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OSCILLATION AND NONOSCILLATION OF QUASILINEAR DIFFERENCE EQUATIONS OF SECOND ORDER

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Abstract. In this paper the authors establish conditions for the oscillatory and nonoscillatory behavior of solutions of second order quasilinear difference equations

$$\Delta(a_{n-1}|\Delta y_{n-1}|^{\alpha-1}\Delta y_{n-1}) + q_n f(y_n) = 0$$

and

$$\Delta(a_{n-1}|\Delta y_{n-1}|^{\alpha-1}\Delta y_{n-1}) + q_n f(y_{n-\lambda}) = 0$$

when $\{q_n\}$, $\{a_n\}$ and the function f satisfy different type of conditions. Examples are inserted to illustrate our results.

1. Introduction

This paper is concerned with the oscillatory and nonoscillatory behavior of quasilinear difference equations of the forms

$$\Delta(a_{n-1}|\Delta y_{n-1}|^{\alpha-1}\Delta y_{n-1}) + q_n f(y_n) = 0, \quad n = 1, 2, \dots \tag{1}$$

and

$$\Delta(a_{n-1}|\Delta y_{n-1}|^{\alpha-1}\Delta y_{n-1}) + q_n f(y_{n-l}) = 0, \quad n = 1, 2, \dots \tag{2}$$

where Δ is the forward difference operator defined by $\Delta y_n = y_{n+1} - y_n$, $\{a_n\}$ is a positive real sequence, $\{q_n\}$ is a real sequence, $f : \mathbf{R} \to \mathbf{R}$ is continuous, nondecreasing and $f(u) > 0$ for $u \neq 0$, $\alpha > 0$ and λ is a positive integer.

By a solution of equation (1) $\{(2)\}$ we mean a nontrivial sequence $\{y_n\}$ satisfying equation (1) $\{(2)\}$ for all $n \geqslant 1$ $\{n \geqslant 1 - l\}$. A solution $\{y_n\}$ is said to be nonoscillatory if it is either eventually positive or eventually negative and osillatory otherwise.

The problem of oscillation and nonoscillation of solutions difference equations has received a great deal of attention in the last few years, e.g. see [1,2,6,8,12] which cover a large number of recent papers. In particular, we refer to [4,5,13–21] where oscillations of equations similar to equations (1) and (2) have been studied.

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Our aim in this paper is to obtain oscillation and nonoscillation results for the equations (1) and (2) when $\{q_n\}$ is nonnegative and $f(u) = |u|^{\alpha-1}u$ or $\{q_n\}$ may change sign infinitely often. Some of our results include, as special cases, known oscillation criteria for second order difference equations. Examples dewelling upon the importance of our theorems are also included.

2. Preliminaries

Here we present some lemmas which are interesting in their own right and will be used in proofs of our main results.

For simplicity, we list the conditions used in the sequel as:

(i) $\{q_n\}$ is a nonnegative real sequence with infinitely many positive terms,

(ii) $\{q_n\}$ may change sign infinitely often for $n \geqslant 1$,

(iii) $-\infty < \sum_{n=1}^{\infty} q_n < \infty$,

(iv) $R_{n,n_0} = \sum_{s=n_0}^{n-1} \frac{1}{a_s^{1/\alpha}}$, $\quad R_{n,n_0} \to \infty$ and $R_n = R_{n,1}$,

(v) $\rho_{n_0} = \sum_{n=n_0}^{\infty} \frac{1}{a_n^{1/\alpha}} < \infty$,

(vi) $\lim_{|u|\to\infty} |f(u)| = +\infty$.

LEMMA 1. *Assume that conditions* (i) *and* (iv) *hold. If* $\{y_n\}$ *is a solution of equation* (1) *such that* $y_n > 0$ *for* $n \geqslant N \geqslant 1$ *then* $\Delta y_n > 0$ *for* $n \geqslant N$.

Proof. Since $\{a_n|\Delta y_n|^{\alpha-1}\Delta y_n\}$ is nonincreasing by equation (1), we see that $\{\Delta y_n\}$ is eventually of constant sign, that is $\Delta y_n > 0$ for $n \geqslant N \geqslant 1$ or there is $N_1 > N$ such that $\Delta y_n < 0$ for $n \geqslant N_1$. If $\Delta y_n < 0$ for $n \geqslant N_1$ we have

$$a_n^{1/\alpha}\Delta y_n \leqslant a_{N_2}^{1/\alpha}\Delta y_{N_2} < 0, \quad \text{for} \quad n \geqslant N_2 > N_1.$$

Dividing the last inequality by $a_n^{1/\alpha}$ and summing from N_2 to n, we obtain

$$y_{n+1} \leqslant y_{N_2} + a_{N_2}^{1/\alpha}\Delta y_{N_2} \sum_{s=N_2}^{n} \frac{1}{a_s^{1/\alpha}}.$$

As $n \to \infty$ we see that $y_n \to -\infty$, a contradiction. This completes the proof of the lemma.

LEMMA 2. *Assume conditions* (ii) *and* (iii), (iv) *and* (vi) *hold. If* $\{y_n\}$ *is a nonoscillatory solution of equation* (1), *then*

$$\frac{a_n|\Delta y_n|^{\alpha-1}\Delta y_n}{f(y_{n+1})} = \sum_{i=n+1}^{\infty} q_i + \sum_{i=n+1}^{\infty} \frac{a_i|\Delta y_i|^{\alpha-1}\Delta y_i \Delta f(y_i)}{f(y_i)f(y_{i+1})}, \quad n \geqslant 1.$$

For the proof see Theorem 1 of Thandapani, Manuel and Agarwal [15] and also see Lemma 2.2 of Zhang and Chen [21].

