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# CONVEXITY AND THE RIEMANN $\xi$—FUNCTION

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*Abstract.* The convexity properties of the kernel $\Phi(t)$ whose Fourier transform is the Riemann $\xi$–function are investigated. In particular, it is shown that $\Phi\left(\sqrt{t}\right)$ is convex for $t > 0$. Also, lower bounds for the Turán differences involving the moments of $\Phi(t)$ are established. The paper concludes with several questions and open problems.

## 1. Introduction

Let

$$H(x) := \frac{1}{8}\xi\left(\frac{x}{2}\right) := \int_0^\infty \Phi(t)\cos(xt)dt\,, \tag{1.1}$$

where

$$\Phi(t) := \sum_{n=1}^{\infty} \pi n^2\left(2\pi n^2 e^{4t} - 3\right)\exp\left(5t - \pi n^2 e^{4t}\right). \tag{1.2}$$

The Riemann Hypothesis is equivalent to the statement that all the zeros of $H(x)$ are real (cf. [T, p. 255]). Today, there are no known explicit necessary and sufficient conditions which a function must satisfy in order that its Fourier transform have only real zeros (see, however, [P1, p. 17]). Nevertheless, the *raison d'être* for investigating the kernel $\Phi(t)$ is that there is an intimate connection (the precise meaning of which is unknown) between the properties of $\Phi(t)$ and the distribution of the zeros of its Fourier transform.

In Section 2 we begin with a brief summary of results pertaining to $\Phi$ (Theorem 2.1) and we highlight some of the lesser known, recently established convexity properties of $\Phi$ (Theorem 2.2 and Theorem 2.3). In particular, we make use of the fact that $\log\Phi(\sqrt{t})$ is concave for $t > 0$ and show that this implies that the moments of $\Phi(t)$ satisfy the Turán inequalities (2.7). These inequalities are some of the simplest necessary conditions which $H(x)$ (see (1.1) above) must satisfy in order that it possess only real zeros. With the aid of Matiyasevich's triple integral representation we establish a new lower bound (Theorem 2.8) for the Turán differences of the moments associated with certain admissible kernels (see Definition 2.7). As a consequence of the foregoing results, we prove (Corollary 2.11) that $\Phi(t)$ is a Pólya frequency function of order 2. (This concept is defined in Section 2.) The main

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result of this paper is that $\Phi(\sqrt{t})$ is convex for $t > 0$ (Theorem 2.12). Therefore, it follows that the Fourier cosine transform of $\Phi(\sqrt{t})$ is positive for all $x \in \mathbb{R}$ (Corollary 2.13). For the sake of clarity of exposition, the proof of Theorem 2.12 is deferred to Section 3. This proof requires not only a detailed analysis of the behavior of $\Phi(t)$ but also some numerical work as well as some fairly complicated (albeit elementary) estimates. In Section 4 we cite some unanswered questions and open problems. In several instances these problems are supplemented by partial results and additional references.

## 2. Convexity properties of $\Phi(t)$

For the reader's convenience we begin with a brief review of the basic properties of $\Phi(t)$ defined by (1.2).

THEOREM 2.1. ([CNV1, Theorem A]) *Consider the function* $\Phi(t)$ *of* (1.2) *and set*

$$\Phi(t) = \sum_{n=1}^{\infty} a_n(t)\,, \tag{2.1}$$

*where*

$$a_n(t) := \pi n^2 \left(2\pi n^2 e^{4t} - 3\right) \exp\left(5t - \pi n^2 e^{4t}\right) \qquad (n = 1, 2, \ldots). \tag{2.2}$$

*Then, the following are valid:*

(i) *for each* $n \geqslant 1$, $a_n(t) > 0$ *for all* $t \geqslant 0$, *so that* $\Phi(t) > 0$ *for all* $t \geqslant 0$;
(ii) $\Phi(z)$ *is analytic in the strip* $-\pi/8 < \operatorname{Im} z < \pi/8$;
(iii) $\Phi(t)$ *is an even function, so that* $\Phi^{(2m+1)}(0) = 0 \quad (m = 0, 1, \ldots)$;
(iv) *for any* $\varepsilon > 0$,

$$\lim_{t\to\infty} \Phi^{(n)}(t) \exp\left[(\pi - \varepsilon)e^{4t}\right] = 0$$

*for each* $n = 0, 1, \ldots$;
(v) $\Phi'(t) < 0$ *for all* $t > 0$.

The proofs of statements (i) – (iv) can be found in Pólya [P1], whereas the proof of (v) is in Wintner [W] (see also Spira [S]). □

In order to indicate the significance of the next theorem, we first recall that

$$H(x) := \frac{1}{2}\int_{-\infty}^{\infty} \Phi(t) e^{ixt} dt = \int_{0}^{\infty} \Phi(t)\cos(xt) dt \tag{2.3}$$

is an entire function of order one ([T, p. 16]) of maximal type (cf. [CNV2, Appendix A]) whose Taylor series about the origin can be written in the form

$$H(z) = \sum_{m=0}^{\infty} \frac{(-1)^m \hat{b}_m}{(2m)!} z^{2m}\,, \tag{2.4}$$

where

$$\hat{b}_m := \int_0^\infty t^{2m}\Phi(t)dt \qquad (m = 0, 1, 2, \dots). \tag{2.5}$$

