On an inequality of Grüss type

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Abstract. *We prove an inequality of Grüss type for p-norm, which for $p = \infty$ gives an estimate similar to a result of Pachpatte [2].*

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1 Introduction

For differentiable functions $f, g\colon [a,b] \to \mathbb{R}$, $a \cdot b > 0$, Pachpatte has in [2] proved, using Pompeiu's mean value theorem [4], the following Grüss-like inequality:

$$\left| \int_a^b f(x)g(x)\,\mathrm{d}x - \tfrac{1}{b^2-a^2}\left(\int_a^b f(x)\,\mathrm{d}x \cdot \int_a^b xg(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x \cdot \int_a^b xf(x)\,\mathrm{d}x \right) \right|$$
$$\leq \|f - \iota f'\|_\infty \int_a^b |g(x)| \left|\tfrac{1}{2} - \tfrac{x}{a+b}\right| \mathrm{d}x + \|g - \iota g'\|_\infty \int_a^b |f(x)| \left|\tfrac{1}{2} - \tfrac{x}{a+b}\right| \mathrm{d}x$$

where $\iota(t) = t$, $t \in [a,b]$.

We are going to prove an estimate for the left-hand side using the p-norm, $1 \leq p \leq \infty$, which will for $p = \infty$ give a result similar to the one in [2].

2 The main result

Before stating the main result, we will introduce some notation and state without proof an elementary lemma (see also [3] for some less obvious details of the proof):

Lemma 1. *For $\frac{1}{p} + \frac{1}{q} = 1$, $1 \leq p, q \leq \infty$, and $0 < a \leq x \leq b$, denote*

$$A(x,p) := \left(\int_a^x \left(\int_t^x \frac{t^q}{u^{2q}}\,\mathrm{d}u \right) \mathrm{d}t \right)^{\frac{1}{q}} + \left(\int_x^b \left(\int_x^t \frac{t^q}{u^{2q}}\,\mathrm{d}u \right) \mathrm{d}t \right)^{\frac{1}{q}}, \tag{1}$$

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where for $p = 1$, i. e. $q = \infty$, the integrals are to be interpreted as the ∞-norms, i. e. as maxima of the function $(u,t) \mapsto \frac{t}{u^2}$ on the corresponding domains of integration. Then

$$A(x,p)=\left(\frac{a^{2-q}-x^{2-q}}{(1-2q)(2-q)}+\frac{x^{2-q}-a^{1+q}x^{1-2q}}{(1-2q)(1+q)}\right)^{\frac{1}{q}}+\left(\frac{b^{2-q}-x^{2-q}}{(1-2q)(2-q)}+\frac{x^{2-q}-b^{1+q}x^{1-2q}}{(1-2q)(1+q)}\right)^{\frac{1}{q}}$$

for $1 < p, q < \infty$, $p, q \neq 2$;

$$A(x,2) = \frac{1}{3}\left(\left(\ln\left(\frac{x}{a}\right)^3 + \frac{a^3}{x^3} - 1\right)^{\frac{1}{2}} + \left(\ln\left(\frac{x}{b}\right)^3 + \frac{b^3}{x^3} - 1\right)^{\frac{1}{2}}\right) = \lim_{p\to 2} A(x,p)\,;$$

$$A(x,\infty) = \frac{a^2+b^2}{2x} + x - (a+b) = \lim_{p\to\infty} A(x,p)\,;$$

$$A(x,1) = \frac{1}{a} + \frac{b}{x^2} = \lim_{p\to 1} A(x,p)\,.$$

Now we state the main result

Theorem 2. *Let the functions $f, g\colon [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) with $0 < a < b$. Then for $\frac{1}{p} + \frac{1}{q} = 1$, with $1 \le p, q \le \infty$, the following inequality holds:*

$$\begin{aligned}&\left|\int_a^b f(x)g(x)\,\mathrm{d}x - \frac{1}{b^2-a^2}\Big(\int_a^b f(x)\,\mathrm{d}x\cdot\int_a^b xg(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x\cdot\int_a^b xf(x)\,\mathrm{d}x\Big)\right|\\ &\le \frac{(b-a)^{\frac{1}{p}}}{b^2-a^2}\left(\|f-\iota f'\|_p\int_a^b x\,|g(x)|\,A(x,p)\,\mathrm{d}x + \|g-\iota g'\|_p\int_a^b x\,|f(x)|\,A(x,p)\,\mathrm{d}x\right) \quad (2)\end{aligned}$$

where $\iota(t) = t$, $t \in [a,b]$, and $A(x,p)$ is as in Lemma 1.

First we prove a lemma:

Lemma 3. *Let the function $\varphi\colon [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) with $a \cdot b > 0$. Then*

$$t\,\varphi(x) - x\,\varphi(t) = x\,t\int_x^t \big(\varphi(u) - u\,\varphi'(u)\big)\,\frac{1}{u^2}\,\mathrm{d}u \quad (3)$$

for all $x, t \in [a,b]$.

