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ON STRONGLY STARLIKENESS OF ORDER ALPHA IN SEVERAL COMPLEX VARIABLES

Gabriela Kohr Cluj-Napoca, Romania and **Piotr Liczberski,** Lodz, Poland

Abstract. In this paper we introduce the concept of strongly starlikeness of order $\alpha > 0$, for holomorphic mappings defined on the unit ball of $\mathbf{C}^n$. We obtain the distortion and the covering theorems for strongly starlike mappings of order $\alpha \in (0, 1]$ and we give a connection between strongly starlikeness and spirallikeness in $\mathbf{C}^n$.

1. Introduction

Let $\mathbf{C}^n$ denote the space of n complex variables $z = (z_1, \ldots, z_n)'$ with the Euclidean inner product $\langle z, w\rangle = \sum_{j=1}^{n} z_j\overline{w}_j$ and the norm $\|z\| = \sqrt{\langle z, z\rangle}$, for all $z \in \mathbf{C}^n$. The open Euclidean ball $\{z \in \mathbf{C}^n : \|z\| < r\}$ is denoted by B_r and the open unit Euclidean ball B_1 is abbreviated by $B_1 = B$. In the case $n = 1$, the open ball B is abbreviated by U and it is called the unit disc. The origin $(0, 0, \ldots, 0)'$ is always denoted by 0. As usual, by $L(\mathbf{C}^n, \mathbf{C}^m)$ we denote the space of all continuous linear operators from $\mathbf{C}^n$ into $\mathbf{C}^m$, with the standard operator norm. The letter I will always represent the identity operator in $L(\mathbf{C}^n, \mathbf{C}^n)$. The class of holomorphic mappings from a domain $G \subset \mathbf{C}^n$ into $\mathbf{C}^n$ is denoted by $H(G)$. A mapping $f \in H(G)$ is said to be locally biholomorphic on G if its Fréchet derivative

$$Df(z) = \left[\frac{\partial f_j(z)}{\partial z_k}\right]_{1\leqslant j,k\leqslant n}$$

as an element of $L(\mathbf{C}^n, \mathbf{C}^n)$ is nonsingular at each point $z \in G$. A mapping $f \in H(G)$ is called biholomorphic on G if its inverse f^{-1} does exist, is holomorphic on a domain Ω and $f^{-1}(\Omega) = G$. If $D^2f(z)$ means the Fréchet derivative of second order of $f \in H(G)$ at $z \in G$, then $D^2f(z)$ is a continuous bilinear operator from $\mathbf{C}^n \times \mathbf{C}^n$ into $\mathbf{C}^n$ and its restriction $D^2f(z)(u, \cdot)$ to $u \times \mathbf{C}^n$ belongs to $L(\mathbf{C}^n, \mathbf{C}^n)$. Also, if $f \in H(G)$ then we denote by $D^kf(z)$ the kth Fréchet derivative of f at $z \in G$. The symbol $'$ means the transpose of vectors and matrices on $\mathbf{C}^n$.

For our purpose, we shall use the following definitions and results.

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Definition 1.1. A holomorphic mapping $f : B \to \mathbf{C}^n$ is said to be starlike on B if f is biholomorphic on B, $f(0) = 0$ and $tf(B) \subseteq f(B)$, for all $t \in [0, 1]$.

LEMMA 1.1. [3], [7], [12]. *Let $f : B \to \mathbf{C}^n$ be a locally biholomorphic mapping with $f(0) = 0$. Then f is starlike iff*

$$\text{Re}\,\langle [Df(z)]^{-1}f(z), z\rangle > 0,$$

for all $z \in B \setminus \{0\}$.

Definition 1.2. Let $f : B \to \mathbf{C}^n$ be a locally biholomorphic mapping on B, normalized by $f(0) = 0$ and $Df(0) = I$. We say that f is strongly starlike of order α, where $\alpha > 0$, if

$$|\arg\langle [Df(z)]^{-1}f(z), z\rangle| < \alpha\frac{\pi}{2}, \tag{1.1}$$

for all $z \in B \setminus \{0\}$.

Note that if $n = 1$ the condition (1.1) is equivalent to

$$\left|\arg\frac{zf'(z)}{f(z)}\right| < \alpha\frac{\pi}{2},\ z \in U,$$

hence, we obtain the usual class of strongly starlike functions of order $\alpha > 0$, on the unit disc U. This class was introduced and studied independently by J. Stankiewicz [10] and D. Brannan - W. Kirwan [2].

On the other hand, if we compare the Definition 1.2 and Lemma 1.1, we deduce that if f is strongly starlike of order $\alpha \in (0, 1]$, then f is also starlike, hence biholomorphic on B.

Obviously, the above class is not empty, because $f(z) = z$, $z \in B$, is strongly starlike of order α, for all $\alpha > 0$. Also, if f is strongly starlike of order $\alpha > 0$, then f is also strongly starlike of order β, for all $\beta \geqslant \alpha$.