The change of variable, $z^2 = -x$ in (2.4), yields the entire function

$$F(x) := \sum_{m=0}^{\infty} \frac{\hat{b}_m}{(2m)!} x^m. \tag{2.6}$$

Then it is easy to see that $F(x)$ is an entire function of order $\frac{1}{2}$ and that the Riemann Hypothesis (see the Introduction) is equivalent to the statement that all the zeros of $F(x)$ are real and negative. Now it is known (cf. Boas [B, p. 24] or Pólya and Schur [PS]) that a *necessary condition* for $F(x)$ to have only real zeros is that the moments $\hat{b}_m$ (in (2.5)) satisfy the *Turán inequalities*, that is,

$$\hat{b}_m^2 - \frac{2m-1}{2m+1}\hat{b}_{m-1}\hat{b}_{m+1} \geqslant 0 \qquad (m = 1, 2, 3, \dots). \tag{2.7}$$

These inequalities (at least for $m \geqslant 2$) have been established (cf. [CNV1] and [CV1]; see also [M]) as a consequence of either one of the two properties ((a) or (b)) of $\Phi(t)$ stated in the following theorem.

THEOREM 2.2. *Let* $\Phi(t)$ *be defined by* (2.1). *Then* $\Phi(t)$ *satisfies the following concavity properties.*

(a) ([CNV1, Proposition 2.1]) *If*

$$K_\Phi(t) := \int_t^\infty \Phi(\sqrt{u})du \qquad (t \geqslant 0),$$

*then* $\log K_\Phi(t)$ *is strictly concave for* $t > 0$, *that is,*

$$\frac{d^2}{dt^2}\log K_\Phi(t) < 0 \quad \textit{for} \quad t > 0.$$

(b) ([CV1, Theorem 2.1]) *The function* $\log \Phi(\sqrt{t})$ *is strictly concave for* $t > 0$.

□

Now a calculation shows that $\log \Phi(\sqrt{t})$ is strictly concave for $t > 0$ if and only if

$$g(t) := t\left[(\Phi'(t))^2 - \Phi(t)\Phi''(t)\right] + \Phi(t)\Phi'(t) > 0 \quad \text{for} \quad t > 0. \tag{2.8}$$

To express (2.8) in another way, we use the fact that $\Phi(t) > 0$ (Theorem 2.1) and so we can write

$$\Phi(t) = e^{-\nu(t)} \qquad (t \in \mathbb{R}). \tag{2.9}$$

Since $\Phi'(t) < 0$ for $t > 0$, we see that

$$g(t) = (\Phi'(t))^2 \frac{d}{dt}\left(\frac{t\Phi(t)}{\Phi'(t)}\right) \tag{2.10}$$

and hence, by (2.8) and (2.9), $g(t) > 0$ if and only if $\dfrac{\nu'(t)}{t}$ is strictly increasing for $t > 0$. Using some of the analysis in [CV1], Newman [N] proved the following stronger result.

THEOREM 2.3. ([N, Theorem 1]) *Let $v(t)$ defined by (2.9). Then $v'''(t) > 0$ for $t > 0$.*

In terms of $\Phi(t)$, Theorem 2.3 says that the function $\dfrac{\Phi'(t)}{\Phi(t)}$ is strictly concave for $t > 0$. In order to see that $v'''(t) > 0$ for $t > 0$ implies that $\dfrac{v'(t)}{t}$ is strictly increasing for $t > 0$, we first note that for $t > 0$,

$$\frac{d}{dt}\left(t^2\frac{d}{dt}\left(\frac{v'(t)}{t}\right)\right) = \frac{d}{dt}\left(tv''(t) - v'(t)\right) = tv'''(t). \tag{2.11}$$

Now set $f(t) := tv''(t) - v'(t)$. Then $f(0) = 0$, since $v'(t)$ is an odd function ($\Phi(t)$ is even by Theorem 2.1). Since $v'''(t) > 0$ for $t > 0$, it follows from (2.11) that $tv''(t) - v'(t) > 0$ for $t > 0$ and hence $\frac{v'(t)}{t}$ is strictly increasing for $t > 0$.

*Remark 2.4.* In [CV1] and [CVV] it was shown that the moments of various kernels, $K(t)$, for which $\log K(\sqrt{t})$ is concave for $t > 0$, satisfy, in addition to the Turán inequalities, some more general moment inequalities. Unaware of the results in [CV1], [CVV] and [N], in an interesting paper Conrey and Gosh [CG, Theorem 1] used the concavity of $\frac{K'(t)}{K(t)}$ for $t > 0$, to deduce the Turán inequalities for certain classes of entire functions. □

We next demonstrate by means of a concrete example that if we only assume that $\log K(t)$ is concave for $t > 0$, then the moments of $K(t)$ need not satisfy the Turán inequalities.

