strict preference. Thus f is well-defined.

To establish the (P) property holds, suppose that $x >_{\pi(i)} y$ for all i. In this case, both $x >_{\pi(1)} y$ and $x >_{\pi(2)} y$. Since $\pi(1)|_{\{x,y\}} = \pi(2)|_{\{x,y\}}$ specifies a strict preference for $x > y$, this preference is reflected in both $L_a(\pi(1))$ and $\pi(2)$, so for all possible such pairs $x \neq y$, $x >_{f(\pi)} y$. Hence, f satisfies (P).

To demonstrate that f satisfies (WI), suppose that for two profiles π and

$\theta \in W(X)$,

$$\pi|_{\{x,y\}} = \theta|_{\{x,y\}}$$

If $\{x, y\} \neq \{a, b\}$, then $f(\pi)|_{\{x,y\}} = \pi(1)|_{\{x,y\}} = \theta(1)|_{\{x,y\}} = f(\theta)|_{\{x,y\}}$. Thus $f(\pi)|_{\{x,y\}} = f(\theta)|_{\{x,y\}}$. (Notice that for $\{x, y\} \neq \{a, b\}$, f actually satisfies (IIA), which implies (WI).) If $\{x, y\} = \{a, b\}$, then we have one of the following three cases:

1. $f(\pi)|_{\{x,y\}} = \pi(2)|_{\{x,y\}} = \theta(2)|_{\{x,y\}} = f(\theta)|_{\{x,y\}}$,
2. $f(\pi)|_{\{x,y\}} = \pi(2)|_{\{x,y\}}$ and $x \sim_{f(\theta)} y$, or
3. $x \sim_{f(\pi)} y$ and $f(\theta)|_{\{x,y\}} = \theta(2)|_{\{x,y\}}$.

While $f(\theta)|_{\{x,y\}}$ and $f(\pi)|_{\{x,y\}}$ do not necessarily agree, no scenario produces a case where they are direct opposites. Thus this function satisfies (WI).

Now that we have established that f meets our desired criteria, we will observe that f is not based on a unique weakly decisive voter. Consider the following profile π^1:

$$\pi^1: \quad \begin{array}{cc} 1 & 2 \\ \hline a & b \\ b & a \\ c & d \\ d & c \end{array}$$

From voter 1 we have the following pairs: $a >_{f(\pi)} c$, $a >_{f(\pi)} d$, $b >_{f(\pi)} c$, $b >_{f(\pi)} d$, and $c >_{f(\pi)} d$. According to voter 2, $b >_{\pi(2)} a$, and including this would give us the linear order $b >_{f(\pi)} a >_{f(\pi)} c >_{f(\pi)} d$. Thus $a >_{\pi(1)} b$ but $b >_{f(\pi)} a$, and $d >_{\pi(2)} c$ but $c >_{f(\pi)} d$. Hence, there is no weakly decisive voter. This implies that the theorem cannot be generalized to include all functions $f : W(X)^n \to ACIT(X)$ satisfying (WI) and (P).

CHAPTER 7
CONCLUSION

Arrow's Theorem is an integral element of social choice theory, which seeks to combine the interests of individuals to represent the preferences of the entire group. Arrow demonstrated that demanding a set of criteria which are seemingly essential to ensuring a fair election actually produces the unintended result of a dictatorial function. In Chapter 2, we discussed this theorem and its implications, as well as a few approaches which have been taken to generalize the theorem and produce a more favorable result. By peeling away the layers of the so-called "fairness criteria" which Arrow required of his functions, social choice researchers hope to move as far away from the dictatorship as possible while still retaining as much "fairness" as possible. We chose to investigate in detail what happens when we weaken the transitivity requirements to include functions whose range is a subset of the acyclic, indifference-transitive relations on the set of alternatives.

In Chapter 3, we extended the IKN Generalization to prove that Arrovian ACIT collective choice functions whose domains satisfy the free-quadruple property are based on a unique weakly decisive voter. While Arrow's Theorem can be extended to include domains satisfying the free-triple property, we provided a counterexample to demonstrate that if the domain of an Arrovian ACIT rule is only required to satisfy the free-triple property, the rule is not necessarily based on a weakly decisive voter. This is very similar to the fact that the IKN Generalization requires at least 4 alternatives, whereas Arrow's classic theorem only requires 3.

