Generalized product of fuzzy subgroups and t-level subgroups

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Abstract. *Ray (Fuzzy Sets and Systems 105(1999)181-183) studied some results of the product of two fuzzy subsets and fuzzy subgroups. In this paper, Ray's results will be generalized. Furthermore, we define a t-level subset and t-level subgroups, and then we study some of their properties.*

Key words: *fuzzy subgroup, level subset, level subgroup, t-fuzzy subgroup, product of fuzzy subset*

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1. Introduction

The concept of fuzzy sets was introduced by Zadeh [11]. Since its inception, the theory of fuzzy sets has developed in many directions and is finding applications in a wide variety of fields. In [8] Rosenfeld used this concept to develop the theory of fuzzy groups. In fact, many basic properties in group theory are found to be carried over to fuzzy groups. Anthony and Sherwood [1] redefined fuzzy subgroups in terms of a t-norm which replaced the minimum operation and they characterized basic properties of t-fuzzy subgroups in [1, 2]. Sherwood [10] defined products of fuzzy subgroups using t-norms and gave some properties of these products.

In this work, we first generalize the results of the product of two fuzzy subsets and fuzzy subgroups which were done by Ray in [7]. We also define a t-level subset and t-level subgroups, and then we study some of their properties.

2. Preliminaries

We record here some basic concepts and clarify notions used in the sequel.

Definition 1 [see [8]]. *A fuzzy subset A of a group G is said to be a fuzzy subgroup of G if for all x, y in G*

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1. $A(xy) \geq min(A(x), A(y))$ *and*

2. $A(x^{-1}) \geq A(x)$,

where the product of x *and* y *is denoted by* xy *and the inverse of* x *by* x^{-1}. It is well known and easy to see that a fuzzy subgroup G satisfies $A(x) \leq A(e)$ and $A(x^{-1}) = A(x)$ for all $x \in G$, where e is the identity element of G.

Definition 2 [see [9]]. *A triangular norm (briefly a t-norm) is a function* $T : [0,1] \times [0,1] \to [0,1]$ *satisfying, for each* p, q, r, s *in* $[0,1]$,

1. $T(p, 1) = p$,

2. $T(p, q) \leq T(r, s)$ *if* $p \leq r$ *and* $q \leq s$,

3. $T(p, q) = T(q, p)$,

4. $T(p, T(q, r)) = T(T(p, q), r))$.

Definition 3 [see [1]]. *Let* S *be a groupoid and* T *a t-norm. A function* $B : S \to [0,1]$ *is a subgroupoid of* S *iff for every* x, y *in* S, $B(xy) \geq T(B(x), B(y))$. *If* S *is a group, a t-fuzzy subgroupoid* B *is a t-fuzzy subgroup of* S *iff for each* $x \in S$, $B(x^{-1}) \geq B(x)$.

Definition 4 [see [6]]. *For each* $i = 1, 2, \ldots, n$, *let* G_i *be a t-fuzzy subgroup in a group* X_i. *Let* T *be a* T*-norm. The* T*-product of* G_i $(i = 1, 2, \ldots, n)$ *is the function* $G_1 \times G_2 \times \cdots \times G_n : X_1 \times X_2 \times \cdots \times X_n \to [0,1]$ *defined by*

$$(G_1 \times G_2 \times \cdots \times G_n)(x_1, x_2, \ldots, x_n) = T(G_1(x_1), G_2(x_2), \ldots, G_n(x_n)).$$

Definition 5 [see [7]]. *For each* $i = 1, 2, \ldots, n$, *let* G_i *be a fuzzy subgroup under a minimum operation in a group* X_i. *The membership function of the product* $G = G_1 \times G_2 \times \cdots \times G_n$ *in* $X = X_1 \times X_2 \times \cdots \times X_n$ *is defined by*

$$(G_1 \times G_2 \times \cdots \times G_n)(x_1, x_2, \ldots, x_n) = min(G_1(x_1), G_2(x_2), \ldots, G_n(x_n)).$$

Definition 6 [see [4]]. *Let* A *be a fuzzy subset of a set* S *and let* $t \in [0,1]$. *The set* $A_t = \{x \in S : A(x) \geq t\}$ *is called a level subset of* A.

Definition 7 [see [5]]. *A fuzzy subgroup* A *of a group* G *is called fuzzy normal if for all* x, y *in* G *it fulfils the following condition:*

$$A(xy) = A(yx).$$

Definition 8 [see [5]]. *A fuzzy subgroup* A *of a group* G *is said to be conjugate to a fuzzy subgroup* B *of* G *if there exists* x *in* G *such that for all* g *in* G

$$A(g) = B(x^{-1}gx).$$

3. Product of fuzzy subgroup and level subgroups

Here we state and prove some results which are generalizid results given in [7].