*Example 2.5.* If we set $K(t) := e^{-t^2/2}(t^4 + 36)$, then it is not difficult to show that

$$\frac{d^2}{dt^2}\log K(t) = \frac{-1296 + 432t^2 - 72t^4 - 4t^6 - t^8}{(36 + t^4)^2} < 0$$

for $t > 0$. Next, let

$$\beta_m := \int_0^\infty t^{2m}K(t)dt \qquad (m = 0, 1, 2, \ldots). \tag{2.12}$$

These moments can be explicitly calculated with the aid of the gamma function:

$$\beta_m = 2^{3/2}2^m\left[9\Gamma\left(\frac{1}{2} + m\right) + \Gamma\left(\frac{5}{2} + m\right)\right],$$

where $\Gamma(x) := \displaystyle\int_0^\infty e^{-t}t^{x-1}dt$, $\operatorname{Re} x > 0$. Let $d_m$ denote the *Turán differences*

$$d_m := \beta_m^2 - \frac{2m-1}{2m+1}\beta_{m-1}\beta_{m+1} \qquad (m = 1, 2, 3, \ldots).$$

Then $d_1 = -84\pi$, so that the Turán inequalities fail to hold for $m = 1$. Consequently, the entire function

$$G(x) := \int_0^\infty e^{-t^2/2}(t^4 + 36)\cos(xt)dt, \tag{2.13}$$

cannot have only real zeros. By evaluating the integral in (2.13), we find that

$$G(x) = \sqrt{\frac{\pi}{2}}\left(39 - 6x^2 + x^4\right)e^{-x^2/2}$$

and hence we infer the stronger conclusion, namely, that $G(x)$ has no real zeros. □

*Remark 2.6.* We call attention to the fact that it is possible to establish fairly general moment inequalities under the weaker assumption that $\log K(t)$ is concave for $t > 0$. Indeed, let $K(t) > 0$ for $t > 0$ and suppose that $K(t)$ is $C^2$ on $[0, \infty)$. For $R > 0$, set

$$\mu_m(R) := \int_0^R t^m K(t)dt \qquad (m = 0, 1, 2, \dots).$$

If $\log K(t)$ is concave for $t > 0$, then

$$\mu_m^2(R) \geqslant \frac{m}{m+1}\mu_{m-1}(R)\mu_{m+1}(R) \qquad (m = 1, 2, 3, \dots). \tag{2.14}$$

Recently, Mitrinović and Pečarić [MP] gave an elegant proof of (2.14) using the Chebyshev integral inequality. □

In order to expedite our presentation, it will be convenient to introduce the following definition.

*Definition 2.7.* A function $K : \mathbb{R} \longrightarrow \mathbb{R}$ is called an *admissible kernel*, if it satisfies the following properties:

(i) $K(t)$ is analytic in the strip $|\operatorname{Im} z| < \tau$ for some $\tau > 0$,
(ii) $K(t) > 0$ for $t \in \mathbb{R}$,
(iii) $K(t) = K(-t)$ for $t \in \mathbb{R}$,
(iv) $K'(t) < 0$ for $t > 0$, and
(v) for some $\varepsilon > 0$ and $n = 0, 1, 2, \dots$,

$$K^{(n)}(t) = O\left(\exp\left(-|t|^{2+\varepsilon}\right)\right) \text{ as } t \longrightarrow \infty. \tag{2.15}$$

We next proceed to establish a lower bound for the Turán differences of the moments of those admissible kernels $K(t)$ for which $\log K(\sqrt{t})$ is concave on $(0, \infty)$.

THEOREM 2.8. *Let $K(t)$ be an admissible kernel. For each $R > 0$, let*

$$\beta_m(R) := \beta_m(R, K) := \int_0^R t^{2m} K(t)dt \qquad (m = 0, 1, 2, \dots). \tag{2.16}$$

*If $\log K(\sqrt{t})$ is concave for $t > 0$, then for $m = 1, 2, 3, \dots$ we have*

$$\begin{aligned} d_m(R) :&= \beta_m^2(R) - \frac{2m-1}{2m+1}\beta_{m-1}(R)\beta_{m+1}(R) \\ &\geqslant \frac{K(R)}{2m+1}R^{2m-1}\left[R^2\beta_m(R) - \beta_{m+1}(R)\right]. \end{aligned} \tag{2.17}$$

*Moreover, $R^2\beta_m(R) - \beta_{m+1}(R) > 0$ (for $R > 0$, $m = 1, 2, 3, \dots$) and*

$$\beta_m^2(\infty) \geqslant \frac{2m-1}{2m+1}\beta_{m-1}(\infty)\beta_{m+1}(\infty), \qquad (m = 1, 2, 3, \dots). \qquad \square \tag{2.18}$$

*Proof.* Our proof will be based on a modification of Matiyasevich's triple integral representation of the Turán differences [M]. First, an integration by parts applied to the integral in (2.16) yields

$$\beta_m(R) = \frac{R^{2m+1}}{2m+1}K(R) - \frac{1}{2m+1}\int_0^R t^{2m+1}K'(t)dt. \tag{2.19}$$

Thus, using (2.19) we obtain

$$\begin{aligned}
&\int_0^R\int_0^R u^{2m}v^{2m}K(u)K(v)\left(v^2-u^2\right)\left(\int_u^v -\frac{d}{dt}\left(\frac{K'(t)}{tK(t)}\right)dt\right)dudv\\
&= \int_0^R\int_0^R u^{2m-1}v^{2m-1}\left(v^2-u^2\right)\left(vK(v)K'(u)-uK(u)K'(v)\right)dudv\\
&= 2\beta_{m+1}(R)\left[R^{2m-1}K(R)-(2m-1)\beta_{m-1}(R)\right] \qquad\qquad (2.20)\\
&\qquad -2\beta_m(R)\left[R^{2m+1}K(R)-(2m+1)\beta_m(R)\right]\\
&= 2(2m+1)\left[d_m(R) - \frac{K(R)}{2m+1}R^{2m-1}\left(R^2\beta_m(R)-\beta_{m+1}(R)\right)\right]\\
&=: I_m(R)\,.
\end{aligned}$$

Since $\log K(\sqrt{t})$ is concave for $t>0$, it follows that $\frac{d}{dt}\left(\frac{-K'(t)}{tK(t)}\right)>0$ for $t>0$ (cf. (2.8) and (2.10)) and hence we see that $I_m(R)>0$ and thus (2.17) holds.