LEMMA 3. *Suppose conditions* (i) *and* (v) *hold. If* $\{y_n\}$ *is a nonoscillatory solution of equation* (1) *then* $\{y_n\}$ *is bounded for* $n \geqslant N$ *together with*

$$\frac{\rho_n^\alpha a_n|\Delta y_n|^{\alpha-1}\Delta y_n}{|y_n|^{\alpha-1}y_n}. \tag{3}$$

Moreover

$$\frac{\rho_n^\alpha a_n|\delta\Delta y_n|^{\alpha-1}\Delta y_n}{|y_n|^{\alpha-1}y_n} \geqslant -1 \quad \textit{for} \quad n \geqslant N \tag{4}$$

and

$$\limsup_{n\to\infty} \frac{\rho_n^\alpha a_n|\Delta y_n|^{\alpha-1}\Delta y_n}{|y_n|^{\alpha-1}y_n} \leqslant 0. \tag{5}$$

Proof. We may assume that $y_n > 0$ for $n \geqslant N \geqslant 1$. Since $\{a_n|\Delta y_n|^{\alpha-1}\Delta y_n\}$ is nonincreasing by equation (1), we see that $\Delta\{y_n\}$ is eventually of fixed sign, that is, $\Delta y_n > 0$ for $n \geqslant N$ or there is $N_1 > N$ such that $\Delta y_n < 0$ for $n > N_1$, and that

$$a_s^{1/\alpha}\Delta y_s \leqslant a_n^{1/\alpha}\Delta y_n \quad \text{for} \quad s \geqslant n \geqslant N.$$

Dividing the last inequality by $a_s^{1/\alpha}$ and summing it from n to $j-1$ gives

$$0 < y_j \leqslant y_n + a^{1/\alpha}\Delta y_n \sum_{s=n}^{j-1} \frac{1}{a_s^{1/\alpha}}, \quad j \geqslant n \geqslant N. \tag{6}$$

If $\Delta y_n > 0$ for $n \geqslant N$, then we have from (6)

$$0 < y_j \leqslant y_n + \alpha_n^{1/\alpha}\Delta y_n \rho_n, \quad j > n > N,$$

which shows that $\{y_n\}$ is bounded for $n \geqslant N$. If $\Delta y_n < 0$ for $n \geqslant N_1$ then $\{y_n\}$ is clearly bounded and letting $j \to \infty$ in (6), we have

$$0 \leqslant y_n + \alpha_n^{1/\alpha}\Delta y_n \rho_n, \quad n \geqslant N.$$

In either case, we obtain

$$\rho_n a_n^{1/\alpha}\frac{\Delta y_n}{y_n} \geqslant -1, \quad n \geqslant N,$$

of which (4) is an immediate consequence.

The relation (5) clearly holds if $\Delta y_n < 0$ for $n \geqslant N_1$, since in this case the function (3) itself is negative for $n \geqslant N_1$. If $\Delta y_n > 0$ for $n \geqslant N$, then there exist positive constants c_1 and c_2 such that $y_n > c_1$ and $a_n|\Delta y_n|^{\alpha-1}\Delta y_n \leqslant c_2$ for $n \geqslant N$, which implies

$$\frac{a_n|\Delta y_n|^{\alpha-1}\Delta y_n}{|y_n|^{\alpha-1}y_n} \leqslant \frac{c_2}{c_1^\alpha}, \quad n \geqslant N.$$

Since $\rho_n \to 0$ as $n \to \infty$, we then conclude that

$$\lim_{n\to\infty} \frac{\rho_n^\alpha a_n|\Delta y_n|^{\alpha-1}\Delta y_n}{|y_n|^{\alpha-1}y_n} = 0.$$

This proves (5) and the proof of the lemma is complete.

Finally, we need the following well known inequality due to Hardy, Littlewood and Polya [7, Theorem 41].

LEMMA 4. *If* $x, y \geqslant 0$ *then* $x^{\nu} - y^{\nu} \leqslant \nu x^{\nu-1}(x-y)$ *for* $\nu \geqslant 1$ *and* $x^{\nu} - y^{\nu} \leqslant \nu y^{\nu-1}(x-y)$ *for* $0 < \nu < 1$.

3. Oscillation and nonoscillation of equation (1)

In this section we establish criteria for the nonoscillation and oscillation of solutions of equation (1) subject to the condition $f(u) = |u|^{\alpha-1}u$.

It is convenient to rewrite the equation (1) in the form

$$\Delta\big(a_{n-1}(\Delta y_{n-1})^{\alpha^*}\big) + q_n y_n^{\alpha^*} = 0, \quad n \geqslant 1 \tag{7}$$

by introducing the notation

$$z^{\alpha^*} = |z|^{\alpha-1} z_n = |z|^{\alpha}\operatorname{sgn} z, \quad \alpha > 0. \tag{8}$$

THEOREM 1. *Assume conditions* (i), (iii) *and* (iv) *hold. Then equation* (7) *has nonoscillatory solution if and only if there is a sequence* $\{u_n\}$ *which satisfies*

$$u_n = \sum_{i=n+1}^{\infty} \frac{|u_{i-1}|\left\{\left[\left(\frac{|u_{i-1}|}{a_{i-1}}\right)^{1/\alpha} + 1\right]^{\alpha} - 1\right\}}{\left[\left(\frac{|u_{i-1}|}{a_{i-1}}\right)^{1/\alpha} + 1\right]^{\alpha}} + \sum_{i=n+1}^{\infty} q_i; \quad \text{for} \quad n \geqslant N \geqslant 1. \tag{9}$$

Proof. Assume that there is a sequence $\{u_n\}$ satisfying (9) for $n \geqslant N$. In view of conditions (i) and (iii) we have $u_n > 0$ for all $n \geqslant N$. Taking difference operator on both sides of (9) shows that $\{u_n\}$ is a solution of

$$\Delta u_{n-1} + \frac{|u_{n-1}|\left\{\left[\left(\frac{|u_{n-1}|}{a_{n-1}}\right)^{1/\alpha} + 1\right]^{\alpha} - 1\right\}}{\left[\left(\frac{|u_{n-1}|}{a_{n-1}}\right)^{1/\alpha} + 1\right]^{\alpha}} + q_n = 0 \tag{10}$$

for $n \geqslant N$ and $y_n = \prod_{s=n+1}^{n} \left[1 + \left(\frac{u_{s-1}}{a_{s-1}}\right)^{1/\alpha^*}\right]$ gives a nonoscillatory solution of equation (7) for $n \geqslant N$, where the meaning of * is defined by (8).