We shall show that between strongly starlikeness of order $\alpha \in (0, 1]$ and spirallikeness, there exists a close connection. For this aim, we recall the notion of spirallikeness in $\mathbf{C}^n$ (see, for details [3], [13]).

Definition 1.3. Suppose $f : B \to \mathbf{C}^n$ is a biholomorphic mapping on B, with $f(0) = 0$ and $Df(0) = I$. Let $A \in L(\mathbf{C}^n, \mathbf{C}^n)$ be a positive linear operator, that means $m(A) > 0$, where

$$m(A) = \min\{\text{Re}\,\langle A(z), z\rangle : \|z\| = 1\}.$$

We say that f is spirallike relative to A if

$$e^{-tA}f(B) \subset f(B), \tag{1.2}$$

for all $t > 0$, where $e^{-tA} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}t^kA^k$.

LEMMA 1.2. [3], [13]. *Let $f : B \to \mathbf{C}^n$ be a locally biholomorphic mapping on B, with $f(0) = 0$, $Df(0) = I$ and let $A \in L(\mathbf{C}^n, \mathbf{C}^n)$, with $m(A) > 0$. Then f is spirallike relative to A iff*

$$\operatorname{Re} \langle [Df(z)]^{-1} Af(z), z\rangle > 0, \quad z \in B \setminus \{0\}.$$

On the other hand, we use in our paper the following result due to Rogosinski.

LEMMA 1.3. [9]. *Let h and H be holomorphic functions on the unit disc U, such that H is convex on U. If $h(z) = 1 + \sum_{k=1}^{\infty} c_k z^k$, $H(z) = 1 + \sum_{k=1}^{\infty} C_k z^k$, $z \in U$, and h is subordinate to H, then $|c_k| \leqslant |C_1|$, for all $k \geqslant 1$.*

In this paper we obtain the distortion and covering theorems for strongly starlike mappings of order $\alpha \in (0, 1]$ on B and we give several interesting properties concerning this class of biholomorphic mappings on the unit ball of $\mathbf{C}^n$.

2. Main results

For our purpose we need to prove the following result.

LEMMA 2.1. *Let $p \in H(B)$ be normalized by $p(0) = 0$, $Dp(0) = I$ and suppose that for all $z \in B \setminus \{0\}$ the relation*

$$|\arg\langle p(z), z\rangle| < \alpha\frac{\pi}{2}, \tag{2.1}$$

is satisfied, where $\alpha \in (0, 1]$.

Then

$$\|z\|^2 \left(\frac{1-\|z\|}{1+\|z\|}\right)^{\alpha} \leqslant \operatorname{Re} \langle p(z), z\rangle \leqslant \|z\|^2 \left(\frac{1+\|z\|}{1-\|z\|}\right)^{\alpha}, \quad z \in B. \tag{2.2}$$

This estimation is sharp.

Proof. Let $z \in B \setminus \{0\}$ be fixed and let $f : U \to \mathbf{C}$, be given by

$$f(\zeta) = \frac{1}{\zeta}\left\langle p\left(\zeta\frac{z}{\|z\|}\right), \frac{z}{\|z\|}\right\rangle, \quad \zeta \in U \setminus \{0\}$$

and

$$f(0) = \lim_{\zeta \to 0} f(\zeta).$$

Since $p \in H(B)$, $p(0) = 0$ and $Dp(0) = I$, then $f \in H(U)$ and $f(0) = 1$.

On the other hand, since

$$f(\zeta) = \frac{1}{\left\|\zeta\frac{z}{\|z\|}\right\|^2}\left\langle p\left(\zeta\frac{z}{\|z\|}\right), \zeta\frac{z}{\|z\|}\right\rangle, \quad \zeta \in U \setminus \{0\},$$

then

$$|\arg f(\zeta)| = \left|\arg\left\langle p\left(\zeta\frac{z}{\|z\|}\right), \zeta\frac{z}{\|z\|}\right\rangle\right| < \alpha\frac{\pi}{2}, \quad \zeta \in U\setminus\{0\},$$

and since $f(0) = 1$, then $|\arg f(\zeta)| < \alpha\frac{\pi}{2}$, $\zeta \in U$, hence $f(\zeta) \prec g(\zeta)$, where

$$g(\zeta) = \left(\frac{1+\zeta}{1-\zeta}\right)^{\alpha}$$

and by $\prec$ we mean the usual subordination. Then, there exists a Schwarz function ω ($\omega \in H(U)$, $\omega(0) = 0$, $\omega(U) \subset U$) such that $f(\zeta) = g(\omega(\zeta))$, $\zeta \in U$.