Proof. Define $\Phi\colon [1/b, 1/a] \to \mathbb{R}$ by $\Phi(t) := t\,\varphi(\frac{1}{t})$. The function Φ is continuous and differentiable on $(1/b, 1/a)$, and for all $x_1, x_2 \in [1/b, 1/a]$ we have

$$\begin{aligned}\Phi(x_1) - \Phi(x_2) &= \int_{x_2}^{x_1} \Phi'(t)\,\mathrm{d}t = \int_{x_2}^{x_1}\left(\varphi\Big(\frac{1}{t}\Big) - \frac{1}{t}\,\varphi'\Big(\frac{1}{t}\Big)\right)\mathrm{d}t \qquad \left[u := \frac{1}{t}\right]\\ &= -\int_{1/x_2}^{1/x_1}\big(\varphi(u) - u\varphi'(u)\big)\,\frac{1}{u^2}\,\mathrm{d}u\,. \quad (4)\end{aligned}$$

Denote $x_1 =: 1/x$ and $x_2 =: 1/t$. Then for all $x, t \in [a,b]$ from (4) we get

$$\frac{1}{x}\,\varphi(x) - \frac{1}{t}\,\varphi(t) = \int_x^t \big(\varphi(u) - u\varphi'(u)\big)\,\frac{1}{u^2}\,\mathrm{d}u$$

which gives (3) and proves the lemma. □

Proof of Theorem 2. By *Lemma 3* we have

$$t\,f(x)-x\,f(t)=x\,t\int_x^t\big(f(u)-u\,f'(u)\big)\frac{1}{u^2}\,\mathrm{d}u$$

$$t\,g(x)-x\,g(t)=x\,t\int_x^t\big(g(u)-u\,g'(u)\big)\frac{1}{u^2}\,\mathrm{d}u\,.$$

Multiplying these identities by $g(x)$ and $f(x)$ respectively, and adding the results, gives

$$\begin{aligned}&2\,tf(x)\,g(x)-x\,g(x)\,f(t)-xf(x)\,g(t)\\&\quad=x\,g(x)\,t\int_x^t\big(f(u)-u\,f'(u)\big)\frac{1}{u^2}\,\mathrm{d}u+x\,f(x)\,t\int_x^t\big(g(u)-u\,g'(u)\big)\frac{1}{u^2}\,\mathrm{d}u\,,\end{aligned}$$

which we integrate on t and obtain

$$\begin{aligned}&(b^2-a^2)\,f(x)\,g(x)-x\,g(x)\int_a^b f(t)\,\mathrm{d}t-xf(x)\int_a^b g(t)\,\mathrm{d}t\\&\quad=x\,g(x)\int_a^b\Big(\int_x^t\big(f(u)-uf'(u)\big)\frac{t}{u^2}\,\mathrm{d}u\Big)\,\mathrm{d}t\\&\qquad+xf(x)\int_a^b\Big(\int_x^t\big(g(u)-u\,g'(u)\big)\frac{t}{u^2}\,\mathrm{d}u\Big)\,\mathrm{d}t\,.\end{aligned}$$

Integrating the last identity on x and taking the absolute value, gives

$$\begin{aligned}&\Big|(b^2-a^2)\int_a^b f(x)\,g(x)\,\mathrm{d}x-\int_a^b xg(x)\,\mathrm{d}x\cdot\int_a^b f(t)\,\mathrm{d}t-\int_a^b xf(x)\,\mathrm{d}x\cdot\int_a^b g(t)\,\mathrm{d}t\Big|\\&\quad\le\Big|\int_a^b\Big(xg(x)\int_a^b\Big(\int_x^t\big(f(u)-uf'(u)\big)\frac{t}{u^2}\,\mathrm{d}u\Big)\,\mathrm{d}t\Big)\,\mathrm{d}x\Big|\\&\qquad+\Big|\int_a^b\Big(xf(x)\int_a^b\Big(\int_x^t\big(g(u)-ug'(u)\big)\frac{t}{u^2}\,\mathrm{d}u\Big)\,\mathrm{d}t\Big)\,\mathrm{d}x\Big|\,.\end{aligned}\tag{5}$$