Let $\{y_n\}$ be a nonoscillatory solution of equation (7) and suppose that $y_n > 0$ for $n \geqslant N$ since a similar argument holds if we suppose $y_n < 0$ for $n \geqslant N$. It can be easily verified that $u_n = a_n\left(\frac{\Delta y_n}{y_n}\right)^{\alpha^*}$ satisfies equation (10) for $n \geqslant N$. Let n be fixed but arbitrary and sum (10) from $n+1$ to j, we obtain

$$u_j - u_n + \sum_{i=n+1}^{j} q_i + \sum_{i=n+1}^{j} \frac{|u_{i-1}|\left\{\left[\left(\frac{|u_{i-1}|}{a_{i-1}}\right)^{1/\alpha} + 1\right]^{\alpha} - 1\right\}}{\left[\left(\frac{|u_{i-1}|}{a_{i-1}}\right)^{1/\alpha} + 1\right]^{\alpha}} = 0, \quad j > n \geqslant N. \tag{11}$$

We claim that

$$\sum_{i=n+1}^{\infty} \frac{|u_{i-1}|\left\{\left[\left(\frac{|u_{i-1}|}{a_{i-1}}\right)^{1/\alpha}+1\right]^{\alpha}-1\right\}}{\left[\left(\frac{|u_{i-1}|}{a_{i-1}}\right)^{1/\alpha}+1\right]^{\alpha}} < \infty. \tag{12}$$

If

$$\sum_{i=n+1}^{\infty} \frac{|u_{i-1}|\left\{\left[\left(\frac{|u_{i-1}|}{a_{i-1}}\right)^{1/\alpha}+1\right]^{\alpha}-1\right\}}{\left[\left(\frac{|u_{i-1}|}{a_{i-1}}\right)^{1/\alpha}+1\right]^{\alpha}} = \infty.$$

then, in view of (11), there is $N_1 \geqslant n$ large enough such that

$$u_j + \sum_{i=N_1+1}^{\infty} \frac{|u_{i-1}|\left\{\left[\left(\frac{|u_{i-1}|}{a_{i-1}}\right)^{1/\alpha}+1\right]^{\alpha}-1\right\}}{\left[\left(\frac{|u_{i-1}|}{a_{i-1}}\right)^{1/\alpha}+1\right]^{\alpha}}$$

$$= u_n - \sum_{i=n+1}^{j} q_i - \sum_{i=n+1}^{N_1} \frac{|u_{i-1}|\left\{\left[\left(\frac{|u_{i-1}|}{a_{i-1}}\right)^{1/\alpha}+1\right]^{\alpha}-1\right\}}{\left[\left(\frac{|u_{i-1}|}{a_{i-1}}\right)^{1/\alpha}+1\right]^{\alpha}} \leqslant -1$$

for $j \geqslant N_1$, or

$$-u_j \geqslant 1 + \sum_{i=N_1+1}^{j} \frac{|u_{i-1}|\left\{\left[\left(\frac{|u_{i-1}|}{a_{i-1}}\right)^{1/\alpha}+1\right]^{\alpha}-1\right\}}{\left[\left(\frac{|u_{i-1}|}{a_{i-1}}\right)^{1/\alpha}+1\right]^{\alpha}}, \quad j \geqslant N_1. \tag{13}$$

It follows that $\Delta y_j < 0$ for $j \geqslant N_1$, a contradiction to Lemma 1. Therefore (12) must hold.

We now let $j \to \infty$ in (11). Using (12) and the summability of $\{q_n\}$, we find that u_j tends to a finite limit u_∞. But u_∞ must be zero, since otherwise (12) would fail to hold. Thus we are led to the equality (9). This completes the proof of the theorem.

THEOREM 2. *Assume conditions* (i), (iii) *and* (iv) *hold. Then all solutions of equation* (7) *are oscillatory if*

$$\lim_{n\to\infty} \sup R_n^{\alpha} \sum_{i=n+1}^{\infty} q_i > 1. \tag{14}$$

Proof. Suppose to the contrary that equation (7) has a nonoscillatory solution $\{y_n\}$. Without loss of generality we may assume that $y_n > 0$ for $n \geqslant N$, since the proof for the case $\{y_n\}$ is eventually negative is similar. By Lemma 1, $\Delta y_n > 0$ for $n \geqslant N$, so that the equation (7) can be written as

$$\Delta(a_{n-1}(\Delta y_{n-1})^{\alpha}) + q_n y_n^{\alpha} = 0, \quad n \geqslant N. \tag{15}$$

Divide (15) by y_n^α and use summation by parts formulae [1, Lemma 1.7.4], we have

$$\frac{a_j(\Delta y_j)^\alpha}{y_j^\alpha} - \frac{a_n(\Delta y_n)^\alpha}{y_n^\alpha} + \sum_{i=n+1}^{j} q_i + \sum_{i=n+1}^{j} \frac{a_{i-1}(\Delta y_{i-1})^\alpha \Delta y_{i-1}^\alpha}{y_i^\alpha y_{i-1}^\alpha} = 0 \qquad (16)$$

for $j \geqslant n+1 \geqslant N$. Let n be fixed but arbitrary and let $j \to \infty$. Because of the summability of $\{q_n\}$, if follows from (16) that (12) holds and the limit $\lim_{j\to\infty} \frac{a_j(\Delta y_j)^\alpha}{y_j^\alpha}$ exists and is finite. This limit must be zero, for, otherwise (12) would fail to hold. Hence we obtain

$$\left(\frac{a_n^{1/\alpha}\Delta y_n}{y_n}\right)^\alpha = \sum_{i=n+1}^{\infty} q_i + \sum_{i=n+1}^{\infty} \frac{a_{i-1}(\Delta y_{i-1})^\alpha \Delta y_{i-1}^\alpha}{y_i^\alpha y_{i-1}^\alpha}, \quad n \geqslant N, \qquad (17)$$

which implies that

$$\frac{(a_n^{1/\alpha}\Delta y_n)^\alpha}{y_n^\alpha} \geqslant \sum_{i=n+1}^{\infty} q_i, \quad n \geqslant N \qquad (18)$$

since the second sum in (17) is positive. From (15), we see that $\{a_n^{1/\alpha}\Delta y_n\}$ is nonincreasing for $n \geqslant N$ and so we have

$$y_n = y_N + \sum_{i=N}^{n-1} \frac{a_i^{1/\alpha}\Delta y_i}{a_i^{1/\alpha}} \geqslant a_n^{1/\alpha}\Delta y_n R_{n,N}, \quad n \geqslant N. \qquad (19)$$

Combining (18) and (19) give $1 \geqslant R_{n,N}^\alpha \sum_{i=n+1}^{\infty} q_i$, $n > N$, from which it follows that $\limsup_{n\to\infty} R_n^\alpha \sum_{i=n+1}^{\infty} q_i \leqslant 1$. This contradicts (14) and the proof is complete.