Next, it is easy to see that

$$\beta_{m+1}(R) = \int_0^R t^{2m+2}K(t)dt < R^2\beta_m(R)\,,$$

so that the lower bound for the Turán difference in (2.17) is positive. Since $\int_0^R t^{2m}K(t)dt \leqslant R^{2m}\int_0^\infty K(t)dt$ and since $K(t)$ is an admissible kernel, it follows that for each fixed $m$ ($m=1,2,3,\dots$),

$$\lim_{R\to\infty}\frac{K(R)}{2m+1}R^{2m-1}\left[R^2\beta_m(R)-\beta_{m+1}(R)\right]=0\,. \tag{2.21}$$

Therefore, (2.17) and (2.21) yield the required Turán inequalities (2.18). □

A glance at the above proof shows that Theorem 2.8 remains valid for kernels which meet less stringent assumptions than those stipulated for admissible kernels. Since $\Phi(t)$ (cf. (2.2)) is an admissible kernel (cf. Theorem 2.1) and $(\log\Phi(\sqrt{t}))''<0$ $(t>0)$, therefore, as an immediate consequence of Theorem 2.9 we have

COROLLARY 2.9. *With $\Phi(t)$ defined by* (2.2), *set*

$$\hat{b}_m(R) := \int_0^R t^{2m}\Phi(t)dt \qquad (R>0, m=0,1,2,\dots). \tag{2.22}$$

*Then the moments $\hat{b}_m(R)$ satisfy* (2.17) *and* (2.18).

The interest in Corollary 2.9 stems, in part, from the fact that a necessary condition for the entire function

$$H_R(x) := \int_0^R \Phi(t)\cos(xt)dt \qquad (R>0) \tag{2.23}$$

to have only real zeros is that the associated moments (2.22) satisfy the Turán inequalities. (See also Problem 1 in Section 4.)

By way of application of the foregoing results we next show that $\Phi(t)$ is a Pólya frequency function of order 2 ($PF_2$). We recall that a function $K : \mathbb{R} - \mathbb{R}$ is a *Pólya frequency function of order* 2, if $K(t) \geqslant 0$ for all $t \in \mathbb{R}$ and

$$K(x_1 - y_1)K(x_2 - y_2) \geqslant K(x_1 - y_2)K(x_2 - y_1), \tag{2.24}$$

whenever $-\infty < x_1 \leqslant x_2 < \infty$ and $-\infty < y_1 \leqslant y_2 < \infty$ (see, for example, Barlow and Proschan [BP, p. 24]).

*Remark 2.10.* For an exhaustive treatment of Pólya frequency functions of any order we refer to Karlin [K]. Pólya frequency functions of order 2 are particularly important and have applications to statistical theory (see [K, p. 32] for many references) and reliability theory [BP].

COROLLARY 2.11. *The kernel* $\Phi(t)$ (see (2.1)) *is a Pólya frequency function of order* 2.

*Proof.* For $t \in \mathbb{R}$, set $\Phi(t) = e^{-v(t)}$ (cf. (2.9)). By Theorem 2.2 $(\log \Phi(\sqrt{t}))'' < 0$ for $t > 0$ and so it follows that $(\log \Phi(t))'' < 0$ for $t > 0$ (cf. (2.8)). Therefore, $v(t)$ is convex ($v''(t) > 0$) for $t > 0$. But $v(t)$ is an *even* ($C^\infty$) function and thus we see that $v(t)$ is convex on $\mathbb{R}$. Since $\Phi(t) \geqslant 0$ it suffices to show that

$$\Phi(x_1 - y_1)\Phi(x_2 - y_2) \geqslant \Phi(x_1 - y_2)\Phi(x_2 - y_1), \tag{2.25}$$

whenever $x_1 \leqslant x_2$ and $y_1 \leqslant y_2$. Next, we set $a := x_1 - y_2$, $b := x_2 - y_1$, $c := x_1 - y_1$ and $d := x_2 - y_2$. Then $a \leqslant c \leqslant b$, $a \leqslant d \leqslant b$ and $a + b = c + d$. (Note that these relations determine $x_1$, $x_2$, $y_1$ and $y_2$ with $x_1 \leqslant x_2$ and $y_1 \leqslant y_2$, up to a translation.) Thus, (2.25) is equivalent to the inequality $\Phi(a)\Phi(b) \leqslant \Phi(c)\Phi(d)$, so that in terms of the function $v$ it suffices to show that

$$v(a) + v(b) \geqslant v(c) + v(d). \tag{2.26}$$

If $a = b$, then $a = c = b = d$ and inequality (2.26) is obvious. Thus, we may assume that $a < b$. Since $v(t)$ is convex on $\mathbb{R}$, we have

$$v(c) \leqslant \frac{b-c}{b-a} v(a) + \frac{c-a}{b-a} v(b) \tag{2.27}$$

and

$$v(d) \leqslant \frac{b-d}{b-a} v(a) + \frac{d-a}{b-a} v(b). \tag{2.28}$$

Finally, we add inequalities (2.27) and (2.28) and then use the relation $a+b = c+d$ to obtain the desired inequality (2.26). $\square$

We now turn to the main result of this paper.

