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FUNCTIONAL DIFFERENTIAL INCLUSIONS INVOLVING DISSIPATIVE AND COMPACT MULTIFUNCTIONS

Tzanko Donchev, Sofia, Bulgaria

Abstract. This paper is a natural exstension of [7]. We examine the main qualitative properties of the solution set of differential inclusion with the lag. The right-hand side is supposed to be one side Lipschitz and with almost continuous convex hull. Afterwards we prove the existence of solutions when the right-hand side is a sum of one side Lipschitz and almost (upper) semicontinuous multifunction, which satisfy compactness condition.

1. Preliminaries

Let E be a Banach space. We consider the following differential inclusions with delay:

$$\dot{x}(t) \in F(t, x_t), \quad x_0 = \phi. \tag{1}$$

Here $x \in E$, $t \in I = [0,1]$ and F is a compact valued map from $I \times X$ into E. $X = C([-\tau, 0])$ is the space of the continuous maps from $[-\tau, 0]$ into E and $x_t(s) = x(t-s)$ for $s \in [-\tau, 0]$. We replace the compactness or Lipschitz condition usually used in litereture by one side Lipschitz one. In the next section we extend in a natural (but not trivial) way the main results of [7]. Namely we prove the existence of solutions generalizing the corresponding result of [7, 8]. We also show that the solution set of (1) is dense in the solution set of

$$\dot{x}(t) \in \overline{co}F(t, x_t), \quad x_0 = \phi. \tag{2}$$

So we generalise the corresponding results of [11, 14]. We also show that the solution set of (2) is R_δ set. In the last section we prove the existence of solution for the system:

$$\dot{x}(t) \in F(t, x_t) + R(t, x_t), \quad x_0 = \phi. \tag{3}$$

Here $F(\cdot,\cdot)$ is one side Lipschitz and $R(\cdot,\cdot)$ satisfies compactness type hypotheses. In that case we extend the main result of [13] presented for differential equations. We also improve Theorem 4 of [2], where F is single valued.

All the concepts not discussed in details can be found in [5]. Let E^* be a dual space of E. We let $P_f(E) = \{A \subset E;\ \text{nonempty, compact}\}$. For $A \subset E$ by A, $(co\,A)$ denote the closed (convex) hull of A. If $x \in E, A \subset E$ then $d(x, A) := \inf_{a \in A} |x - a|$,

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$\rho(C,A) = \sup_{c\in C} d(c,A)$ and $D_H(A,B) = \max\{\rho(A,B),\rho(B,A)\}$ – the Hausdorff distance. When $x,y \in E$ define $[x,y]_+ := \lim\limits_{h\to 0^+} h^{-1}\{|x+hy|-|x|\}$ the right derivative of $|x|$ in direction y. The map $[\cdot,\cdot]_+$ is upper semicontinuous as a function from $E\times E$ into R and moreover $|[x,y]_+ - [x,z]_+| \leqslant |y-z|$. (see [10] p.7). We will consider continuous multifunctions $F : M \to P_f(E)$, when the later is provided with the Hausdorff metrics and M is a metric space (commonly $M = I\times X$). Let U be the united ball in X centered in origin.

Definition 1. The multifunction $F : M \to P_f(E)$ is called lower semicontinuous (LSC) at x when for every open set $V \cap F(x) \neq \emptyset$ there exists a neighbourhood $A \ni x$ such that $F(y)\cap V \neq \emptyset$ for all $y \in A$. F is almost LSC (ALSC) when to every $\varepsilon > 0$ there exists a compact I_ε with $meas(I\setminus I_\varepsilon) < \varepsilon$, such that F is LSC on $I_\varepsilon \times E$. (When F is continuous on $I_\varepsilon\times E$ it is called almost continuous). The multifunction F is said to be upper semicontinuous (USC) at x, when to $\varepsilon > 0$ there exists $\delta > 0$ such that $\rho(F(x),F(y)) < \varepsilon$, when $|x-y| < \delta$.

In the paper we will use also the Hausdorff (ball) measure of noncompactness.

$$\beta(B) = \inf\{r > 0 : B \text{ can be covered by finitely many balls of radius } \leqslant r\}.$$

If $B \subset \Omega \subset X$ and centers are chosen from Ω instead of X, then we write $\beta_\Omega(B)$ and have $\beta(B) \leqslant \beta_\Omega(B) \leqslant 2\beta(B)$.