On the other hand, since $\alpha \in (0, 1]$, then g is a convex function and it is well known that the following relations hold:

$$\left(\frac{1-|\zeta|}{1+|\zeta|}\right)^{\alpha} \leqslant \operatorname{Re} g(\zeta) \leqslant \left(\frac{1+|\zeta|}{1-|\zeta|}\right)^{\alpha}, \quad \zeta \in U.$$

Therefore, we obtain the following relation

$$\left(\frac{1-|\omega(\zeta)|}{1+|\omega(\zeta)|}\right)^{\alpha} \leqslant \operatorname{Re} f(\zeta) \leqslant \left(\frac{1+|\omega(\zeta)|}{1-|\omega(\zeta)|}\right)^{\alpha}, \quad \zeta \in U,$$

and taking into account the maximum principle on the unit disc, we deduce the following inequality

$$\left(\frac{1-|\zeta|}{1+|\zeta|}\right)^{\alpha} \leqslant \operatorname{Re} f(\zeta) \leqslant \left(\frac{1+|\zeta|}{1-|\zeta|}\right)^{\alpha}, \quad \zeta \in U.$$

Now if we set $\zeta = \|z\|$ in the above inequality, we obtain the desired relation (2.2) for $z \in B\setminus\{0\}$.

On the other hand, the above inequality is also satiesfied for $z = 0$, therefore it remains to show that this relation is sharp.

For this aim, let $p_0 : B \to \mathbf{C}^n$,

$$p_0(z) = \left(z_1\left(\frac{1+z_1}{1-z_1}\right)^{\alpha}, \dots, z_n\left(\frac{1+z_n}{1-z_n}\right)^{\alpha}\right)', \quad z = (z_1, \dots, z_n)' \in B.$$

Then $p_0 \in H(B)$, $p_0(0) = 0$, $Dp_0(0) = I$ and

$$\langle p_0(z), z\rangle = \sum_{j=1}^{n} |z_j|^2 \left(\frac{1+z_j}{1-z_j}\right)^{\alpha}.$$

Let $z \in B\setminus\{0\}$ and let m be denote the number of nonzero components of z, then $1 \leqslant m \leqslant n$ and

$$\langle p_0(z), z\rangle = \sum_{\substack{j=1\\ z_j\neq 0}}^{n} |z_j|^2 \left(\frac{1+z_j}{1-z_j}\right)^{\alpha}.$$

Let $H = \{w \in \mathbf{C} : \ |\arg w| < \alpha\frac{\pi}{2}\}$, then H is a convex set in $\mathbf{C}$ and because $\left|\arg\left(\frac{1+z_j}{1-z_j}\right)^\alpha\right| < \alpha\frac{\pi}{2}$, for all $j \in \{1, \dots, n\}$, then it is clear that

$$\left|\arg\left[|z_j|^2\left(\frac{1+z_j}{1-z_j}\right)^\alpha\right]\right| < \alpha\frac{\pi}{2}, \quad j \in \{1, \dots, n\}, z_j \neq 0,$$

hence

$$w_j = |z_j|^2\left(\frac{1+z_j}{1-z_j}\right)^\alpha \in H,$$

for all $j \in \{1, \dots, n\}$, $z_j \neq 0$, so, taking into account the convexity of H, we deduce that

$$\frac{1}{m}\sum_{\substack{j=1 \\ z_j\neq 0}}^{n} w_j \in H, \text{ i.e. } |\arg\langle p_0(z), z\rangle| < \alpha\frac{\pi}{2}.$$

Now, let $z = (r, 0, \dots, 0)' \in B$, $r \in [0, 1)$, then

$$\text{Re}\,\langle p_0(z), z\rangle = r^2\left(\frac{1+r}{1-r}\right)^\alpha = \|z\|^2\left(\frac{1+\|z\|}{1-\|z\|}\right)^\alpha,$$

and for $z = (-r, 0, \dots, 0)'$, $r \in [0, 1)$,

$$\text{Re}\,\langle p_0(z), z\rangle = r^2\left(\frac{1-r}{1+r}\right)^\alpha = \|z\|^2\left(\frac{1-\|z\|}{1+\|z\|}\right)^\alpha.$$

Hence, the equalities are attained for some $z \in B$.

The proof is complete.

Next, by using the result of Lemma 2.1, we can give the following distortion theorem.

THEOREM 2.1. *Under the assumptions of Lemma 2.1, we have*

$$\left|\frac{1}{k!}\langle D^k p(0)(z, \dots, z), z\rangle\right| \leqslant 2\alpha\|z\|^{k+1}, \tag{2.3}$$

for all $z \in \mathbf{C}^n$ and $k \geqslant 2$. This estimation is sharp for $k = 2$.

Proof. It is obvious to see that the above inequalities hold for $z = 0$, hence it suffices to show our result for $z \in \mathbf{C}^n \setminus \{0\}$ and $k \in \mathbf{N}$, $k \geqslant 2$.

In the following we consider the same functions f and g as in the proof of Lemma 2.1. Then g is convex on U and $g'(0) = 2\alpha$.