THEOREM 2.12. *With* $\Phi(t)$ *defined by* (2.1), *the function* $\Phi(\sqrt{t})$ *is strictly convex for* $t > 0$, *that is,*

$$\frac{d^2}{dt^2} \Phi(\sqrt{t}) > 0 \qquad \textit{for} \qquad t > 0.$$

In light of the foregoing analysis of the nature of the kernel $\Phi(t)$, the assertion of Theorem 2.12 seems plausible at least for sufficiently large values of $t$. In contrast, the behavior of $\Phi(\sqrt{t}), t > 0$, near the origin is considerably more subtle. On account of the detailed calculations required, the proof of Theorem 2.12 is deferred to Section 3. By way of clarification, we also note that if $K(t)$ is a logarithmically concave admissible kernel, then, in general $K(\sqrt{t})$ need *not* be convex for $t > 0$. Indeed, let $K(t) := e^{-t^4}$. Then $\log K(\sqrt{t}) = -t^2$ is concave for $t > 0$, but $K(\sqrt{t}) = e^{-t^2}$ is not convex for $t > 0$. The kernels $K(t; n, a) := e^{-at^{2n}}$, $(a > 0, n > 1)$ may be thought of as paradigms of admissible kernels. Having said this, we hasten to add that they do not epitomize the class of admissible kernels since they are entire functions. Finally, it is not difficult to demonstrate that for any $a > 0$ and $n > 1$, the kernel $K(\sqrt{t}; n, a)$ is *not* convex for $t > 0$. Therefore, it would be of interest to see some explicit examples of admissible kernels, $K(t)$, such that $K(t)$ is not an entire function but that $K(\sqrt{t})$ is convex for $t > 0$.

We conclude this section by noting that one consequence of Theorem 2.12 is that the entire function represented by the Fourier cosine transform of $\Phi(\sqrt{t})$ cannot have any real zeros. More precisely, we have

COROLLARY 2.13. *For any* $x \in \mathbb{R}$,

$$\int_0^\infty \Phi(\sqrt{t}) \cos(xt) dt > 0 \,, \tag{2.29}$$

*where* $\Phi(t)$ *is defined by* (2.1).

*Proof.* Let $K(t) := \Phi(\sqrt{t})$, $t \geqslant 0$. By virtue of the properties of $\Phi(t)$ (see Theorem 2.1) it is not difficult to show that $K(t)$, $K'(t)$ and $K''(t)$ are all integrable on $[0, \infty)$. The endpoint $t = 0$ is a removable singularity of $K'(t)$ and $K''(t)$. In particular, by L'Hôspital's rule we have $\lim_{t \to 0^+} K'(t) = \frac{1}{2} \lim_{t \to 0^+} \Phi''(\sqrt{t}) = \frac{1}{2}\Phi''(0)$. If $x = 0$, inequality (2.29) is clear. If $x \neq 0$, two integration by parts yield

$$\begin{aligned} x^2 \int_0^\infty K(t) \cos(xt) dt &= x^2 \lim_{\varepsilon \to 0^+} \int_\varepsilon^\infty K(t) \cos(xt) dt \\ &= -\frac{1}{2}\Phi''(0) - \lim_{\varepsilon \to 0^+} \int_\varepsilon^\infty K''(t) \cos(xt) dt \\ &> -\frac{1}{2}\Phi''(0) - \lim_{\varepsilon \to 0^+} \int_\varepsilon^\infty K''(t) dt = -\frac{1}{2}\Phi''(0) + \frac{1}{2}\Phi''(0) = 0 \,, \end{aligned}$$

where we have used the fact that $K''(t) > 0$ for $t > 0$ (cf. Theorem 2.12). □

## 3. Proof of Theorem 2.12

We begin with the observation that for $t > 0$

$$4t^{3/2} \frac{d^2}{dt^2} \Phi(\sqrt{t}) = \sqrt{t}\, \Phi''(\sqrt{t}) - \Phi'(\sqrt{t}) > 0 \tag{3.1}$$

if and only if

$$g(t) := t^2 \frac{d}{dt}\left(\frac{\Phi'(t)}{t}\right) = t\Phi''(t) - \Phi'(t) > 0\,, \tag{3.2}$$

where $\Phi(t)$ is defined by (2.1). In order to establish (3.2), we consider the following three intervals

$$\begin{cases} I_1 := (0, 0.096]\,, \\ I_2 := [0.09, 0.12]\,, \\ I_3 := [0.12, \infty) \end{cases} \tag{3.3}$$

and show that $g(t) > 0$ for $t \in I_j, j = 1, 2, 3$.