Recall that the subset B of a metric space is said to be R_δ, when it is an intersection of a decreasing sequence of compact absolute retracts B_n. The set $A \subset X$ is called contractible if there exists a continuous $h : [0,1]\times A \to A$ and $x_* \in A$ such that $h(0,x) = x$ and $h(1,x) = x_*$ on A. From Proposition 5.1 of [1] we know that B is R_δ set iff it is an intersection of decreasing sequence of closed contractible B_n with $\lim\limits_{n\to\infty}\beta(B_n) = 0$.

We consider differential inclusions (1) under the following assumptions.

A1. $F : I\times X \to P_f(E)$ and $\overline{co}F(\cdot,\cdot)$ is almost continuous.

A2. There exists an integrable $L(\cdot)$ such that $|F(t,\alpha)| \leqslant L(t)\{1+|\alpha|\}$.

Denote $X_0 = \{\alpha \in X : \max|\alpha(s)| = |\alpha(0)|, s \in [-\tau_1, 0]$.

A3. There exists a Kamke function u such that for every $\alpha - \beta \in X_\alpha$ we have

If $f_x \in F(t,\alpha)$ then there exists $f_y \in F(t,\beta)$ such that

$$[\alpha(0)-\beta(0), f_x - f_y]_+ \leqslant u(t, |\alpha-\beta|).$$

Recall that the almost continuous function $u : I\times R^+ \to R$ is said to be Kamke function when it is bounded on bounded sets $u(t,0)\equiv 0$ and the unique solution of $\dot{s}(t) = u(t,s)$, $s(0) = 0$ is $s(t)\equiv 0$.

Remark 1. One can suppose that $L(t)\equiv 1$ preserving A1–A3.

Indeed the map $t \to \int_0^t L(s)\,ds$ is continuous and strictly increasing. Let $\theta(\cdot)$ be its inverse. Define $\widetilde{F}(t,\alpha) = \dfrac{1}{L(\theta(t))}F(\theta(t),\alpha)$ for $(t,\alpha) \in \tilde{I}\times X$, where $\tilde{I} = [0, \int_I L(t)\,dt]$. It is easy to see that $\dot{x} \in F(t,x_t)$ iff $\dot{y}(t) \in \widetilde{F}(t,y_t)$ for $y = x(\theta(t))$.

Furthermore it is not difficult also to see that $\tilde{u}(t,r) = \dfrac{1}{L(\theta(t))}u(\theta(t),r)$ is a Kamke function iff $u(t,r)$ is.

Therefore if $\dot{x}(t) \in \overline{co}F(t, x_t + U) + f(t)U$, then $\dfrac{d}{dt}|x(t)| \leqslant 2 + f(t) + |x|$. Consequently $|x(t)| \leqslant e^t(|x_0| + 2 + \int_0^t f(s)\,ds)$. Here $\int_0^1 f(t)\,dt \leqslant 1$ and $f(t) \geqslant 0$. One can suppose that $F(\cdot,\cdot)$ is bounded by a constant M since A2 and the fact that we will consider only $x(\cdot)$ satisfying the conditions above. We suppose also that $|u(t,s)| \leqslant M$, since it is bounded on the bounded sets.

Definition 2. The absolutely continuous (AC) function $x : I \to E$ is said to be

a) ε-solution when $d(\dot{x}(s), F(s, x_s)) < \varepsilon + \lambda_\varepsilon(t)$ for a.e. $s \in I$, where $\int_0^1 \lambda_\varepsilon(t)\,dt \leqslant \varepsilon$ and $2M \geqslant \lambda_\varepsilon(t) \geqslant 0$.

b) quasitrajectory when there exists a sequence $\{x^i(\cdot)\}_{i=1}^\infty$ such that $|x^i(t) - x(t)| \to 0$ uniformly on I and $d(\dot{x}^i(t), F(t, x_t^i)) \to 0$ for a.e. $t \in I$.

c) polygonal when there exists a countable family of semiopen intervals $I_q \subset I$ such that $[0,1) = \bigcup_{q=1}^{\infty} I_q$, $I_p \cap I_q = \emptyset$ if $q \neq p$ and $\dot{x}(\cdot)$ is a constant on every I_q.

PROPOSITION 1. *For every $\delta > 0$ there exists a decreasing sequence of positive numbers $\{\varepsilon_i\}_{i=1}^\infty$ such that $\sum_{i=0}^{\infty} r_i(t) \leqslant \delta$, where $r_i(\cdot)$ are AC with $\dot{r}_i(t) \leqslant u(t, r_i(t)) + 4(\varepsilon_i + \lambda_i(t))$, $r_i(0) = 0$. Here $0 \leqslant \lambda_i(t) \leqslant 2M$ for a.e. t and $\int_0^1 \lambda_i(t)\,dt \leqslant \varepsilon_i$.*

The proof is omitted since is very similar of the one of Lemma 3 of [6].