Hence, from Lemma 1.3, we conclude that

$$\left|\frac{f^{(k)}(0)}{k!}\right| \leqslant 2\alpha, \text{ for all } k \geqslant 1. \tag{2.4}$$

It is obvious to see that f has the following Taylor expansion on the unit disc:

$$f(\zeta) = 1 + \left\langle D^2p(0)\left(\frac{z}{\|z\|}, \frac{z}{\|z\|}\right), \frac{z}{\|z\|}\right\rangle \frac{\zeta}{2!} + \cdots$$
$$+ \left\langle D^kp(0)\left(\frac{z}{\|z\|}, \ldots, \frac{z}{\|z\|}\right), \frac{z}{\|z\|}\right\rangle \frac{\zeta^{k-1}}{k!} + \cdots,$$

for $\zeta \in U$, hence,

$$f^{(k-1)}(0) = \frac{1}{k}\left\langle D^kp(0)\left(\frac{z}{\|z\|}, \ldots, \frac{z}{\|z\|}\right), \frac{z}{\|z\|}\right\rangle, \ k \geqslant 1.$$

Therefore, by using the relation (2.4) and the above equalities, we obtain the estimation (2.3). Since z was arbitrarily chosen, we conclude that the relations (2.3) hold for all $z \in \mathbf{C}^n$.

It remains to show that the estimation (2.3) is sharp in the case of $k = 2$.

To this end, we consider the same mapping $p_0 : B \to \mathbf{C}^n$, as in the proof of Lemma 2.1. From above, we see that $p_0 \in H(B)$, $p_0(0) = 0$, $Dp_0(0) = I$ and

$$|\arg\langle p_0(z), z\rangle| < \alpha\frac{\pi}{2}, \quad z \in B \setminus \{0\}.$$

On the other hand, since the linear operator $D^2p_0(0)(z, \cdot)$ has the following matrix representation

$$D^2p_0(0)(z, \cdot) = \left(\sum_{m=1}^{n} \frac{\partial^2 p_0^k(0)}{\partial z_j \partial z_m} z_m\right)_{1 \leqslant j,k \leqslant n},$$

where $p_0(z) = (p_0^1(z), ..., p_0^n(z))'$, $z \in B$, then after a straightforward calculation, we obtain

$$D^2p_0(0)(z, z) = 4\alpha(z_1^2, \ldots, z_n^2)', \quad z = (z_1, \ldots, z_n)' \in \mathbf{C}^n,$$

thus,

$$\langle D^2p_0(0)(z, z), z\rangle = 4\alpha \sum_{j=1}^{n} |z_j|^2 z_j.$$

Now, if we let $z = (r, 0, \ldots, 0)' \in \mathbf{C}^n$, where $r \geqslant 0$, then

$$|\langle D^2p_0(0)(z, z), z\rangle| = 4\alpha r^3 = 4\alpha\|z\|^3,$$

thus, if $k = 2$, the estimation (2.3) is sharp.

The proof is complete.

Remark 2.1. In the case of $\alpha = 1$, the result of Theorem 2.1, was recently obtained by the author [6].

The main result of our paper is the following growth theorem for strongly starlike mappings of order $\alpha \in (0, 1]$.

THEOREM 2.2. *If $f : B \to \mathbf{C}^n$ is strongly starlike of order $\alpha \in (0, 1]$ on B, then*

$$\|z\| \exp \int_0^{\|z\|} \left[\left(\frac{1-t}{1+t} \right)^{\alpha} - 1 \right] \frac{dt}{t} \leqslant \|f(z)\| \leqslant \|z\| \exp \int_0^{\|z\|} \left[\left(\frac{1+t}{1-t} \right)^{\alpha} - 1 \right] \frac{dt}{t}, \tag{2.5}$$

for all $z \in B$ and this estimation is sharp.

Proof. We use in the proof similar reasons as in [1].

Since f is strongly starlike of order $\alpha \in (0, 1]$ on B, then f is also starlike. Hence, if $0 < r < 1$ and $\|z_1\| = r$ with $\|f(z_1)\| = \max\{\|f(z)\| : \|z\| = r\}$, then there exists a ray from zero to $f(z_1)$, contained on $f(\overline{B}_r)$, so the preimage of this ray is on $\overline{B}_r$.

We denote by $z(s)$ this curve, parametrized by arc length. Also, let $p(z) = [Df(z)]^{-1} f(z)$, $z \in B$, then $p \in H(B)$, $p(0) = 0$, $Dp(0) = I$ and using the hypothesis, we deduce that p satisfies the assumptions of Lemma 2.1, therefore the inequality (2.2) holds.