Remark 1. If $a_n \equiv 1$ then Theorems 1 and 2 are discrete analogue of Theorem 1 of Kusano and Yoshida [9] and Theorem 2.2 of Kusano, Yuki and Akio [10] respectively.

Example 1. Consider the difference equation

$$\Delta\left(\frac{1}{2^{n-1}}|\Delta y_{n-1}|^{\alpha-1}\Delta y_{n-1}\right) + 3(2^{\alpha-n})|y_n|^{\alpha-1}y_n = 0, \quad n \geqslant 1 \qquad (20)$$

where $\alpha > 0$. All conditions of Theorem 2 are satisfied and hence every solution of equation (20) is oscillatory. In fact $\{y_n\} = \{(-1)^n\}$ is such a solution of equation (20).

Remark 2. The results obtained in [5, 14–19] cannot be applied to equation (20) to get our results.

THEOREM 3. *Suppose conditions* (i) *and* (v) *hold. If*

$$\sum_{n=1}^{\infty} \rho_n^{\alpha+1} q_n = \infty \qquad (21)$$

then every solution of equation (7) is oscillatory.

Proof. Let $\{y_n\}$ be a nonoscillatory solution of equation (7). Define

$$u_n = \frac{a_n|\Delta y_n|^{\alpha-1}\Delta y_n}{|y_n|^{\alpha-1}y_n}$$

then $\{u_n\}$ is a solution of (10) for $n \geqslant N$, we now multiply (10) by $\rho_n^{\alpha+1}$ and summing from $N+1$ to n, we obtain

$$\rho_n^{\alpha+1}u_n - \rho_N^{\alpha+1}u_N + \sum_{s=N+1}^{n} u_s(-\Delta\rho_s^{\alpha+1}) + \sum_{s=N+1}^{n} \rho_s^{\alpha+1}q_s + \sum_{s=N+1}^{n} \frac{\rho_s^{\alpha+1}|u_{s-1}|\left\{\left[\left(\frac{|u_{s-1}|}{a_{s-1}}\right)^{1/\alpha}+1\right]^{\alpha}-1\right\}}{\left[\left(\frac{|u_{s-1}|}{a_{s-1}}\right)^{1/\alpha}+1\right]^{\alpha}} = 0. \quad (22)$$

In view of boundedness of $\rho_n^{\alpha}u_n$ (cf. Lemma 3), we see that

$$\rho_n^{\alpha+1}u_n = \rho_n(\rho_n^{\alpha}u_n) \to 0 \quad \text{as} \quad n \to \infty.$$

By mean value theorem

$$\Big|\sum_{s=N+1}^{\infty} u_s(-\Delta\rho_s^{\alpha+1})\Big| \leqslant \sum_{s=N+1}^{\infty} |u_s|(\alpha+1)\frac{\rho_s^{\alpha}}{a_s^{1/\alpha}} = \sum_{s=N+1}^{\infty} \frac{\alpha+1}{a_s^{1/\alpha}}|\rho_s^{\alpha}u_s| < \infty.$$

By Lemma 4, we have

$$\left|\sum_{s=N+1}^{\infty} \frac{\rho_s^{\alpha+1}|u_{s-1}|\left\{\left[\left(\frac{|u_{s-1}|}{a_{s-1}}\right)^{1/\alpha}+1\right]^{\alpha}-1\right\}}{\left[\left(\frac{|u_{s-1}|}{a_{s-1}}\right)^{1/\alpha}+1\right]^{\alpha}}\right| \leqslant \begin{cases} \displaystyle\sum_{s=N+1}^{\infty} \frac{|\rho_s^{\alpha+1}|\,|u_{s-1}|\alpha\left(\frac{|u_{s-1}|}{a_{s-1}}\right)^{1/\alpha}}{\left(\frac{|u_{s-1}|}{a_{s-1}}\right)^{1/\alpha}+1} & \text{if } \alpha \geqslant 1 \\ \displaystyle\sum_{s=N+1}^{\infty} \frac{|\rho_s^{\alpha+1}|\,|u_{s-1}|\alpha\left(\frac{|u_{s-1}|}{a_{s-1}}\right)^{1/\alpha}}{\left[\left(\frac{|u_{s-1}|}{a_{s-1}}\right)^{1/\alpha}+1\right]^{\alpha}} & \text{if } 0 < \alpha < 1 \end{cases}$$

Since $\left(\frac{|u_{s-1}|}{a_{s-1}}\right)^{1/\alpha} + 1 > 1$, the above inequality yields

$$\left|\sum_{s=N+1}^{\infty} \frac{\rho_s^{\alpha+1}|u_{s-1}|\left\{\left[\left(\frac{|u_{s-1}|}{a_{s-1}}\right)^{1/\alpha}+1\right]^{\alpha}-1\right\}}{\left[\left(\frac{|u_{s-1}|}{a_{s-1}}\right)^{1/\alpha}+1\right]^{\alpha}}\right| \leqslant \sum_{s=N+1}^{\infty} \frac{1}{a_s^{1/\alpha}}|\rho_s^{\alpha}u_s|^{\frac{\alpha+1}{\alpha}} < \infty.$$

Therefore, letting $n \to \infty$, we find that

$$\rho_N^{\alpha+1} u_N = \sum_{n=N+1}^{\infty} \rho_n^{\alpha+1} q_n + \sum_{n=N+1}^{\infty} u_n(-\Delta\rho_n^{\alpha+1}) + \sum_{s=N+1}^{\infty} \frac{\rho_n^{\alpha+1}|u_{n-1}|\left\{\left[\left(\frac{|u_{n-1}|}{a_{n-1}}\right)^{1/\alpha}+1\right]^{\alpha}-1\right\}}{\left[\left(\frac{|u_{n-1}|}{a_{n-1}}\right)^{1/\alpha}+1\right]^{\alpha}}.$$

From the last equation, we obtain

$$\sum_{n=N+1}^{\infty} \rho_n^{\alpha+1} q_n < \infty$$

a contradiction to (21). This completes the proof of the theorem.

Remark 3. Theorem 3 is discrete analogue of Theorem 2.4 of Kusano and Naito [11].

Example 2. Theorem 3 implies that all solutions of the difference equation

$$\Delta(n^{2\alpha}|\Delta y_{n-1}|^{\alpha-1}\Delta y_{n-1}) + n^{2\alpha+1}|y_n|^{\alpha-1}y_n = 0, \quad n \geqslant 1 \tag{23}$$

are oscillatory.