LEMMA 1. *Denote $G(t,\alpha) = \overline{co}F(t,\alpha)$. Under A1, A2 for every $\varepsilon > 0$ there exists a finite subdivision $\Delta = \{t_i\}_{i=1}^N$ such that if $x(\cdot)$ is AC function with $x_0 = \phi$ and $\dot{x}(t) \in G(t_i, x_i)$ then $D_H(G(t, x_t), G(t_i, x_i)) \leqslant \varepsilon + \lambda_\varepsilon(t)$ for a.e. $t \in [t_i, t_{i+1}]$. Here $i = 0, 1, \ldots, N$ and $x_i = x_{t_i}$, $0 \leqslant \lambda_\varepsilon(t) \leqslant 2M$ and $\int_I \lambda_\varepsilon(t)\,dt \leqslant \varepsilon$. Furthermore the quasitrajectory sets of (1) and (2) coincide.*

This lemma is proved in [7] for continuous $G(\cdot,\cdot)$ and the problem without time lag (see Lemma 1 and 3 in [7]). The proof given there holds also in our case except for trivial modifications.

2. The main qualitative properties

In this section we prove that the solution set of (2) can be approximated by the solution set of the discrete inclusion:

$$\dot{x}(t) \in \overline{co}F(t, x_i), \quad t \in [\tau_i, \tau_{i+1}],\ x_0 = \phi,\ x_i = \lim_{t \to \tau_i^-} x_t \tag{4}$$

where x_i denotes x_{τ_i}, $i = 1, \ldots, N$. Since F is compact valued the last result implies that the solution set of (2) is $C(I, E)$ compact. So we can use similar approach as in case of compactness type conditions.

THEOREM 1. *Denote by R_N the solution set of (4), when $\Delta_N = 0 = t_0 < \ldots < t_N = 1$ and by R_{RP} the solution set of (2). Then under A1 - A3 there exists a sequence Δ_N such that* $\lim_{N\to\infty} D_H(R_{RP}, R_N) = 0$.

Proof. Fix $\varepsilon > 0$. Let $D_H(G(t,x_t), G(t_i,x_i)) \leqslant \varepsilon + \lambda_\varepsilon(t)$, for every $t \in [t_i, t_{i+1})$ and every solution $x(\cdot)$ of (4). We claim that for every solution $y(\cdot)$ of (2) there exists a solution $z(\cdot)$ of (4) such that $|y(t) - z(t)| \leqslant r(t)$, where $\dot{r}(t) \leqslant u(t,r) + \varepsilon + \lambda_\varepsilon(t)$, $r(0) = 0$. We are going to prove the claim. Suppose first the needed $z(\cdot)$ exists on $[0, t_i]$. Given $\mu > 0$ we consider the multifunction

$$\Gamma_\mu(t,\alpha) = \begin{cases} G(t,\alpha) & \alpha - x_t \notin X_0 \\ \{v \in G(t_i, z_i) : |v - \dot{y}(t)| = d(\dot{y}(t), G(t_i, z_i))\} & \alpha = y_t \\ \mathrm{cl}\{v \in G(t_i, z_i) : [y(t) - \alpha(0), \dot{y}(t) - v]_+ & \\ \qquad < u(t, |y(t) - \alpha(0)|) + D_H(G(t_i, z_i), G(t,\alpha)) + \mu\} & \textit{elsewhere.} \end{cases}$$

Now it is not difficult to show that $\Gamma(\cdot,\cdot)$ is nonempty compact valued and almost lower semicontinuous. Therefore the differential inclusion

$$\dot{u}(t) \in \Gamma_\mu(t, u_t); \quad u_{t_i} = z_i$$

has a solution $z_\mu(\cdot)$ defined on $[t_i, t_{i+1}]$. Thus $[z_\mu(t) - y(t),\ \dot{z}_\mu(t) - y(t)]_+ \leqslant u(t, |z_\mu(t) - y(t)|) + \mu + \varepsilon + \lambda_\varepsilon(t)$. Consequently $|y(t) - z_\mu(t)| \leqslant r_\mu(t)$, where $\dot{r}_\mu(t) \leqslant u(t, r_\mu) + \varepsilon + \mu + \lambda_\varepsilon(t)$.

Let $\mu \to 0$. Since $\{z_\mu(\cdot)\}_{\mu>0}$ is a compact net there exists a density point $z(\cdot)$ as $\mu \to 0$. It is easy to show that $z(\cdot)$ is the needed solution of (4), satisfying the claim condition. By induction one can show that this $z(\cdot)$ exists also on $[0,1]$.