Because $\operatorname{Re} \langle p(z), z \rangle = \|p(z)\| \cdot \|z\| \cos\theta$, where θ denotes the angle between the vectors z and $p(z)$, then from (2.2) we deduce that

$$\frac{\cos\theta}{\|z\|} \left(\frac{1+\|z\|}{1-\|z\|} \right)^{\alpha} \geqslant \frac{1}{\|p(z)\|} \geqslant \frac{\cos\theta}{\|z\|} \left(\frac{1-\|z\|}{1+\|z\|} \right)^{\alpha}, \ z \in B \setminus \{0\}. \tag{2.6}$$

On the other hand, from [1, p.17], the following equation holds

$$\frac{df(z(s))}{ds} = \frac{1}{\|p(s)\|} f(z(s)).$$

If $g(s) = \|f(z(s))\|^2$, then

$$\frac{dg}{ds} = 2\operatorname{Re} \left\langle \frac{df(z(s))}{ds}, f(z(s)) \right\rangle = \frac{2g(s)}{\|p(s)\|},$$

hence,

$$\frac{dg}{g} = \frac{2ds}{\|p(s)\|},$$

and in view of (2.6), we obtain

$$\frac{2\cos\theta(s)}{\|z(s)\|} \left(\frac{1+\|z(s)\|}{1-\|z(s)\|} \right)^{\alpha} \geqslant \frac{d\log g}{ds} \geqslant \frac{2\cos\theta(s)}{\|z(s)\|} \left(\frac{1-\|z(s)\|}{1+\|z(s)\|} \right)^{\alpha}, \tag{2.7}$$

where $\theta(s)$ means the angle between $z(s)$ and $\frac{dz(s)}{ds}$.

It is obvious to see that $\frac{d\|z(s)\|}{ds} = \cos\theta(s)$, provided $z(s)$ is not at the origin.

Integrating in the both sides of (2.7) from s_0 to s_1, and taking into account the definition of $g(s)$, we deduce

$$\|f(z(s_0))\| \exp \int_{\|z(s_0)\|}^{\|z(s_1)\|} \left(\frac{1+t}{1-t} \right)^{\alpha} \frac{dt}{t} \geqslant \|f(z(s_1))\|$$

$$\geqslant \|f(z(s_0))\| \exp \int_{\|z(s_0)\|}^{\|z(s_1)\|} \left(\frac{1-t}{1+t} \right)^{\alpha} \frac{dt}{t}.$$

Since $f(0) = 0$ and $Df(0) = I$, if we let s_0 be a small positive ε, then $\|z(s_0)\| = \varepsilon + o(\varepsilon)$ and $\|f(z(s_0))\|^2 = \varepsilon^2 + o(\varepsilon^2)$, hence from the above inequality, we obtain

$$\sqrt{\varepsilon + o(\varepsilon)} \exp \int_{\varepsilon+o(\varepsilon)}^{\|z(s_1)\|} \left(\frac{1+t}{1-t}\right)^{\alpha} \frac{dt}{t} \geqslant \|f(z(s_1))\| \geqslant \tag{2.8}$$

$$\geqslant (\varepsilon + o(\varepsilon)) \exp \int_{\varepsilon+o(\varepsilon)}^{\|z(s_1)\|} \left(\frac{1-t}{1+t}\right)^{\alpha} \frac{dt}{t}.$$

On the other hand, since

$$\frac{1}{z}\left(\frac{1+z}{1-z}\right)^{\alpha} = \frac{1}{z} + q(z), \quad z \in U \setminus \{0\},$$

where q is holomorphic on U, with $q(0) = 2\alpha$ and also,

$$\frac{1}{z}\left(\frac{1-z}{1+z}\right)^{\alpha} = \frac{1}{z} + r(z), \quad z \in U \setminus \{0\},$$

where r is holomorphic on U, with $r(0) = -2\alpha$, then, letting $\varepsilon \to 0$ into (2.8) and taking into account these remarks, we deduce that

$$\|z(s_1)\| \exp \int_0^{\|z(s_1)\|} q(t)dt \geqslant \|f(z(s_1))\| \geqslant \|z(s_1)\| \exp \int_0^{\|z(s_1)\|} r(t)dt.$$

Picking s_1 so that $z(s_1) = z_1 = z$, we obtain the relation (2.5).

It remains to show that our estimation is sharp.

For this aim, we consider the following mapping $f_0 : B \to \mathbf{C}^n$, given by

$$f_0(z) = \left(z_1 \exp \int_0^{z_1} \left[\left(\frac{1+t}{1-t}\right)^{\alpha} - 1\right] \frac{dt}{t}, \ldots, z_n \exp \int_0^{z_n} \left[\left(\frac{1+t}{1-t}\right)^{\alpha} - 1\right] \frac{dt}{t}\right)',$$

for all $z = (z_1, \ldots, z_n)' \in B$.

Then, it is easy to see that f_0 is locally biholomorphic on B, $f_0(0) = 0$, $Df_0(0) = I$ and after simple computations we obtain

$$\langle [Df_0(z)]^{-1} f_0(z), z\rangle = \sum_{j=1}^{n} |z_j|^2 \left(\frac{1-z_j}{1+z_j}\right)^{\alpha},$$

for all $z = (z_1, \ldots, z_n)' \in B$.