For the first addend on the right-hand side of (5) we have

$$\begin{aligned}&\Big|\int_a^b\Big(xg(x)\int_a^b\Big(\int_x^t\big(f(u)-uf'(u)\big)\frac{t}{u^2}\,\mathrm{d}u\Big)\,\mathrm{d}t\Big)\,\mathrm{d}x\Big|\\&\qquad\le\int_a^b\Big(\Big|xg(x)\Big|\cdot\Big|\int_a^b\Big(\int_x^t\big(f(u)-uf'(u)\big)\frac{t}{u^2}\,\mathrm{d}u\Big)\,\mathrm{d}t\Big|\Big)\,\mathrm{d}x\,,\end{aligned}\tag{6}$$

and for the second factor under the integral on x (i.e. the outermost integral) in (6), we find

$$\begin{aligned}&\Big|\int_a^b\Big(\int_x^t\big(f(u)-uf'(u)\big)\frac{t}{u^2}\,\mathrm{d}u\Big)\,\mathrm{d}t\Big|\\&\le\int_a^b\Big|\int_x^t\Big|\big(f(u)-uf'(u)\big)\,\frac{t}{u^2}\Big|\,\mathrm{d}u\Big|\,\mathrm{d}t\\&=\int_a^x\Big|\int_t^x\Big|\big(f(u)-uf'(u)\big)\,\frac{t}{u^2}\Big|\,\mathrm{d}u\Big|\,\mathrm{d}t+\int_x^b\Big|\int_x^t\Big|\big(f(u)-uf'(u)\big)\frac{t}{u^2}\Big|\,\mathrm{d}u\Big|\,\mathrm{d}t\,.\end{aligned}\tag{7}$$

Application of the Hölder's inequality shows that the last line in (7) is

$$\begin{aligned}
&\le \Big(\int_a^x\Big(\int_t^x \big|f(u)-uf'(u)\big|^p\,\mathrm{d}u\Big)\,\mathrm{d}t\Big)^{\frac1p}\cdot\Big(\int_a^x\Big(\int_t^x \frac{t^q}{u^{2q}}\,\mathrm{d}u\Big)\,\mathrm{d}t\Big)^{\frac1q}\\
&\quad+\Big(\int_x^b\Big(\int_x^t \big|f(u)-uf'(u)\big|^p\,\mathrm{d}u\Big)\,\mathrm{d}t\Big)^{\frac1p}\cdot\Big(\int_x^b\Big(\int_x^t \frac{t^q}{u^{2q}}\,\mathrm{d}u\Big)\,\mathrm{d}t\Big)^{\frac1q}\\
&\le \Big(\int_a^b\Big(\int_a^b \big|f(u)-uf'(u)\big|^p\,\mathrm{d}u\Big)\,\mathrm{d}t\Big)^{\frac1p}\\
&\quad\times\bigg(\Big(\int_a^x\Big(\int_t^x \frac{t^q}{u^{2q}}\,\mathrm{d}u\Big)\,\mathrm{d}t\Big)^{\frac1q}+\Big(\int_x^b\Big(\int_x^t \frac{t^q}{u^{2q}}\,\mathrm{d}u\Big)\,\mathrm{d}t\Big)^{\frac1q}\bigg). \qquad (8)
\end{aligned}$$

The first factor in (8) equals $(b-a)^{\frac1p}\,\|f-\iota\, f'\|_p$, and, by *Lemma 1*, the second factor equals $A(x,p)$. Plugging this into (6) shows that for the first addend on the right-hand side in (5) we have

$$\begin{aligned}
&\Big|\int_a^b\Big(xg(x)\int_a^b\Big(\int_x^t\big(f(u)-uf'(u)\big)\frac{t}{u^2}\,\mathrm{d}u\Big)\,\mathrm{d}t\Big)\,\mathrm{d}x\Big|\\
&\qquad\le\int_a^b\big|xg(x)\big|\cdot(b-a)^{\frac1p}\,\|f-\iota\, f'\|_p\cdot A(x,p)\,\mathrm{d}x\\
&\qquad=(b-a)^{\frac1p}\,\|f-\iota\, f'\|_p\cdot\int_a^b x\,\big|g(x)\big|\,A(x,p)\,\mathrm{d}x\,. \qquad (9)
\end{aligned}$$

An analogous inequality holds for the second addend in (5), so putting these two inequalities into (5) and dividing by b^2-a^2 gives the required inequality (2), proving the theorem. □

Inequality (2) has a particularly simple form in case $p=1$, so we state it as a corollary:

Corollary 4. *Let the functions $f,g\colon[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) with $0<a<b$. Then the following inequality holds:*

$$\begin{aligned}
&\Big|\int_a^b f(x)g(x)\,\mathrm{d}x-\frac{1}{b^2-a^2}\Big(\int_a^b f(x)\,\mathrm{d}x\cdot\int_a^b xg(x)\,\mathrm{d}x+\int_a^b g(x)\,\mathrm{d}x\cdot\int_a^b xf(x)\,\mathrm{d}x\Big)\Big|\\
&\le\frac{1}{a+b}\Big(\|f-\iota f'\|_1\int_a^b|g(x)|\Big(\frac{x}{a}+\frac{b}{x}\Big)\,\mathrm{d}x+\|g-\iota g'\|_1\int_a^b|f(x)|\Big(\frac{x}{a}+\frac{b}{x}\Big)\,\mathrm{d}x\Big)
\end{aligned}$$

where $\iota(t)=t$, $t\in[a,b]$.

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