Let $z(\cdot)$ be AC with $d(\dot{z}(t), G(t,z_t)) < \varepsilon + \lambda_\varepsilon(t)$. It is not difficult to show that to $\delta > 0$ there exists a polygonal solution $y(\cdot)$ such that $|z(t) - y(t)| \leqslant \delta$ and $d(\dot{y}(t), G(t,y_t)) \leqslant \varepsilon + \lambda_\varepsilon(t) + \delta + \lambda_\delta(t)$,

Given $\varepsilon > 0$ and $\delta > 0$. We claim that for every polygonal $z(\cdot)$ with $d(\dot{z}(t), G(t,z_t)) \leqslant \varepsilon + \lambda_\varepsilon(t)$ there exists a polygonal $y(\cdot)$ with $d(\dot{y}(t), G(t,y_t)) \leqslant \delta + \lambda_\delta(t)$ such that $|z(t) - y(t)| \leqslant r(t)$. Furthemore $\dot{r}(t) = u(t, r(t)) + 2(\varepsilon + \delta + \lambda_\varepsilon(t) + \lambda_\delta(t))$ and $r(0) = |z_0 - y_0|$.

Let $\mu > 0$ be arbitrary and let $u(\cdot,\cdot)$ be continuous on $I_\mu \times \mathbf{R}^+$. where $meas(I_\mu) > 1 - \mu$. If such $y(\cdot)$ exists on $[0,T)$ with $T < 1$, then y exists also on $[0,T]$ since is Lipschitz. Furthemore $T \in [\kappa, \nu)$, which are successive points of the corresponding to $z(\cdot)$ subdivision, i.e. $z(t) = z(\kappa) + (t - \kappa)\dot{z}(\kappa)$ on this interval. If $z_T - y_T \in X_0$ we choose $f \in G(T, y_T)$ such that $[z(T) - y(T), \dot{z}(T) - f]_+ \leqslant d\big(\dot{z}(t), G(T, z(T))\big) + u\big(T, |z(T) - y(T)|\big)$. If $z_T - y_T \notin X_0$ we choose $f \in G(T, y_T)$ arbitrary. Consequently there exists $\tau > T$ such that setting $y(t) = y(T) + (t-T)f$ one has $[z(t) - y(t), \dot{z}(t) - \dot{y}(t)]_+ \leqslant u\big(t, |z(t) - y(t)|\big) + \frac{3}{2}\big(\varepsilon + \delta + \mu + \lambda_\mu(t) + \lambda_\varepsilon(t) + \lambda_\delta(t)\big)$ on $[T,\tau) \cap I_\mu$ or $z_t - y_y \notin X_0$ on $[T,\tau)$. Since μ is arbitrary one can suppose that $\dot{r}(t) = u\big(t, r(t)\big) + 2\big(\varepsilon + \delta + \lambda_\varepsilon(t) + \lambda_\delta(t)\big)$. By simple application of Zorn lemma one can prove that $y(\cdot)$ exists on the whole I. Taking into account Proposition 1 and choosing appropriate sequence ε_i we can conclude that there exists a solution $x(\cdot)$ of (1) such that $|x(t) - z(t)| \leqslant r_\varepsilon(t)$ where $\dot{r}_\varepsilon(t) = u\big(t, r_\varepsilon(t)\big) + 3\big(\varepsilon + \lambda_\varepsilon(t)\big)$, $r_\varepsilon(0) = 0$.

From Lemma 1 one can conclude that $\lim D_H(R_{RP}, R_N) = 0$ for appropriate sequence Δ_N of subdivisions of I since $\lim_{\varepsilon \to 0} r_\varepsilon(t) = 0$. □

COROLLARY 1. *Let $H(t, \alpha) \subset G(t, \alpha)$ be almost USC. Under A1–A3 the solution set of*

$$\dot{x}(t) \in H(t, x_t), \quad x_0 = \phi, \tag{5}$$

is nonempty R_δ set.