Furthermore, it suffices to use same kind of arguments as in the proof of Lemma 2.1, to show that f_0 is strongly starlike of order α.

On the other hand, let $z = (r, 0, \ldots, 0)' \in B$, where $r \in [0, 1)$, then $\|z\| = r$ and

$$f_0(z) = \left(r \exp \int_0^r \left[\left(\frac{1+t}{1-t}\right)^{\alpha} - 1\right] \frac{dt}{t}, 0 \cdots, 0\right)',$$

hence,

$$\|f_0(z)\| = r\exp\int_0^r \left[\left(\frac{1+t}{1-t}\right)^\alpha - 1\right]\frac{dt}{t} = \|z\|\exp\int_0^{\|z\|} \left[\left(\frac{1+t}{1-t}\right)^\alpha - 1\right]\frac{dt}{t}.$$

Also, for $z = (-r, 0, \dots, 0)' \in B$, where $r \in [0, 1)$, then

$$f_0(z) = \left(-r\exp\int_0^{-r} \left[\left(\frac{1+t}{1-t}\right)^\alpha - 1\right]\frac{dt}{t}, 0, \cdots, 0\right)',$$

therefore,

$$\|f_0(z)\| = r\exp\int_0^r \left[\left(\frac{1-x}{1+x}\right)^\alpha - 1\right]\frac{dx}{x} = \|z\|\exp\int_0^{\|z\|} \left[\left(\frac{1-t}{1+t}\right)^\alpha - 1\right]\frac{dt}{t}.$$

Hence the equalities are attained for some $z \in B$.
The proof is complete.

A direct consequence of the above result is the following covering theorem.

COROLLARY 2.1. *If $f : B \to \mathbf{C}^n$ is strongly starlike of order $\alpha \in (0, 1]$, then $f(B)$ contains the ball of radius $\rho = \rho(\alpha)$ and centered at zero, where*

$$\rho(\alpha) = \exp\int_0^1 \left[\left(\frac{1-t}{1+t}\right)^\alpha - 1\right]\frac{dt}{t}.$$

An interesting result can be obtained for $\alpha = \frac{1}{2}$ in Theorem 2.2 and Corollary 2.1, respectively.

In this case, a straightforward calculation yields the following estimation.

COROLLARY 2.2. *If $f : B \to \mathbf{C}^n$ is strongly starlike of order $\frac{1}{2}$ on B, then*

$$\frac{2\|z\|}{1+\sqrt{1-\|z\|^2}}\exp\left[2\arctan\sqrt{\frac{1-\|z\|}{1+\|z\|}} - \frac{\pi}{2}\right] \leqslant \|f(z)\| \leqslant$$

$$\leqslant \frac{2\|z\|}{1+\sqrt{1-\|z\|^2}}\exp\left[2\arctan\sqrt{\frac{1+\|z\|}{1-\|z\|}} - \frac{\pi}{2}\right],$$

for all $z \in B$ and $f(B) \supset B_{\rho(\frac{1}{2})}$, where

$$\rho\left(\frac{1}{2}\right) = 2\exp\left(-\frac{\pi}{2}\right).$$

The result is sharp.

THEOREM 2.3. *If $f : B \to \mathbf{C}^n$ is strongly starlike of order $\alpha \in (0, 1]$ on B, then*

$$\|[Df(z)]^{-1}\| \geqslant \left(\frac{1-\|z\|}{1+\|z\|}\right)^{\alpha} \exp\int_0^{\|z\|}\left[1-\left(\frac{1+t}{1-t}\right)^{\alpha}\right]\frac{dt}{t}, \quad z \in B.$$

Proof. Since f is strongly starlike of order $\alpha \in (0, 1]$, then from Lemma 2.1, we deduce that

$$\operatorname{Re}\langle [Df(z)]^{-1}f(z), z\rangle \geqslant \|z\|^2\left(\frac{1-\|z\|}{1+\|z\|}\right)^{\alpha}, \quad z \in B.$$

On the other hand, by using the properties of the linear operator norm, we obtain

$$\operatorname{Re}\langle [Df(z)]^{-1}f(z), z\rangle \leqslant \|[Df(z)]^{-1}\| \cdot \|f(z)\| \cdot \|z\|,$$

and combining the above relations together with the inequality (2.5), we obtain our conclusion.

Remark 2.2. For $\alpha = 1$ in Theorem 2.2, we obtain the well known covering and growth result for starlike mappings on the unit ball of $\mathbf{C}^n$.

COROLLARY 2.3. [1], [8]. *If $f : B \to \mathbf{C}^n$ is starlike on B, normalized by $f(0) = 0$ and $Df(0) = I$, then*

$$\frac{\|z\|}{(1+\|z\|)^2} \leqslant \|f(z)\| \leqslant \frac{\|z\|}{(1-\|z\|)^2}, \quad z \in B$$

and $f(B) \supset B_{\frac{1}{4}}$.