Theorem 1. *Let $A_1, A_2, \dots, A_n$ be fuzzy subsets of the sets $G_1, G_2, \dots, G_n$, respectively, and let $t \in [0,1]$. Then $(A_1 \times A_2 \times \dots \times A_n)_t = A_{1_t} \times A_{2_t} \times \dots \times A_{n_t}$.*

Proof. For the element $(a_1, a_2, \dots, a_n) \in (A_1 \times A_2 \times \dots \times A_n)_t$, using *Definition 6* we can write $(A_1 \times A_2 \times \dots \times A_n)(a_1, a_2, \dots, a_n) = min(A_1(a_1), A_2(a_2), \dots, A_n(a_n)) \geq t$. This gives us $A_1(a_1) \geq t, A_2(a_2) \geq t, \dots, A_n(a_n) \geq t$ and $a_1 \in A_{1_t}, a_2 \in A_{2_t}, \dots, a_n \in A_{n_t}$. Thus $(a_1, a_2, \dots, a_n) \in A_{1_t} \times A_{2_t} \times \dots \times A_{n_t}$. Let $(a_1, a_2, \dots, a_n) \in A_{1_t} \times A_{2_t} \times \dots \times A_{n_t}$. Then $a_i \in A_{i_t}$, for $i = 1, 2, \dots, n$, $A_1(a_1) \geq t, A_2(a_2) \geq t, \dots, A_n(a_n) \geq t$. That is, $min(A_1(a_1), A_2(a_2), \dots, A_n(a_n)) \geq t$. This gives us $(a_1, a_2, \dots, a_n) \in (A_1 \times A_2 \times \dots \times A_n)_t$. Finally we get $(A_1 \times A_2 \times \dots \times A_n)_t$ $= A_{1_t} \times A_{2_t} \times \dots \times A_{n_t}$. □

Theorem 2. *Let $A_1, A_2, \dots, A_n$ be fuzzy subgroups of the groups $G_1, G_2, \dots, G_n$, respectively. If $A_1, A_2, \dots, A_n$ are fuzzy normal, then $A_1 \times A_2 \times \dots \times A_n$ is fuzzy normal.*

Proof. Firstly, we must show that $A_1 \times A_2 \times \dots \times A_n$ is a fuzzy subgroup of $G_1 \times G_2 \times \dots \times G_n$. For all elements $(x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n) \in G_1 \times G_2 \times \dots \times G_n$. We get

$$\begin{aligned}
&(A_1 \times A_2 \times \dots \times A_n)((x_1, x_2, \dots, x_n)(y_1, y_2, \dots, y_n)) \\
&= (A_1 \times A_2 \times \dots \times A_n)(x_1y_1, x_2y_2, \dots, x_ny_n) \\
&= min(A_1(x_1y_1), A_2(x_2y_2), \dots, A_n(x_ny_n)) \\
&\geq min(min(A_1(x_1), A_1(y_1)), min(A_2(x_2), A_2(y_2)), \dots, min(A_n(x_n), A_n(y_n))) \\
&= min(min(A_1(x_1), A_2(x_2), \dots, A_n(x_n)), min(A_1(y_1), A_2(y_2), \dots, A_n(y_n))) \\
&= min((A_1 \times A_2 \times \dots \times A_n)(x_1, x_2, \dots, x_n), (A_1 \times A_2 \times \dots \times A_n)(y_1, y_2, \dots, y_n)).
\end{aligned}$$

Also,

$$\begin{aligned}
&(A_1 \times A_2 \times \dots \times A_n)((x_1, x_2, \dots, x_n)^{-1}) \\
&= (A_1 \times A_2 \times \dots \times A_n)(x_1^{-1}, x_2^{-1}, \dots, x_n^{-1}) \\
&= min(A_1(x_1^{-1}), A_2(x_2^{-1}), \dots, A_n(x_n^{-1})) \\
&= min(A_1(x_1), A_2(x_2), \dots, A_n(x_n)) \\
&= (A_1 \times A_2 \times \dots \times A_n)(x_1, x_2, \dots, x_n).
\end{aligned}$$

Thus $A_1 \times A_2 \times \dots \times A_n$ is a fuzzy subgroup of $G_1 \times G_2 \times \dots \times G_n$. Now, let us show that $A_1 \times A_2 \times \dots \times A_n$ is a fuzzy normal. For $(x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n) \in G_1 \times G_2 \times \dots \times G_n$,

$$\begin{aligned}
&(A_1 \times A_2 \times \dots \times A_n)((x_1, x_2, \dots, x_n)(y_1, y_2, \dots, y_n)) \\
&= min(A_1(x_1y_1), A_2(x_2y_2), \dots, A_n(x_ny_n)) \\
&= min(A_1(y_1x_1), A_2(y_2x_2), \dots, A_n(y_nx_n)) \\
&= (A_1 \times A_2 \times \dots \times A_n)((y_1, y_2, \dots, y_n)(x_1, x_2, \dots, x_n)).
\end{aligned}$$