In the following we establish oscillation criteria for equation (1) when $f^{1/\alpha}$ is of superlinear type.

THEOREM 4. *Assume conditions* (ii)–(iv) *and* (vi) *hold. If*

$$0 < \int_{\varepsilon}^{\infty} \frac{du}{f^{1/\alpha}(u)}, \quad \int_{-\varepsilon}^{\infty} \frac{du}{f^{1/\alpha}(u)} < \infty, \quad \textit{for any} \quad \varepsilon > 0, \tag{24}$$

$$\alpha \textit{ is a ratio of odd positive integers,} \tag{25}$$

and

$$\lim_{n\to\infty} \sum_{s=1}^{n} \left(\frac{1}{a_s}\sum_{i=s+1}^{\infty} q_i\right)^{1/\alpha} = \infty \tag{26}$$

then every solution of equation (1) *is oscillatory.*

Proof. Assume that equation (1) has a nonoscillatory solution $\{y_n\}$ and we may suppose that $\{y_n\}$ is eventually positive. Under our assumption Lemma 2 is true. Since f is nondecreasing, the second sum in Lemma 2 is nonnegative. Hence

$$\frac{|\Delta y_{n-1}|^{\alpha-1}\Delta y_n}{f(y_{n+1})} \geqslant \frac{1}{a_n}\sum_{i=n+1}^{\infty} q_i$$

or

$$\frac{\Delta y_n}{f^{1/\alpha}(y_{n+1})} \geqslant \left(\frac{1}{a_n}\sum_{i=n+1}^{\infty} q_i\right)^{1/\alpha}.$$

Summing the last inequality from $N \geqslant 1$ to n, we get

$$\sum_{s=N}^{n} \frac{\Delta y_s}{f^{1/\alpha}(y_{s+1})} \geqslant \sum_{s=N}^{n} \left(\frac{1}{a_s} \sum_{i=s+1}^{\infty} q_i \right)^{1/\alpha}. \tag{27}$$

Define $r(t) = y_n + (t-n)\Delta y_n$, $n \leqslant t \leqslant n+1$. If $\Delta y_n \geqslant 0$ then $y_n \leqslant r(t) \leqslant y_{n+1}$ and

$$\frac{\Delta y_n}{f^{1/\alpha}(y_{n+1})} \leqslant \frac{r'(t)}{f^{1/\alpha}(r(t))} \leqslant \frac{\Delta y_n}{f^{1/\alpha}(y_n)}. \tag{28}$$

If $\Delta y_n < 0$, then $y_{n+1} \leqslant y_n$ and (28) also holds. From (27) and (28) we obtain

$$\int_{r(N)}^{\infty} \frac{du}{f^{1/\alpha}(u)} \geqslant \int_{N}^{N+1} \frac{dr(t)}{f^{1/\alpha}(r(t))} \geqslant \sum_{s=N}^{n} \left(\frac{1}{a_s} \sum_{i=s+1}^{\infty} q_i \right)^{1/\alpha}. \tag{29}$$

Let $G(s) = \int_s^{\infty} \frac{du}{f^{1/\alpha}(u)}$, then (29) implies that

$$\sum_{s=N}^{n} \left(\frac{1}{a_s} \sum_{i=s+1}^{\infty} q_i \right)^{1/\alpha} \leqslant G(r(N)),$$

which contradicts condition (26). Similarly, one can prove that equation (1) does not possess eventually negative solutions. This completes the proof.

Remark 4. When $\alpha = 1$ and $a_n = 1$ then Theorem 4 reduces to Theorem 3.1 of Zhang and Chen [21].

Example 3. The difference equation

$$\Delta(n^{-\alpha}(\Delta y_{n-1})^{\alpha}) + \frac{2^{\alpha}((n+1)^{\alpha} + n^{\alpha})}{n^{\alpha}(n+1)^{\alpha}} y_n^{\beta} = 0, \quad n \geqslant 1 \tag{30}$$

where $\beta > \alpha > 1$ and α, β are ratio of odd positive integers, satisfying all conditions of Theorem 4, and hence all of its solutions are oscillatory. In fact, $\{y_n\} = \{(-1)^n\}$ is such a solution of equation (30).

We conclude this section with another oscillation criteria for the equation (1) when $f(u) = |u|^{\beta} \operatorname{sgn} u$.

THEOREM 5. *With respect to the difference equation* (1), *assume that* $\alpha < \beta$ *and conditions* (ii)–(iv) *hold. If* $Q_n = \sum_{s=n+1}^{\infty} q_s \geqslant 0$ *for all* $n \geqslant N \geqslant 1$ *and*

$$\sum_{n=N}^{\infty} \left[\left(\frac{Q_n}{a_n} \right)^{1/\alpha} + \left(\frac{1}{a_n} \sum_{s=n+1}^{\infty} \left(\frac{Q_s^{\alpha+1}}{a_s^{1/\alpha}} \right) \right)^{1/\alpha} \right] = \infty \tag{31}$$

then all solutions of equation (1) are oscillatory.

Proof. Assume that equation (1) has a nonoscillatory solution $\{y_n\}$ and we may suppose that $\{y_n\}$ is eventually positive. From the proof of the Theorem 4, we have

$$\frac{\Delta y_n}{y_{n+1}^{\beta/\alpha}} \geqslant \left(\frac{Q_n}{a_n} \right)^{1/\alpha} \geqslant 0.$$

Summing the above inequality from N to n and using (28) we obtain

$$\sum_{s=N}^{n}\left(\frac{Q_s}{a_s}\right)^{1/\alpha} < \infty \tag{32}$$

since $\alpha < \beta$. Again from Lemma 2, we have

$$\frac{a_n(\Delta y_n)^\alpha}{y_{n+1}^\beta} \geqslant \sum_{i=n+1}^{\infty} \frac{a_i(\Delta y_i)^\alpha \Delta y_i^\beta}{(y_i^\beta)(y_{i+1}^\beta)}, \quad \text{for} \quad n \geqslant N,$$

and hence

$$\frac{(\Delta y_n)^\alpha}{y_{n+1}^\beta} \geqslant \frac{1}{a_n}\sum_{i=n+1}^{\infty} Q_i \frac{\Delta y_i^\beta}{y_i^\beta}, \quad n \geqslant N. \tag{33}$$