Proof. The fact that the solution set S of (5) is nonempty compact follows from Theorem 1. From Lemma 2.2 of [5] we know that there exists a sequence $H_n(t, \alpha)$ of locally Lipschitz multifunctions such that $H_n(t, \alpha) \subset \overline{co}H(t + \frac{1}{3^n} I \cap A_{kj}; \alpha + \frac{1}{3^n} U)$ where $meas(I \setminus \bigcup_{k=1}^{\infty} A_k) = 0$ and $G(\cdot, \cdot)$ is continuous, while $H(\cdot, \cdot)$ is USC on $A_k \times X$ for every k. Furthermore from Lemma 5.2 of [1] we know that there exists a single valued $f_n(t, \alpha) \subset H_n(t, \alpha)$ such that $f_n(\cdot, \alpha)$ is strongly measurable, while $f(t, \cdot)$ is locally Lipschitz. Let $\widetilde{z}(\cdot, \tau, \alpha)$ be the unique solution of

$$\dot{z}(t) = f_n(t, u_t) \quad \text{on} \quad [\tau, 1], \quad z_\tau = \alpha$$

which depends continuously on (τ, α). Let S_n be the solution set of (5) with H_n instead of H and let $h : [0, 1] \times S_n \to S_n$ be defined by $h(s, u)(t) = z(t)$ if $t \in [0, s]$ and $h(s, u)(t) = \hat{z}(t, s, u_s)$ if $t \in (s, 1]$. Obviously $S \subset S_{n+1} \subset S_n$. We have $\beta(S_n) \leqslant D_H(S, S_n) \to 0$ as $n \to \infty$ since S is compact. Moreover S_n are closed contractible. Taking into account $S = \bigcap_{n \geqslant 1} S_n$ and $\lim_{n \to \infty} \beta(S_n) = 0$ one has that S is R_δ set. □

COROLLARY 2. *The solution set of* (1) *is nonempty connected and dense subset of R_{RP}.*

Proof. Since $\overline{ext}F(t, \alpha) \subset F(t, \alpha)$ we can suppose that $F(\cdot, \cdot)$ is ALSC. Let $\delta > 0$ and let $f(\cdot)$ with $|f(t)| \leqslant M$ be a positive measurable function. We claim that for every AC $y(.)$ with $d(\dot{y}(t), F(t, y(t))) < f(t)$ there exists a solution $x(\cdot)$ of (1) such that $|x(t) - y(t)| \leqslant r(t)$, where $\dot{r}(t) \leqslant u(t, r) + f(t) + \delta$.

Suppose first $f(\cdot), \dot{y}(\cdot)$ and $u(\cdot, \cdot)$ are continuous. Define the map:

$$\Gamma_\mu(t, \alpha) = \begin{cases} F(t, \alpha) & \alpha - y_t \notin X_0 \\ \{v \in F(t, \alpha) : |v - \dot{y}(t)| = d(\dot{y}(t), G(t, \alpha))\} & \alpha = y_t \\ \mathrm{cl}\{v \in F(t, \alpha) : [y(t) - \alpha(0), \dot{y}(t) - v]_+ & \\ \qquad < u(t, |y(t) - \alpha(0)|) + \delta + f(t)\} & \textit{elsewhere.} \end{cases}$$

Let $\xi \in \Gamma(t, \alpha)$ and let $(t_i, \alpha_i) \to (t, \alpha)$. Since $u(\cdot, \cdot), y(\cdot), \dot{y}(\cdot)$ and $f(\cdot)$ are continuous and $[\cdot, \cdot]_+$ is USC, one has that there exists ξ_n such that

$$[y(t_n) - \alpha_n(0), \dot{y}(t_n) - \xi_n]_+ < u(t_n, |y(t_n) - \alpha_n(0)|) + \delta + f(t_n)$$

moreover $\xi_n \to \xi$ as $n \to \infty$. Thus $\Gamma(\cdot, \cdot)$ is ALSC. When $f(\cdot), y(\cdot)$ are not continuous one can use Lusin's theorem. Recall also that $u(\cdot, \cdot)$ is almost continuous. The existence of solutions of $\dot{x}(t) \in \Gamma(t, x_t)$, $x(0) = x_0$ can be proved with the help of Theorem 3 of [4] as in [1] (see also [8]). Every solution $x(\cdot)$ of the last differential inclusion satisfies claim's inequality. The proof of the connectedness of the solution set of (2) is standard. The connectedness of the solution set of (1) can be proved as in [7]. □

3. Main result

In this section we prove the existence result for differential inclusion, which right-hand side is a sum of one side Lipschitz and satisfying compactness conditions multifunction.

The following lemma is proved in [9].

LEMMA 2. *Let Y be a Banach space, $\emptyset \neq D \subset Y$ be compact convex. If $F : D \to 2^D$ be USC with nonempty R_δ values, then F admits a fixed point.*

Consider the system (3), where F satisfies A1–A3 and R is nonempty compact valued satisfying the assumptions

B1. $|R(t,\alpha)| \leqslant c(t)\{1 + |\alpha|\}$ for every α, where $c(t)$ is an integrable function.