Thus $A_1 \times A_2 \times \dots \times A_n$ is fuzzy normal. □

Theorem 3. *Let fuzzy subgroups $A_1, A_2, \dots, A_n$ of groups $G_1, G_2, \dots, G_n$ conjugate to fuzzy subgroups $B_1, B_2, \dots, B_n$ of groups $G_1, G_2, \dots, G_n$, respectively. Then the fuzzy subgroup $A_1 \times A_2 \times \cdots \times A_n$ of the group $G_1 \times G_2 \times \cdots \times G_n$ is conjugate to the fuzzy subgroup $B_1 \times B_2 \times \cdots \times B_n$ of $G_1 \times G_2 \times \cdots \times G_n$.*

Proof. By *Definition 8*, if a fuzzy subgroup of G_i conjugates to a fuzzy subgroup B_i of G_i, then there exists x_i in G_i such that for all g_i in G_i, $A_i(g_i) = B_i(x_i^{-1} g_i x_i)$, for $i = 1, 2, \dots, n$. Thus we have

$$\begin{aligned} &(A_1 \times A_2 \times \cdots \times A_n)(g_1, g_2, \dots, g_n) \\ &= min(A_1(g_1), A_2(g_2), \dots, A_n(g_n)) \\ &= min(B_1(x_1^{-1} g_1 x_1), B_2(x_2^{-1} g_2 x_2), \dots, B_n(x_n^{-1} g_n x_n)) \\ &= (B_1 \times B_2 \times \cdots \times B_n)(x_1^{-1} g_1 x_1, x_2^{-1} g_2 x_2, \dots, x_n^{-1} g_n x_n). \end{aligned}$$

□

Theorem 4. *Let $A_1, A_2, \dots, A_n$ be fuzzy subsets of the groups $G_1, G_2, \dots, G_n$, respectively. Suppose that $e_1, e_2, \dots, e_n$ are identity elements of $G_1, G_2, \dots, G_n$, respectively. If $A_1 \times A_2 \times \cdots \times A_n$ is a fuzzy subgroup of $G_1 \times G_2 \times \cdots \times G_n$, then for at least one $i = 1, 2, \dots, n$, the following statement must hold*

$$(A_1 \times \cdots \times A_{i-1} \times A_{i+1} \times \cdots \times A_n)(e_1, \dots, e_{i-1}, e_{i+1}, \dots, e_n) \geq A_i(x_i) \qquad (1)$$

for all $x_i \in G_i$.

Proof. Let $A_1 \times A_2 \times \cdots \times A_n$ be a fuzzy subgroup of $G_1 \times G_2 \times \cdots \times G_n$. By contraposition, suppose that, for non of $i = 1, 2, \dots, n$, the statement (1) holds. Then we can find $a_1, a_2, \dots, a_n$ in $G_1, G_2, \dots, G_n$, respectively, such that

$$A_i > (A_1 \times \cdots \times A_{i-1} \times A_{i+1} \times \cdots \times A_n)(e_1, \dots, e_{i-1}, e_{i+1}, \dots, e_n).$$

We have

$$\begin{aligned} &(A_1 \times A_2 \times \cdots \times A_n)(a_1, a_2, \dots, a_n) \\ &= min(A_1(a_1), A_2(a_2), \dots, A_n(a_n)) \\ &> min_i((A_1 \times \cdots \times A_{i-1} \times A_{i+1} \times \cdots \times A_n)(e_1, \dots, e_{i-1}, e_{i+1}, \dots, e_n)) \\ &= min_i(min(A_1(e_1), \dots, A_{i-1}(e_{i-1}), A_{i+1}(e_{i+1}) \dots, A_n(e_n))) \\ &= min(A_1(e_1), A_2(e_2), \dots, A_n(e_n)) \\ &= (A_1 \times A_2 \times \cdots \times A_n)(e_1, e_2, \dots, e_n). \end{aligned}$$

Thus $A_1 \times A_2 \times \cdots \times A_n$ is not a fuzzy subgroup of $G_1 \times G_2 \times \cdots \times G_n$. Hence for at least one $i = 1, 2, \dots, n$, the inequality

$$A_i > (A_1 \times \cdots \times A_{i-1} \times A_{i+1} \times \cdots \times A_n)(e_1, \dots, e_{i-1}, e_{i+1}, \dots, e_n)$$

is satisfied for all x_i in G_i. □

Theorem 5. *Let $A_1, A_2 \dots, A_n$ be fuzzy subsets of the groups $G_1, G_2 \dots, G_n$, respectively, such that*

$$A_i(x_i) \leq (A_1 \times \cdots \times A_{i-1} \times A_{i+1} \times \cdots \times A_n)(e_1, \dots, e_{i-1}, e_{i+1}, \dots, e_n)) \qquad (2)$$

for all x_i in G_i, e_i being the identity element of G_i. If $A_1 \times A_2 \times \cdots \times A_n$ is a subgroup of $G_1 \times G_2 \times \cdots \times G_n$, then A_i is a fuzzy subgroup of G_i.