By mean value theorem

$$\Delta y_n^\beta = \xi^{\beta-1}\Delta y_n \quad \text{when} \quad y_n < \xi < y_{n+1}$$

and therefore

$$\Delta y_n^\beta \geqslant \beta y_n^{\beta-1}\Delta y_n. \tag{34}$$

Using (34) in (33) we obtain

$$\begin{aligned}\frac{(\Delta y_n)^\alpha}{y_{n+1}^\beta} &\geqslant \frac{1}{a_n}\sum_{i=n+1}^{\infty} \frac{Q_s \beta y_s^{\beta-1}\Delta y_s}{y_s^\beta} \geqslant \frac{1}{a_n}\beta \sum_{i=n+1}^{\infty}\frac{Q_s\Delta y_s}{y_{s+1}} \\ &\geqslant \frac{\beta}{a_n}\sum_{s=n+1}^{\infty} Q_s\left(\frac{Q_s}{a_s}\right)^{1/\alpha} y_{s+1}^{\frac{\beta-\alpha}{\alpha}}, \quad n \geqslant N \\ &\geqslant \frac{\beta}{a_n} y_{N+1}^{\frac{\beta-\alpha}{\alpha}} \sum_{s=n+1}^{\infty} Q_s\left(\frac{Q_s}{a_s}\right)^{1/\alpha} \quad \text{for} \quad n \geqslant N,\end{aligned}$$

or

$$\frac{\Delta y_n}{y_{n+1}^{\beta/\alpha}} \geqslant \left(\beta y_{N+1}^{\frac{\beta-\alpha}{\alpha}}\right)^{1/\alpha}\left[\frac{1}{a_n}\sum_{s=n+1}^{\infty}\frac{Q_s^{\frac{\alpha+1}{\alpha}}}{a_s^{1/\alpha}}\right]^{1/\alpha} \quad \text{for} \quad n \geqslant N.$$

As above we obtain

$$\left(\beta y_{N+1}^{\frac{\beta-\alpha}{\alpha}}\right)^{1/\alpha}\left[\sum_{s=N}^{n}\left(\frac{1}{a_s}\sum_{t=s+1}^{n}\frac{Q_t^{\frac{1+\alpha}{\alpha}}}{a_t^{1/\alpha}}\right)\right] < \infty \tag{35}$$

since $\alpha < \beta$. Combining (35) and (32) give a contradiction to (31). This completes the proof.

Remark 5. Theorem 5 is a discrete generalization of an oscillation theorem of Butler [3] for the superlinear discrete Emden–Fowler equation (1) (α=1 and $\beta > 1$) subject to the condition $Q_n \geqslant 0$.

4. Oscillation theorems for equation (2)

Here, we shall obtain sufficient conditions for all solutions of equation (2) to be oscillatory.

THEOREM 6. *Assume conditions* (i) *and* (iv) *hold. If*

$$\lim_{|u|\to\infty} \inf\left|\frac{f(u)}{u^\alpha}\right| \geqslant d > 0, \tag{36}$$

there exists a positive sequence $\{h_n\}$ *such that*

$$\lim_{n\to\infty} \sum_{s=1}^{n} h_s q_s = +\infty, \tag{37}$$

and

$$\lim_{n\to\infty} \sum_{s=1}^{n} \frac{a_s(\Delta h_{s+1})^{\alpha+1}}{h_{s+1}^\alpha} < +\infty, \tag{38}$$

then every solution of equation (2) is oscillatory.

Proof. Let $\{y_n\}$ be a nonoscillatory solution of equation (2), say $y_n > 0$ for $n \geqslant N \geqslant 1$, since the proof for the case $y_n < 0$ for $n \geqslant N$ is similar. Let

$$z_{n-1} = \frac{a_{n-1}(\Delta y_{n-1})^\alpha}{y_{n-l}^\alpha}, \quad n \geqslant N + l$$

since $\Delta y_{n-1} > 0$ by Lemma 1. Then

$$\Delta z_{n-1} = -q_n \frac{f(y_{n-l})}{y_{n-l}^\alpha} - \frac{a_n(\Delta y_n)^\alpha}{y_{n+1-l}^\alpha y_{n-l}^\alpha} \Delta y_{n-l}^\alpha$$

and hence

$$\Delta z_{n-1} + \frac{z_n \Delta y_{n-l}^\alpha}{y_{n-l}^\alpha} + q_n \frac{f(y_{n-l})}{y_{n-l}^\alpha} = 0, \quad \text{for} \quad n \geqslant N + l. \tag{39}$$

Since $\Delta y_n > 0$ for $n \geqslant N + 1$, $\lim_{n\to\infty} y_n$ exists (finite or infinite). If $\lim_{n\to\infty} y_n = b$, then $\lim_{n\to\infty} \frac{f(y_{n-l})}{y_{n-l}^\alpha} = \frac{f(b)}{b^\alpha} > 0$. If $\lim_{n\to\infty} y_n = +\infty$ then, in view of condition (36), $\lim_{n\to\infty} \inf \frac{f(y_{n-l})}{y_{n-l}^\alpha} = d_1 \geqslant d > 0$. Let $b^* = \min\left(\frac{d}{2}, \frac{f(b)}{b^\alpha}\right)$, from (39) we have

$$\Delta z_{n-1} + z_n \frac{\Delta y_{n-l}^\alpha}{y_{n-l}^\alpha} + b^* q_n \leqslant 0, \quad n \geqslant N + l. \tag{40}$$

From the mean value theorem, we have

$$\Delta y_{n-l}^\alpha \geqslant \begin{cases} \alpha y_{n-l}^{\alpha-1} \Delta y_{n-l} & \text{if } \alpha \geqslant 1 \\ \alpha y_{n+1-l}^{\alpha-1} \Delta y_{n-l} & \text{if } 0 < \alpha < 1 \end{cases} \tag{41}$$

for $n \geqslant N + l$. Hence from (40) and (41), we obtain

$$\Delta z_{n-1} + z_n \alpha \frac{\Delta y_{n-l}}{y_{n-l}} + b^* q_n \leqslant 0, \quad \alpha \geqslant 1. \tag{42}$$

and

$$\Delta z_{n-1} + z_n \alpha \frac{\Delta y_{n+1-l}^{\alpha-1} \Delta y_{n-l}}{y_{n-l}^{\alpha}} + b^* q_n \leqslant 0, \quad 0 < \alpha < 1. \tag{43}$$