Taking into account Remark 1 we set $c(\cdot) = 1$ if needed. Hence we suppose without loss of generality that $|F(t,\alpha) + R(t,\alpha)| \leqslant M$, i.e. the right-hand side of (3) is bounded.

B2. There exists a Kamke function $w(\cdot,\cdot)$ such that

$$\beta\big(R(t,A)\big) \leqslant w\big(\beta(A)\big), \quad \text{for every bounded } A \subset X.$$

Furthermore we assume that R satisfies at least one of the following assumptions:

(1) a) $R(\cdot,\alpha)$ admits strongly measurable selector and $R(t,\cdot)$ has a closed graph.

(2) b) $R(\cdot,\cdot)$ is almost LSC.

PROPOSITION 2. *Let Y be a Banach space and $\emptyset \neq \Omega_n \subset Y$ with $\Omega_n \subset \Omega_{n+1}$ for $n \geqslant 1$ be such that $\beta(\Omega_n \cap A) = 0$ for every bounded A and all $n \geqslant 1$. Let $\Omega = \overline{\bigcup_{n\geqslant 1} \Omega_n}$ and let $B = \{x_k : k \geqslant 1\}$ be bounded. Then*

$$\beta_\Omega(B) = \lim_{n\to\infty} \overline{\lim_{k\to\infty}}\, d(x_k, \Omega_n).$$

This is Proposition 3 of [2] and can be proved just as Proposition 9.2 in [5] except for trivial modifications.

Remark 2. Let $C \subset X$ be compact. It is not difficult to show that under the conditions of the proposition above one has also

$$\beta_\Omega(B) = \lim_{n\to\infty} \overline{\lim_{k\to\infty}}\, \rho(A_k, \Omega_n)$$

where $B = \bigcup_{k\geqslant 1} A_k$ and A_k are compacts.

In the following lemmas we consider (3) with F replaced by H.

LEMMA 3. *Let $V(\cdot)$. $W(\cdot)$ be strongly measurable bounded. Let R_V and R_W be the solution sets of* (3) *with $R(\cdot,\cdot)$ replaced by $V(t)$, $W(t)$ respectively. If A1–A3 hold then $D_H(R_V(t), R_W(t)) \leqslant r(t)$, where $r(0) = 0$ and $\dot r(t) = u(t, r(t)) + D_H(V(t), W(t))$.*

Proof. Given $\mu > 0$. For $x(\cdot) \in R_V$ we define the multifunction

$$S_\mu(t,\alpha)=\begin{cases} H(t,\alpha)+W(t) & \alpha-x_t\notin X_0\\ \{v\in H(t,x_t)+W(t):|v-\dot x(t)|=d(\dot x(t),H(t,x_t)+W(t))\} & u=y_t\\ \mathrm{cl}\{v\in H(t,\alpha)+W(t):[x(t)-\alpha(0),\dot x(t)-v]_+ & \\ \qquad < u(t,|x(t)-\alpha(0)|)+D_H(V(t),W(t))+\mu\} & \text{elsewhere.}\end{cases}$$

Obviously $S_\mu(\cdot,\cdot)$ is almost LSC nonempty compact valued. Therefore there exists $y^\mu(\cdot) \in R_W$ such that $y_0^\mu = \phi$ and $\dot y^\mu(t) \in S_\mu(t, y_t^\mu)$. Hence $|x(t) - y_t^\mu(t)| \leqslant r(t)$, where $r(0) = 0$ and $\dot r(t) = u(t, r(t)) + D_H(W(t), V(t)) + \mu$. Consequently one can replace μ by 0 since R_W is compact and $\mu > 0$ is arbitrary. □

Consider first the case B2 a). Denote by $Lip^M(A)$ the set of all Lipschitz with a constant M functions from $[-\tau, 0]$ into A. We need of the following proposition.

PROPOSITION 3. *For every $M > 0$ one has $\beta(A) = \beta(Lip^M(A))$, where the second measure of noncompactness is in X.*