The result is sharp.

A particular interest for us is the case of $\alpha = \frac{1}{2}$. We give a distortion result for mappings which are strongly starlike of order $\frac{1}{2}$ on B.

THEOREM 2.4. *If $f : B \to \mathbf{C}^n$ is strongly starlike of order $\frac{1}{2}$ on B, then*

$$|\langle D^2f(0)(z, z), z\rangle| \leqslant 2\|z\|^3, \quad z \in \mathbf{C}^n.$$

This estimation is sharp.

Also, for $k \geqslant 2$, and $z \in \mathbf{C}^n$, $\|z\| = 1$, the following estimation holds

$$\left\|\frac{1}{k!}D^kf(0)(z, \ldots, z)\right\| \leqslant \tag{2.9}$$

$$\leqslant 2\exp\left\{\frac{1}{2(k-1)} + 2\arctan\sqrt{\frac{k-1+\sqrt{(k-1)^2+1}}{1-k+\sqrt{(k-1)^2+1}}} - \frac{\pi}{2}\right\}.$$

Proof. Since f is strongly starlike of order $\frac{1}{2}$ on B, then

$$|\arg\langle [Df(z)]^{-1}f(z), z\rangle| < \frac{\pi}{4}, \quad z \in B \setminus \{0\}.$$

Let $p : B \to \mathbf{C}^n$, $p(z) = [Df(z)]^{-1}f(z)$, $z \in B$, then $p \in H(B)$, $p(0) = 0$ and $Dp(0) = I$. Also, $|\arg\langle p(z), z\rangle| < \frac{\pi}{4}$, $z \in B \setminus \{0\}$, hence, from Theorem 2.1, we deduce that

$$|\langle D^2p(0)(w, w), w\rangle| \leqslant 2\|w\|^3, \quad w \in \mathbf{C}^n. \tag{2.10}$$

On the other hand, using the Taylor expansion of f and p on B, we have

$$f(z) = z + \frac{1}{2}D^2f(0)(z, z) + \cdots + \frac{1}{k!}D^kf(0)(z, \ldots, z) + \cdots$$

and

$$p(z) = z + \frac{1}{2}D^2p(0)(z, z) + \cdots + \frac{1}{k!}D^kp(0)(z, \ldots, z) + \cdots,$$

for all $z \in B$. Since $f(z) = Df(z)p(z)$, $z \in B$, using the above relations and identifying the coefficients of second order, we deduce the following relation

$$D^2f(0)(z, z) = -D^2p(0)(z, z), \quad z \in B.$$

From this it is clear that the following equality holds

$$D^2f(0)(z, z) = -D^2p(0)(z, z),\ z \in \mathbf{C}^n.$$

Therefore, from (2.10) and the above equality we conclude that

$$|\langle D^2f(0)(z, z), z\rangle| \leqslant 2\|z\|^3, \quad z \in \mathbf{C}^n.$$

To prove this inequality is sharp it is sufficient to consider the following strongly starlike mapping of order $1/2$:

$$f(z) = \left(z_1 \exp\int_0^{z_1}\left[\left(\frac{1+t}{1-t}\right)^{1/2} - 1\right]\frac{dt}{t}, \ldots, z_n \exp\int_0^{z_n}\left[\left(\frac{1+t}{1-t}\right)^{1/2} - 1\right]\frac{dt}{t}\right)',$$

for all $z = (z_1, \ldots, z_n)' \in B$.

By using the proof of Theorem 2.2, we conclude that f is starlike of order $\frac{1}{2}$ on B and by a straightforward calculation, we obtain

$$\langle D^2f(0)(z, z), z\rangle = 2\sum_{j=1}^{n}|z_j|^2 z_j, \text{ for all } z = (z_1, \ldots, z_n)' \in \mathbf{C}^n.$$

Hence, for $z = (r, 0, \ldots, 0)' \in \mathbf{C}^n$, where $r \geqslant 0$, then

$$|\langle D^2f(0)(z, z), z\rangle| = 2r^3 = 2\|z\|^3,$$

that means our estimation is sharp.