In Chapter 4, we restricted our functions to those with four alternatives and two voters, whose domain is $L(X)^2$, and we developed a computer algorithm which can be used to generate all such functions satisfying the properties of (IIA) and (P). We determined that there are exactly 92 functions satisfying these constraints. The framework and algorithm for this software is detailed in this chapter. A more general algorithm is described which can be implemented for any number of alternatives or voters. This is a novel approach which could potentially be adapted to other variations of collective choice functions.

In Chapter 5 we introduced the Partial Principal Ultrafilter Property, which we used to give a complete characterization of ACIT functions with linear order domains. In the future, we are interested in the possibility of extending this approach to characterize functions whose domains satisfy the free-quadruple property. We also used this property to mathematically verify the function counts determined by the software described in Chapter 4, and to develop a recursive formula which counts the number of functions based on the size of X. Due to the complexity of the recursion, we have not yet succeeded in finding a closed formula for this sequence, and the sequence does not seem to occur in other contexts. Further work could also be done to derive a recursive or closed formula for a general number of voters as well as alternatives.

Finally, in Chapter 6 we addressed the question which inspired our interest in ACIT collective choice functions. As it has been shown that, with at least four alternatives, either weakening the range of Arrovian functions or weakening the independence condition to that of (WI) produces a function which must only be based on a weakly decisive voter, we wanted to investigate what happens when both conditions are weakened simultaneously. We now know that, for $D(X) \subseteq W(X)^n$, a function $f : D(X) \to ACIT(X)$ satisfying (WI) and (P) is not necessarily based on a weakly decisive voter. While we do not yet have a complete characterization of such functions, we feel that this is a problem which merits further investigation.

In conclusion, we feel that while ACIT relations are interesting in their own right, further study of acyclic, indifference-transitive collective choice functions is greatly merited. We are intrigued by their unique properties, and we feel that there is a great deal of potential in this particular subset of social choice theory.

REFERENCES

[1] K.J. Arrow, *Social Choice and Individual Values*, 1st ed., John Wiley & Sons, Inc., New York, 1951.

[2] D. Austen-Smith and J.S. Banks, *Positive Political Theory I: Collective Preference (Michigan Studies in Political Analysis)*, University of Michigan Press, December 2000.

[3] N. Baigent, *Twitching weak dictators*, Journal of Economics **47** (1987), no. 4, 407–411.

[4] D.H. Blair and R. Pollack, *Acyclic collective choice rules*, Econometrica : journal of the Econometric Society, an internat. society for the advancement of economic theory in its relation to statistics and mathematics. (1982).

[5] D.H. Blair and R.A. Pollack, *Collective rationality and dictatorship: The scope of the Arrow theorem*, Journal of Economic Theory. (1979).

[6] J.H. Blau, *Semiorders and collective choice*, Journal of economic theory. (1979).

[7] D.E. Campbell, *Equity, efficiency, and social choice*, Oxford University Press, Incorporated, 1992.

[8] D.E. Campbell and J.S. Kelly, *Weak independence and veto power*, Economics Letters **66** (2000), no. 2, 183–189.

[9] C. Coban and M.R. Sanver, *Social choice without the Pareto principle under weak independence*, Social Choice and Welfare (2014), 1–9.

[10] A.F. Gibbard, *Intransitive social indifference and the Arrow dilemma*, Review of Economic Design **18** (2014), no. 1, 3–10.

[11] A.S. Guha, *Neutrality, monotonicity, and the right of veto*, Econometra **40** (1972), 821–826.

[12] J. Iritani, T. Kamo, and R. Nagahisa, *Vetoer and tie-making group theorems for indifference-transitive aggregation rules*, Social Choice and Welfare **40** (2013), 155–171.

[13] A. Mas-Colell and A.F. Sonnenschein, *General possibility theorems for group decisions*, The review of economic studies. (1972).

[14] A.D. Miller and S. Rachmilevitch, *Arrow's theorem without transitivity*, unpublished manuscript (2014).

[15] P. Tang and F. Lin, *Computer-aided proofs of Arrow's and other impossibility theorems*, Artif. Intell. **173** (2009), no. 11, 1041–1053.

[16] J.A. Weymark, *Arrow's theorem with social quasi-orderings*, Public Choice **42** (1984), no. 3, 235–246.

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