LEMMA 3.1. *With $g(t)$ defined by* (3.2), *we have*

$$g(t) > 0 \qquad \textit{for} \qquad t \in I_1 = (0, 0.096]\,.$$

*Proof.* Since $g(0) = 0$ and $g'(t) = t\Phi'''(t)$, it suffices to show that $g'(t) > 0$ for $t \in I_1$. To this end, we proceed to show that $\Phi^{(4)}(t) > 0$ for $t \in I_1$. First, it is known (see [CNV1, p. 534, inequality (3.36)]) that

$$\Phi^{(4)}(t) > a_1^{(4)}(t) \qquad \text{for all} \qquad t \geqslant 0, \tag{3.4}$$

where $a_1(t)$ is defined by (2.2) with $n = 1$. Now an explicit calculation shows that

$$a_1^{(4)}(t) = \pi p_5(\pi e^{4t}) \exp\left(5t - \pi e^{4t}\right),$$

where $p_5(y) := 512y^5 - 8,448y^4 + 41,408y^3 - 68,096y^2 + 30,930y - 1,875$. In addition, in [CV1, p. 185] it was proved that $p_5(y)$ has 5 distinct positive zeros. These zeros are $x_1 := 0.071\ldots$, $x_2 := 0.604\ldots$, $x_3 := 1.996\ldots$, $x_4 := 4.617\ldots$ and $x_5 := 9.209\ldots$. Consequently, $p_5(y) > 0$ on $(x_3, x_4)$. But $\pi e^{4t}$ falls in this interval provided $0 \leqslant t < \frac{1}{4}\log\left(\frac{x_4}{\pi}\right) = 0.096\ldots$. Hence, it follows that

$$a_1^{(4)}(t) > 0 \qquad \text{for} \qquad t \in I_1\,. \tag{3.5}$$

By (3.4) and (3.5), $\Phi^{(4)}(t) > 0$ for $t \in I_1$. Since $g'(t) = t\Phi'''(t)$ and $\Phi'''(0) = 0$, we conclude that $\Phi'''(t) > 0$ and $g'(t) > 0$ for $t \in I_1$. This shows that $g(t) > 0$ for $t \in I_1$ and the proof of the lemma is complete. □

LEMMA 3.2. *With $g(t)$ defined by* (3.2), *we have*

$$g(t) > 0 \quad \textit{for} \quad t \in I_2 = [0.09, 0.12]\,.$$

*Proof.* As in the proof of Lemma 3.1, we consider $g'(t) = t\Phi'''(t)$ and show that $\Phi'''(t) > 0$ on $I_2$. To this end, we use (2.1) and (2.2) and write

$$\Phi'''(t) = \sum_{n=1}^{\infty} a_n'''(t) = a_1'''(t) + \Phi_1'''(t)\,, \tag{3.6}$$

where

$$a_1'''(t) = \pi p_4(\pi e^{4t}) \exp\left(5t - \pi e^{4t}\right), \tag{3.7}$$

$$\Phi_1'''(t) = \sum_{n=2}^{\infty} \pi n^2 p_4(\pi n^2 e^{4t}) \exp\left(5t - \pi n^2 e^{4t}\right) \tag{3.8}$$

and

$$p_4(y) := -128y^4 + 1,440y^3 - 4,232y^2 + 3,270y - 375\,. \tag{3.9}$$

Then the idea of the proof is to show that (cf. (3.6))

$$\Phi'''(t) \geqslant a_1'''(t) - \max_{t\in I_2} |\Phi_1'''(t)| > 0 \qquad (t \in I_2)\,. \tag{3.10}$$

We first obtain an upper bound for $|\Phi_1'''(t)|$ $(t \in I_2)$. To begin with, for $y \geqslant 3$ we have the elementary estimate

$$\begin{aligned} |p_4(y)| &\leqslant y^4\left(1 + \frac{1,440}{128y} + \frac{4,232}{128y^2} + \frac{3,270}{128y^3} + \frac{375}{y^4}\right) \\ &\leqslant 10y^4 \qquad (y \geqslant 3). \end{aligned} \tag{3.11}$$

Let $y := \pi e^{4t}$, so that $y \geqslant 3$ for all $t \geqslant 0$. Then

$$\begin{aligned} |\Phi_1'''(t)| &\leqslant \sum_{n=2}^{\infty} \pi n^2 |p_4(n^2 y)| \exp\left(5t - n^2 y\right) \\ &\leqslant 10 e^{5t} \sum_{n=2}^{\infty} \pi n^2 \left(n^2 y\right)^4 e^{-n^2 y} \\ &= 10\pi^5 e^{21t} \sum_{n=2}^{\infty} n^{10} e^{-n^2 y}. \end{aligned} \tag{3.12}$$

Since $x^{10}e^{-\pi x^2}$ is strictly decreasing for $x \geqslant 2$, by the integral test

$$\begin{aligned} \sum_{n=2}^{\infty} n^{10} e^{-n^2\pi} &= 2^{10} e^{-4\pi} + \sum_{n=3}^{\infty} n^{10} e^{-n^2\pi} \\ &\leqslant 2^{10} e^{-4\pi} + \int_2^{\infty} x^{10} e^{-\pi x^2}\, dx \\ &\leqslant 0.00357\ldots + 0.00042\ldots \\ &< 0.0041. \end{aligned} \tag{3.13}$$