Next, let $z \in \mathbf{C}^n$, $\|z\| = 1$. The inequality (2.9) follows from Cauchy's estimations:

$$\frac{1}{k!}D^kf(0)(z, \ldots, z) = \int\limits_{|\zeta|=r}\frac{f(\zeta z)}{\zeta^{k+1}}d\zeta,\ 0 < r < 1.$$

Taking into account the result of Corollary 2.2 and the above equality, we obtain

$$\|\frac{1}{k!}D^k f(0)(z,\ldots,z)\| \leqslant \frac{2}{r^{k-1}} \exp\left[2\arctan\sqrt{\frac{1+r}{1-r}} - \frac{\pi}{2}\right].$$

Since this inequality holds for all $r \in (0, 1)$, it follows that

$$\|\frac{1}{k!}D^k f(0)(z,\ldots,z)\| \leqslant 2 \min_{0<r<1}\left\{\frac{1}{r^{k-1}} \exp\left[2\arctan\sqrt{\frac{1+r}{1-r}} - \frac{\pi}{2}\right]\right\}.$$

A straightforward calculation yields

$$\min_{0<r<1}\left\{\frac{1}{r^{k-1}} \exp\left[2\arctan\sqrt{\frac{1+r}{1-r}} - \frac{\pi}{2}\right]\right\} \leqslant$$

$$\leqslant \exp\left\{\frac{1}{2(k-1)} + 2\arctan\sqrt{\frac{\sqrt{(k-1)^2+1}+(k-1)}{\sqrt{(k-1)^2+1}-(k-1)}} - \frac{\pi}{2}\right\},$$

for all $k \geqslant 2$, in consequence we obtain the desired formula (2.9).

This completes the proof.

We close this paper with the following connection between strongly starlikeness of order $\alpha \in (0, 1]$ and spirallikeness in $\mathbf{C}^n$. For $n = 1$ see the result of J. Stankiewicz [11].

THEOREM 2.5. *Let $\alpha \in (0, 1]$ and $f : B \to \mathbf{C}^n$ be a locally biholomorphic mapping on B, normalized by $f(0) = 0$ and $Df(0) = I$. Then f is strongly starlike of order α if and only if f is spirallike relative to $A = e^{i\beta}I$, for all $\beta \in \mathbf{R}$, $|\beta| \leqslant (1-\alpha)\frac{\pi}{2}$.*

Proof. First, we suppose that f is strongly starlike of order $\alpha \in (0, 1]$. Let β be a real number, with $|\beta| \leqslant (1-\alpha)\frac{\pi}{2}$ and let $A = e^{i\beta}I$. Then it is obvious to see that $m(A) > 0$. In view of Lemma 1.3 it suffices to show that

$$|\arg\langle[Df(z)]^{-1}Af(z), z\rangle| < \frac{\pi}{2}, \quad z \in B\setminus\{0\},$$

i.e.

$$|\beta + \arg\langle[Df(z)]^{-1}f(z), z\rangle| < \frac{\pi}{2}, \quad z \in B\setminus\{0\}.$$

Since $|\beta| \leqslant (1-\alpha)\frac{\pi}{2}$ and f is strongly starlike of order α, then

$$|\beta + \arg\langle[Df(z)]^{-1}f(z), z\rangle| < |\beta| + \alpha\frac{\pi}{2} < \frac{\pi}{2}, \quad z \in B\setminus\{0\}.$$

Thus f is spirallike relative to $A = e^{i\beta}I$, for all $\beta \in \mathbf{R}$, $|\beta| \leqslant (1-\alpha)\frac{\pi}{2}$.

Conversely, suppose that f is spirallike relative to $A = e^{i\beta}I$, for all $\beta \in \mathbf{R}$, $|\beta| \leqslant (1-\alpha)\frac{\pi}{2}$. It is obvious to see that for $\beta = (1-\alpha)\frac{\pi}{2}$ and then $\beta = -(1-\alpha)\frac{\pi}{2}$, respectively, we obtain the following inequalities

$$(1-\alpha)\frac{\pi}{2} + \arg\langle[Df(z)]^{-1}f(z), z\rangle < \frac{\pi}{2},$$

and

$$-(1-\alpha)\frac{\pi}{2} + \arg\langle [Df(z)]^{-1}f(z), z\rangle > -\frac{\pi}{2},$$

for all $z \in B \setminus \{0\}$, hence

$$|\arg\langle [Df(z)]^{-1}f(z), z\rangle| < \frac{\alpha\pi}{2}, \quad z \in B \setminus \{0\},$$

too, which completes the proof.

In the following let S^*_α denote the class of strongly starlike mappings of order $\alpha \in (0, 1]$ on B and also, let $\check{S}_\beta$ the class of spirallike mappings relative to $A = e^{i\beta}I$, where $\beta \in \mathbf{R}$, $|\beta| \leqslant (1-\alpha)\frac{\pi}{2}$, then, in view of the above result, we obtain

Remark 2.3. For every $\alpha \in (0, 1]$, the following relation holds:

$$S^*_\alpha = \bigcap_{|\beta| \leqslant (1-\alpha)\frac{\pi}{2}} \check{S}_\beta$$

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Babeş-Bolyai University
Faculty of Mathematics
1, M. Kogălniceanu Str.
3400 Cluj-Napoca
Romania
e-mail:gkohr@math.ubbcluj.ro

Technical University of Lodz
Institute of Mathematics
Al. Politechniki 11
90-924 Lodz
Poland
e-mail:piliczb@lodz1.p.lodz.pl