Since $\{y_n\}$ is nondecreasing, we have from (42) and (43)

$$\Delta z_{n-l} + \alpha \frac{z_n \Delta y_{n-l}}{y_{n+1-l}} + b^* q_n \leqslant 0 \tag{44}$$

for $\alpha > 0$ and $n \geqslant N + l + 1$. Since $\Delta(a_{n-1}(\Delta y_{n-1})^n) \leqslant 0$, we have $a_{n-1}(\Delta y_{n-1})^{\alpha} \geqslant a_{n-1}(\Delta y_{n-1})^{\alpha}$ and so (44) gives

$$\Delta z_{n-l} + \alpha \frac{z_n z_n^{1/\alpha}}{a_{n-l}^{1/\alpha}} + b^* q_n \leqslant 0 \quad n \geqslant N_1 \geqslant N + l + 1. \tag{45}$$

Multiply (45) by h_n and summing, we obtain

$$\begin{aligned} b^* \sum_{s=N_1}^{n} h_s q_s &\leqslant - \sum_{s=N}^{n} h_s \Delta z_{s-1} - \alpha \sum_{s=N_1}^{n} \frac{h_s z_s^{\frac{\alpha+1}{\alpha}}}{a_{s-1}^{1/\alpha}} = -h_{n+1} z_n + h_{N_1} z_{N_1-1} \\ &\quad - \sum_{s=N_1}^{n} \frac{\alpha h_s}{a_{s-1}^{1/\alpha}} \left[z_s^{\frac{\alpha+1}{\alpha}} - \frac{\Delta h_s a_{s-1}^{1/\alpha} z_s}{\alpha h_s} + \frac{1}{\alpha} \left(\frac{\Delta h_s}{(\alpha+1) h_s} \right)^{\alpha+1} a_{s-1}^{\frac{\alpha+1}{\alpha}} \right] \\ &\qquad + \sum_{s=N_1}^{n} \frac{(\Delta h_s)^{\alpha+1}}{(\alpha+1)^{\alpha+1} h_s^{\alpha}} a_{s-1}. \end{aligned}$$

Using Lemma 4 we obtain

$$b^* \sum_{s=N_1}^{n} h_s q_s \leqslant h_{N_1} z_{N_1-1} + \sum_{s=N_1}^{n} \frac{a_{s-l}(\Delta h_s)^{\alpha+1}}{(\alpha+1)^{\alpha+1} h_s^{\alpha}}.$$

This gives contradiction to conditions (37) and (38) and the proof is complete.

Remark 6. When $\alpha = 1$, Theorem 6 reduces to Theorem 1 of Yang and W. Zhang [20].

Example 4. The difference equation

$$\Delta(n^{\alpha}(\Delta y_{n-l})^{\alpha}) + 2^{\alpha}(n^{\alpha} + (n+1)^{\alpha}) y_{n-l}^{\beta} = 0, \quad n \geqslant 1 \tag{46}$$

where $\beta \geqslant \alpha$ and β, α are ratio of odd positive integers and l is an even positive integer, satisfy all conditions of Theorem 6 if we take $h_n = \frac{1}{n^{\alpha+1}}$, and hence all of its solutions are oscillatory. In fact, $\{y_n\} = \{(-1)^n\}$ is such a solution of equation (46).

Next assume that conditions (i) and (iii) are satisfied. Let

$$h_0(n) = \sum_{i=n+1}^{\infty} q_i \quad \text{and} \quad \left|\frac{f(u)}{u^\alpha}\right| \geqslant M > 0 \quad \text{for} \quad |u| \geqslant \delta > 0.$$

Define

$$h_j(n) = M^{\frac{\alpha+1}{\alpha}} \alpha \sum_{i=n+1}^{\infty} \frac{h_{j-1(i)}^{1+1/\alpha}}{a_{i-l}^{1/\alpha}} + Mh_0(n), \quad j = 1, 2, \ldots$$

and

$$E_n = \sum_{i=1}^{n} \frac{1}{a_{i-l}^{1/\alpha}}$$

THEOREM 7. *Assume conditions* (i), (iii) *and* (iv) *hold. Further assume that*

(vii) $$\left|\frac{f(u)}{u^\alpha}\right| \geqslant M > 0 \quad \textit{for} \quad |u| > \delta > 0$$

and that one of the following conditions hold.

(c_1) $\sum_{n=1}^{\infty} q_n = +\infty$, *that is* $h_0(0)$ *does not exist;*

(c_2) *there exists a positive integer* j *such that* $h_k(n)$ *is defined for* $k = 0, 1, 2, \ldots, j-1$ *but* $h_j(0)$ *does not exist;*

(c_3) *there exists a nonnegative integer* j *such that* $h_j(0)$ *exists and*

$$\limsup_{n\to\infty} E_n^{\alpha/\alpha+1} h_j(n) > (M/\alpha)^{\alpha/\alpha+1}.$$

Then every solution of equation (2) is oscillatory.

Proof. Assume that $\{y_n\}$ is a nonoscillatory solution of equation (2) with $y_n > 0$ for $n > N + l$. By Lemma 1, $\Delta y_n > 0$ for $n \geqslant N + l$. Now define

$$W_{n-1} = \frac{a_{n-1}(\Delta y_{n-1})^\alpha}{y_{n-l}^\alpha}$$

for $n \geqslant N + l$. Then from the proof of Theorem 6, we obtain

$$\Delta W_{n-1} + \alpha \frac{W_n^{1+1/\alpha}}{a_{n-l}^{1/\alpha}} + Mq_n \leqslant 0, \quad n \geqslant N_1 \geqslant N + l + 1. \tag{47}$$

Summing the last inequality from N_1 to n, we obtain

$$W_n + \alpha \sum_{s=N_1}^{n} \frac{W_s^{1+1/\alpha}}{a_{s-l}^{1/\alpha}} + M \sum_{s=N_1}^{n} q_s \leqslant W_{N_1-1}. \tag{48}$$

If $\sum_{s=N_1}^{\infty} \frac{W_s^{1+1/\alpha}}{a_{s-l}^{1/\alpha}} = +\infty$ then $\lim_{n\to\infty} \left(1 + \frac{W_n}{\alpha \sum_{s=N_1}^{n} \frac{W_s^{1+1/\alpha}}{a_{s-l}^{1/\alpha}}}\right) \leqslant 0$ or

$$\limsup_{n\to\infty} \frac{W_n}{\alpha \sum_{s=N_1}^{n} \frac{W_s^{1+1/\alpha}}{a_{s-l}^{1/\alpha}}} \leqslant -1$$

which contradicts $W_n > 0$. Hence $\sum\limits_{s=N_1}^{\infty} \frac{W_s^{1+1/\alpha}}{a_{s-l}^{1/\alpha}} < \infty$. On the other hand from (47), we have $\Delta W_{n-1} \leqslant 0$ which imply $\lim\limits_{n\to\infty} W_n = c \geqslant 0$. If $c > 0$ then it would imply that $\sum\limits_{n=N_1}^{\infty} \frac{1}{a_{n-l}^{1/\alpha}} < \infty$, a contradiction. Hence $\lim\limits_{n\to\infty} W_n = 0$.