Thus, for $0.09 \leqslant t \leqslant 0.12$, inequalities (3.12) and (3.13) yield the following upper bound

$$|\Phi_1'''(t)| \leqslant 10\pi^5 e^{21(0.12)}(0.0041) < 156\,. \tag{3.14}$$

Next, we find a lower bound for $a_1'''(t)$ (see (3.7)) on the interval $I_2$. The zeros of the polynomial $p_4(y)$ (cf. (3.9)) are [CV1, p. 187]): $y_1 := 0.138\ldots$, $y_2 := 0.981\ldots$, $y_3 := 3.046\ldots$ and $y_4 := 7.083\ldots$, while the zeros of the derivative $p_4'(y)$ are $x_1 := 0.512\ldots$, $x_2 := 2.165\ldots$ and $x_3 := 5.759\ldots$. Hence, with the aid of the calculus we deduce that $p_4(y) > 0$ on the interval $(y_3, y_4)$ and that $p_4(y)$ is strictly increasing on $(x_2, x_3)$. Now for $t \in I_2$, $\pi e^{4t}$ lies in the interval $(4.5, 5.1)$. Also, $\exp\left(5t - \pi e^{4t}\right)$ is strictly decreasing on $I_2$ and so we obtain the following lower bound for $a_1'''(t)$ on $I_2$:

$$\begin{aligned} a_1'''(t) &= \pi p_4\left(\pi e^{4t}\right) \exp\left(5t - \pi e^{4t}\right) \\ &> \pi p_4(4.5) \exp\left(5(0.12) - \pi e^{4(0.12)}\right) > 263\,. \end{aligned} \tag{3.15}$$

Combining the inequalities (3.14) and (3.15) we have

$$\begin{aligned}\Phi'''(t) &> a_1'''(t) - \max_{I_2} \left|\Phi_1'''(t)\right| \qquad (t \in I_2)\\ &> 263 - 156 > 0\,.\end{aligned}$$

Since $g'(t) = t\Phi'''(t) > 0$ for $t \in I_2$ and since $g(0.09) > 0$ (cf. Lemma 3.1), we conclude that $g(t) > 0$ on $I_2$. □

Finally, to complete the proof of Theorem 2.12, it remains to show that $g(t) > 0$ on the unbounded interval $I_3$ (cf. (3.3)).

LEMMA 3.3. *With $g(t)$ defined by* (3.2), *we have*

$$g(t) > 0 \quad for \quad t \in I_3 = [0.12, \infty)\,.$$

*Proof.* By the properties of $\Phi(t)$ (cf. Theorem 2.1) we can readily justify the term–by–term differentiation of the series (2.1) and thus we have for $t \in \mathbb{R}$

$$\begin{aligned}g(t) &= t\Phi''(t) - \Phi'(t)\\ &= \sum_{n=1}^{\infty} t\,a_n''(t) - a_n'(t), \end{aligned} \tag{3.16}$$

where $a_n(t)$ is defined by (2.2). Next, a calculation shows that, with $y := \pi n^2 e^{4t}$,

$$ta_n''(t) - a_n'(t) = \pi n^2 \exp(5t - y) R(y)\,, \tag{3.17}$$

where

$$R(y) = tp_3(y) + (-p_2(y))\,, \tag{3.18}$$

$$p_3(y) = 32y^3 - 224y^2 + 330y - 75 \tag{3.19}$$

and

$$-p_2(y) = 8y^2 - 30y + 15\,. \tag{3.20}$$

Now the zeros of $p_3(y)$ are $x_1 := 0.277\ldots$, $x_2 := 1.67\ldots$ and $x_3 := 5.049\ldots$ while the zeros of $p_2(y)$ are $t_1 := 0.594\ldots$ and $t_2 := 3.15\ldots$. Hence, $p_3(y) > 0$ and $-p_2(y) > 0$ for $y \geqslant x_3 = 5.045\ldots$ and so we see that (cf. (3.18)) $R(y) > 0$ for all $y \geqslant x_3$. Since $y = \pi n^2 e^{4t} \geqslant x_3 = 5.049\ldots$ for all $n \geqslant 1$ and for $t \geqslant \frac{1}{4}\log\left(\frac{x_3}{\pi}\right) = 0.118\ldots$, it follows that (cf. (3.17)) $ta_n''(t) - a_n'(t) \geqslant 0$ for $n \geqslant 1$ and $t \geqslant 0.118\ldots$ . This together with (3.16) shows that $g(t) > 0$ for $t \in I_3$, and so the proof of the lemma is complete. □

In conclusion, by Lemmas 3.1–3.3, $g(t) > 0$ (cf. (3.2)) for $t > 0$ and hence $\Phi(\sqrt{t})$ is convex for $t > 0$. This completes the proof of Theorem 2.12.

## 4. Questions and open problems

Throughout this section, $\Phi(t)$ will denote the kernel defined by (2.1). The analysis of the properties of $\Phi(t)$ and its Fourier transform has led to many unanswered questions and open problems. Here we will confine our attention to only a few of those problems which seem interesting or significant (in the sense that the solution of the problem may shed some light on the distribution of zeros of certain entire functions related to the Riemann $\xi$–function) or may even be tractable.

*Problem 1.* Let

$$\begin{aligned} H_R(x) := &\int_0^R \Phi(t)\cos(xt)dt \quad (R > 0) \\ = &\sum_{k=0}^{\infty} \frac{(-1)^k \hat{b}_k(R)}{(2k)!} x^{2k}. \qquad \square \end{aligned} \tag{4.1}$$

Is there a positive number $R$ for which the entire function $H_R(x)$ has *some nonreal* zeros?