Proof. Let $\beta(A) = r$. Fix $\varepsilon > 0$ and suppose that $\widetilde{A}$ is a finite set in E such that $\widetilde{A} + \left(r + \frac{\varepsilon}{2}\right)U \supset A$. Let $f(\cdot)$ be M Lipschitz and let $f(t_1) = f_1, f(t_2) = f_2$. Suppose $|l_1 - f_1| < \varepsilon$ and $|l_2 - f_2| < \varepsilon$. Consider $g(t) = l_1 + \dfrac{l_2 - l_1}{t_2 - t_1}(t - t_1)$. It is not difficult to see that $\max\limits_{t\in[t_1,t_2]} |g(t) - f(t)| \leqslant \varepsilon + \frac{M}{2}(t_2 - t_1)$. Now we divide $[-\tau, 0]$ on $\left[\dfrac{4M\tau}{\varepsilon}\right] + 1$ intervals with equal lenghts. Here $[x]$ is the greatest integer $\leqslant x$. Let Δ be subdivision of I with points $\{t_i\}_{i=1}^N$. Denote by $Sp(\widetilde{A})$ be the set of all piecewise linear function with values in $\widetilde{A}$ for every t_i. As shown $D_H(Sp(\widetilde{A}), Lip^M(A)) < r + \varepsilon$. Therefore $\beta(A) \leqslant \beta(Lip^M(A)) \leqslant \beta(A) + \varepsilon$. Since ε is arbitrary the proposition has been proved. □

Let $v(\cdot)$ be strongly measurable bounded multifunction. Denote by $R_H(v)$ the solution set of

$$\dot x(t) \in H(t, x_t) + v(t), \quad x_0 = \phi. \tag{6}$$

LEMMA 4. *If $\{v_k\}_{k=1}^\infty \subset L_1(I, E)$ is unirfomly bounded, then*

$$\beta\left[R_H\left\{\bigcup_{k=1}^{\infty} v_k\right\}(t)\right] \leqslant \int_0^t \left(s, \beta\left\{\bigcup_{k=1}^{\infty} v_k(s)\right\}\right) ds.$$

Proof. Suppose first $C \subset E$ is compact. Then the solution set of (6) is $C(I, E)$ compact. I.e. $R_H(C)(t)$ is compact for all t. Therefore $\beta\left[R_h\left\{\bigcup\limits_{k=1}^{\infty} v_k(t)\right\}\right] = 0$ if

$\bigcup\limits_{k=1}^{\infty} v_k(t) \subset C$ for a.e. $t \in I$. We may also assume that closed linear hull E_0 of $\bigcup\limits_{n\geqslant 1} v_n(I)$ is separable since all $v_k(\cdot)$ are strongly measurable. Let $E_n \subset E$ be finite dimensional subspaces such that $E_0 = \overline{\bigcup\limits_{n\geqslant 1} E_n}$. Denote by $W_n = \{v \in L_1(I, E_n) : |v(s)| \leqslant M$ a.e. on $I\}$. Furthermore $\Omega_n = \{R_H(w)(t) : w \in W_n\}$. Since $(MU) \cap E_n)$ is compact one has $\beta(\Omega_n) = 0$. Denote $\Omega = \overline{\bigcup\limits_{n\geqslant 1} \Omega_n}$. By Remark 2 we have

$$\beta(\{R_H(w_k)(t) : k \geqslant 1\}) \leqslant \beta_\Omega(\{R_H(w_k)(t) : k \geqslant 1\}) = \lim_{n\to\infty} \overline{\lim_{k\to\infty}} \rho(R_H(w_k)(t), \Omega_n)$$

$$\rho(R_H(w_k)(t), \Omega_n) \leqslant \inf\{D_H(R_H(w_k)(t), R_H(w)(t)) : w \in W_n\}$$
$$= \int_0^t u(s, \rho(w_k(s), E_n))\, ds.$$

The multifunction $\Gamma(s) = \{x \in W_n : |w_k(s) - x| \leqslant \rho(w_k(s), E_n)\}$ is strongly measurable and hence admits strongly measurable selector $w(s)$ with $|w(s)| \leqslant 2M$. Using Fatou's lemma and the dominated convergence theorem we get

$$\beta\big[(R_H\{\bigcup_{k=1}^{\infty} v_k\}(t))\big] \leqslant \beta_\Omega\big[(R_H\{\bigcup_{k=1}^{\infty} v_k\}(t))\big] = \lim_{n\to\infty} \overline{\lim_{k\to\infty}} \rho(R_H(w_k)(t), \Omega_n)$$
$$= \int_0^t u(s, \rho(w_k(s), E_n))\, ds = \int_0^t u\big(s, \beta\big(\{\bigcup_{k=1}^{\infty} v_k\}(t) : k \geqslant 1\big)\big)\, ds.$$

□

Denote $J = I \cup [-\tau, 0]$. Now we are ready to prove the main result in the paper, which extends the main result of [13] and Theorem 4 of [2].