(c_1) If $h_0(0) = +\infty$, then from (48) we have

$$W_n \leqslant W_{N_1-1} - M \sum_{s=N_1}^{n} q_s.$$

This gives a contradiction.

(c_2) If $h_k(n)$ is defined for $k = 0, 1, 2, \ldots, j-1$, letting $n \to \infty$ in (48), we have

$$W_n \geqslant \alpha \sum_{s=n+1}^{\infty} \frac{W_s^{1+1/\alpha}}{a_{s-l}^{1/\alpha}} + M h_0(n). \tag{49}$$

and hence $W_n \geqslant M h_0(n)$. If $j = 1$ then $\sum\limits_{s=n+1}^{\infty} \frac{h_0^{1+1/\alpha}(s)}{a_{s-l}^{1/\alpha}} \leqslant \frac{1}{M^{1+1/\alpha}} \sum\limits_{s=n+1}^{\infty} \frac{W_s^{1+1/\alpha}}{a_{s-l}^{1/\alpha}} < \infty$ which contradicts $h_1(0) = +\infty$. If $j > 1$, then

$$h_1(n) = M^{1+1/\alpha} \alpha \sum_{i=n+1}^{\infty} \frac{h_0(i)}{a_{i-l}^{1/\alpha}} + M h_0(n) \leqslant \alpha \sum_{i=n+1}^{\infty} \frac{W_i^{1+1/\alpha}}{a_{i-l}^{\frac{1}{\alpha}}} + M h_0(n) \leqslant W_n.$$

In a similar fashion, we can prove $W_n \geqslant h_k(n)$ for $k = 1, 2, \ldots, j-1$. Therefore

$$\sum_{i=n+1}^{\infty} \frac{h_{j-1}^{1+1/\alpha}(i)}{a_{i-l}^{1/\alpha}} \leqslant \sum_{i=n+1}^{\infty} \frac{W_i^{1+1/\alpha}}{a_{i-l}^{1/\alpha}} < \infty$$

which contradicts (c_2).

(c_3) Since $\sum\limits_{n=N_1}^{\infty} \frac{W_n^{1+1/\alpha}}{a_{n-l}^{1/\alpha}} < \infty$, we have for sufficiently large n: $\sum\limits_{s=N_1}^{n} \frac{W_s^{1+1/\alpha}}{a_{s-l}^{1/\alpha}} < \frac{M}{\alpha}$. Moreover, $\{W_n\}$ is decreasing, we have from the above inequality $W_n^{1+1/\alpha} \sum\limits_{s=N_1}^{n} \frac{1}{a_{s-l}^{1/\alpha}} < \frac{M}{\alpha}$ and hence $\left[\sum\limits_{s=N_1}^{n} \frac{1}{a_{s-l}^{1/\alpha}}\right]^{\alpha/\alpha+1} < \frac{(M/\alpha)^{\alpha+1}}{W_n}$. Then, for n sufficiently large $h_j(n) \leqslant W_n \leqslant \frac{(M/\alpha)^{\alpha/\alpha+1}}{[E_n - E_{N_1}]^{\alpha/\alpha+1}}$ or

$$E_n^{\alpha/\alpha+1} h_j(n) \leqslant \frac{E_n^{\alpha/\alpha+1} (M/\alpha)^{\alpha/\alpha+1}}{[E_n - E_{N_1}]^{\alpha/\alpha+1}}.$$

Therefore

$$\lim_{n\to\infty} \sup E_n^{\alpha/\alpha+1} h_j(n) \leqslant (M/\alpha)^{\alpha/\alpha+1},$$

which contradicts (c_3). Thus the proof of the theorem is complete.

Remark 7. If $\{q_n\}$ is eventually negative in equation (2), then the oscillation and nonoscillation of solutions are discussed in Wong and Agarwal [19].

We conclude this paper with the following example.

Example 6. Consider the difference equation

$$\Delta\left(\frac{1}{n^2}(\Delta y_{n-1})^{\alpha}\right) + \frac{2^{\alpha}(2n^2+2n+1)}{n^2(n+1)^2} y_{n-l}^{\beta} = 0, \quad n > 1 \tag{50}$$

where β and α are quotient of odd positive integers. Clearly conditions (i), (iii),(iv) and (vii) hold. Furthermore,

$$h_0(n) = \sum_{s=n+1}^{\infty} q_s \geqslant 2^{\alpha} \sum_{s=n+1}^{\infty} \frac{2s^2+2s}{s^2(s+1)^2} = 2^{\alpha+1} \sum_{s=n+1}^{\infty} \frac{1}{s(s+1)} = \frac{2^{\alpha+1}}{n+1} \geqslant 0$$

and

$$h_1(0) = \frac{M^{\frac{\alpha+1}{\alpha}} \alpha \sum\limits_{s=1}^{\infty} \left(\frac{2^{\alpha+1}}{s+1}\right)^{\frac{\alpha+1}{\alpha}}}{\frac{1}{(s-l)^{1/\alpha}}} + h_0(0)$$

$$= M^{\frac{\alpha+1}{\alpha}} \alpha\, 2^{\frac{(\alpha+1)^2}{\alpha}} \sum_{s=1}^{\infty} \left(\frac{(s-l)^{1/\alpha}}{(s+1)^{\frac{\alpha+1}{\alpha}}} + h_0(1) = \infty.$$

Hence condition (c_2) is satisfied with $j = 1$. It follows from Theorem 7 that all solutions of equation (50) are oscillatory. One such solution is $\{y_n\} = \{(-1)^n\}$.

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