*Remark.* For any $R$, $0 < R < \infty$, the entire function $H_R(x)$ has at most a *finite* number of nonreal zeros (cf. [CSV, proof of Corollary 2.7]). Since $\Phi''(t) < 0$ on the interval $[0, 0.11]$, it follows that for $0 < R \leqslant 0.11$, $H_R(x)$ has only real zeros [CV2, Theorem 3.6].

*Problem 2.* We have seen in Section 2 (cf. Theorem 2.8 and Corollary 2.9) that the Turán inequalities associated with $H_R(x)$ (cf. (4.11)), that is,

$$\hat{b}_m^2(R) \geqslant \frac{2m-1}{2m+1}\hat{b}_{m-1}(R)\hat{b}_{m+1}(R) \quad (m = 1, 2, 3, \ldots; R > 0) \tag{4.2}$$

are satisfied. In order that $H_R(x)$ possess only real zeros, it must satisfy the stronger set of necessary conditions known as the *Laguerre inequalities* [CVV, p. 122]:

$$L_p[H_R](x) := \left(H_R^{(p)}(x)\right)^2 - H_R^{(p-1)}(x)H_R^{(p+1)}(x) \geqslant 0 \tag{4.3}$$

for all $x \in \mathbb{R}$ and $p = 1, 2, 3, \ldots$. Does $H_R(x)$ satisfy the Laguerre inequalities? For $R = \infty$, set $H(x) := H_\infty(x)$. Then is it true that

$$L_1[H](x) \geqslant 0 \quad \text{for all} \quad x \in \mathbb{R}? \tag{4.4}$$

*Remark.* The verification of the special case (4.4) itself would be significant (see [CSV, Theorems I and III]). Of course, should inequality (4.4) fail to hold for some $x$, then the Riemann Hypothesis would be false. We remark that since for $|x| < 1.09\cdots \times 10^9$ the zeros of $H(x)$ are real, it follows that [CSV, p. 398] with $p = 1$ the Laguerre inequalities (4.4) are satisfied for $|x| < 1.09\cdots \times 10^9$. Nevertheless, it is curious that to date so little progress has been made in proving the Laguerre inequalities (4.4) (cf. [CVV, p. 122]). Therefore, it would be desirable to discover a property of the kernel $\Phi(t)$ (if there is one) which will imply the inequalities (4.4), perchance in the spirit of the proof of Theorem 2.8. Let

$$K(t) := \int_{-\infty}^{\infty} \Phi(t+s)\Phi(t-s)s^2\, ds. \tag{4.5}$$

Then by virtue of the properties of $\Phi$ (cf. Theorem 2.1) it is not difficult to see that $\Phi(t)$ is an even integrable function. Since $\log K(\sqrt{t})$ is concave for $t > 0$, $\log \Phi(t)$ is concave for $t > 0$ (cf. (2.8)) and so it follows that $K'(t) < 0$ for $t > 0$. In addition, using standard Fourier transform techniques, we can establish that

$$L_1[H](\frac{x}{2}) = 2\int_0^\infty K(t)\cos(xt)dt. \tag{4.6}$$

*Problem 3.* According to Corollary 2.13,

$$F(x) := \int_0^\infty \Phi(\sqrt{t})\cos(xt)dt > 0 \quad \text{for all} \quad x \in \mathbb{R}. \tag{4.7}$$

What can be said about the location of the (necessarily nonreal) zeros of $F(x)$?

*Problem 4.* In Section 2 we noted that $\Phi(t) = c_0 + c_1t^2 + c_2t^4 + \dots \left(|t| < \frac{\pi}{8}\right)$ is an admissible kernel. Thus, $\Phi(\sqrt{t}) = c_0 + c_1 t + c_2t^2 + \dots$ is analytic for $|t| < \left(\frac{\pi}{8}\right)^2$ and consequently the Mellin transform of $\Phi(\sqrt{t})$, that is,

$$M(z) := \int_0^\infty t^{z-1}\Phi(\sqrt{t})dt \tag{4.8}$$

represents (via the Prym decomposition) a meromorphic function whose only poles are the simple poles $z = -k$ $(k = 0, 1, 2, \dots)$. Does $M(z)$ have only real negative zeros?

*Remark.* In a beautiful paper, Pólya [P2, Satz II] gave, using the theory of multiplier sequences, an elegant proof of a general result from which we can deduce that if $M(z)$ has only real negative zeros, then $\int_0^\infty \Phi(t)\cos(xt)dt$ has only real zeros!

*Problem 5.* We know that $\Phi(t)$ is a Pólya frequency function of order 2 (cf. Corollary 2.11). Is $\Phi(t)$ a Pólya frequency function of order $k$, for some $k > 2$?

*Remark.* For a comprehensive treatment of Pólya frequency functions (of any order) we refer to Karlin [K].

*Problem 6.* Conrey and Gosh [CG] have shown that the Taylor coefficients of certain generalized $\xi$–functions (for example, with the notation in [T, p. 16], $\xi\left(\frac{k}{4} + it\right)$, where $k$ is a positive parameter) satisfy the Turán inequalities. These functions can also be represented as the Fourier transform of certain kernels $K(t)$ (see [CG, p. 411]). Which of the convexity properties investigated in this paper remain valid for the kernels $K(t)$? What are the de Bruijn–Newman constants (for the definition see, for example, [CNV2]) associated with these $\xi$–functions? Do these entire functions satisfy the Laguerre inequalities (4.3)?

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