THEOREM 2. *Under the assumptions A1–A4, B1–B2 the problem (3) admits a nonempty solution set.*

Proof. Due to growth condition there exists a closed bounded convex set $S_0 \subset C(J, E)$ such that $\overline{co}\, R_H(S_0) \subset S_0$. Here $R_H(S_0) = \bigcup\limits_{x \in Lip^M(S_0)} R_H(V_x(t))$ and $R_H(V_x(t))$ is the solution set

$$\dot{y}(t) \in H(t, y_t) + R(t, x_t), \quad x_0 = y_0 = \phi.$$

We let $S_{n+1} = \overline{co} R_H(S_n)$ for $n \geqslant 0$. Denote $S = \bigcap\limits_{n=1}^{\infty} S_n$. Obviously S is equicontinuous (Lipschitz with a constant M). Denote $p_n(t) = \beta(S_n(t))$. Suppose that for given $\varepsilon > 0$ there exists a sequence $\{v_k(\cdot)\} \subset S_n$ such that $\beta(R_H[S_n(t)]) \leqslant 2\beta\{R_H[v_k(t); k \geqslant 1]\} + \varepsilon$. Then we obtain $p_{n+1}(t) \leqslant 2\int_0^t u(s, \beta(P_n(s))\, ds + \varepsilon$. Since this is true for all $\varepsilon > 0$ and $p_n(t) \to p_\infty(t)$ on I we get $0 \leqslant p(t) \leqslant p_\infty(t)$, where $p(t) = \beta(S(t))$. Furthermore $p_\infty(t) \leqslant 2\int_0^t u(s, (\beta(p_\infty(s)))\, ds, p_\infty(0) = 0$. Hence $p_\infty(t) \equiv 0$ and $p(t) = 0$. I.e. $S \neq \emptyset$ is a convex compact set. Thus $R_H(\cdot)$ is nonempty compact R_δ valued with $R_H : S \to S$. From Lemma 3 we know that $R_H(\cdot)$ is continuous. Therefore there exists a fixed point $x(\cdot)$. Consequently $\dot{x}(t) \in H(t, x_t) + R(t, x_t), x_0 = \phi$.

Now we have to show that for bounded $B \subset X$ and $\varepsilon > 0$ there exists a sequence $\{\alpha_k\} \subset B$ such that $\beta(B) \leqslant 2\beta(\{x_k; k \geqslant 1\}) + \varepsilon$. It suffices to consider $\beta(B) > 0$ and $\varepsilon \in (0, \beta(B))$. Let $r = \beta(B) - \varepsilon$ and $x_1 \in B$. Then there exists $x_2 \in B \setminus rU + x_1$ since otherwise $B \subset x_1 + rU$ and $\beta(B) \leqslant r$ — contradiction. Let $x_1, \dots, x_n \in B$ be such that $|x_i - x_j| \geqslant r$ for $i \neq j$. The same arguments yields $x_{n+1} \in B$ such that $|x_i - x_{n+1}| \geqslant r$ for $i = 1, \dots, n$. By induction there exists a sequence $\{x_k\}_{k=1}^{\infty} \subset B$ such that $|x_i - x_j| \geqslant r$ for $i \neq j$. This implies $\beta(\{x_k\}_{k=1}^{\infty}) \geqslant \frac{r}{2}$ and hence $\beta(B) \leqslant 2\beta(\{x_k\}_{k=1}^{\infty}) + \varepsilon$.

Let now $R(\cdot, \cdot)$ be almost LSC. By virtue of Theorem 3 of [4] there exist Γ^{M+1} continuous selections $f_n(t, \alpha) \in F(t, \alpha)$ and $r_n(t, \alpha) \in R(t, \alpha)$ on A_n, where $\overline{ext}F(\cdot, \cdot)$ and $R(\cdot, \cdot)$ are LSC on $A_n \times X$ for every n and moreover $meas(I \setminus \bigcup_{n=1}^{\infty} A_n) = 0$. We let $f(t, \alpha) = \overline{co} \bigcap_{\varepsilon > 0} f_n(t, \alpha + \varepsilon U)$ and $r(t, \alpha) = \overline{co} \bigcap_{\varepsilon > 0} r_n(t, \alpha + \varepsilon U)$ for $t \in A_n$. Obviously for every $v : I \to E$ strongly measurable with compact values $\beta(R_f(v)) \leqslant \beta(R_H(v))$, where $R_f(v)$ is the solution set of $\dot{x}(t) \in f(t, x_t) + v(t)$, $x_0 = \phi$. Thus by arguments used above there exists a solution $y(\cdot)$ of

$$\dot{x}(t) \in f(t, x_t) + r(t, x_t), \quad x_0 = \phi.$$

One can show that $y(\cdot)$ is a solution of (3) as Theorem 6.2 of [5]. □

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Depart. of Math.
University of Mining and Geology
1100 Sofia
Bulgaria