Proof. Let $A_1, \times A_2 \times \cdots \times A_n$ be a fuzzy subgroup of $G_1 \times G_2 \times \cdots \times G_n$ and x_i, y_i in G_i. Then $(e_1, \ldots, e_{i-1}, x_i, e_{i+1}, \ldots, e_n), e_1, \ldots, e_{i-1}, y_i, e_{i+1}, \ldots, e_n) \in G_1 \times G_2 \times \cdots \times G_n$. Now, using (2), we get

$$\begin{aligned} A_i(x_i y_i) &= min(A_i(x_i y_i), (A_1 \times \cdots \times A_{i-1} \times A_{i+1} \times \cdots \times A_n) \\ &\quad (e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_n)(e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_n)) \\ &= (A_1 \times \cdots \times A_i \times \cdots \times A_n)((e_1, \ldots, x_i, \ldots, e_n)(e_1, \ldots, y_i, \ldots, e_n)) \\ &\geq min((A_1 \times \cdots \times A_i \times \cdots \times A_n)(e_1, \ldots, x_i, \ldots, e_n), \\ &\quad (A_1 \times \cdots \times A_i \times \cdots \times A_n)(e_1, \ldots, y_i, \ldots, e_n)) \\ &= min(min(A_i(x_i), (A_1 \times \cdots \times A_{i-1} \times A_{i+1} \times \cdots \times A_n) \\ &\quad (e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_n)), \\ &\quad min(A_i(y_i), (A_1 \times \cdots \times A_{i-1} \times A_{i+1} \times \cdots \times A_n) \\ &\quad (e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_n))) \\ &= min(A_i(x_i), A_i(y_i)). \end{aligned}$$

Also,

$$\begin{aligned} A_i(x_i^{-1}) &= min(A_i(x_i^{-1}), (A_1 \times \cdots \times A_{i-1} \times A_{i+1} \times \cdots \times A_n) \\ &\quad (e_1^{-1}, \ldots, e_{i-1}^{-1}, e_{i+1}^{-1}, \ldots, e_n^{-1})) \\ &= (A_1 \times \cdots \times A_i \times \cdots \times A_n)(e_1^{-1}, \ldots, x_i^{-1}, \ldots, e_n^{-1}) \\ &= A_1 \times \cdots \times A_i \times \cdots \times A_n)(e_1, \ldots, x_i, \ldots, e_n)^{-1} \\ &\geq A_1 \times \cdots \times A_i \times \cdots \times A_n)(e_1, \ldots, x_i, \ldots, e_n) \\ &= min(A_i(x_i), (A_1 \times \cdots \times A_{i-1} \times A_{i+1} \times \cdots \times A_n) \\ &\quad (e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_n)) \\ &= A_i(x_i). \end{aligned}$$

Hence A_i is a fuzzy subgroup of G. This completes the proof. □

Theorem 6. *Let $A_1, A_2, \ldots, A_n$ be fuzzy subsets of the groups $G_1, G_2, \ldots, G_n$, respectively, such that $(A_1 \times \cdots \times A_{i-1} \times A_{i+1} \times \cdots \times A_n)(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \leq A_i(e_i)$ for all $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in G_1 \times \cdots \times G_{i-1} \times G_{i+1} \times \cdots \times G_n$, e_i being the identity element of G_i. If $A_1 \times A_2 \times \cdots \times A_n$ is a fuzzy subgroup of $G_1 \times G_2 \times \cdots \times G_n$, then $A_1 \times \cdots \times A_{i-1} \times A_{i+1} \times \cdots \times A_n$ is a fuzzy subgroup of $G_1 \times \cdots \times G_{i-1} \times G_{i+1} \times \cdots \times G_n$.*

The proof of *Theorem 6* is proved similar to the proof of *Theorem 5*.

Remark 1. *The theorems in this section are generalizations of results of Ray [7].*

4. t-level subgroups

In this section, we introduce a definition of a t-level subset of a fuzzy subset and then we give some of its important algebraic results.

Definition 9. *Let A be a fuzzy subset of a set G, T a t-norm and $r \in [0,1]$. Then we define a t-level subset of a fuzzy subset A as*

$$A_r^T = \{x \in G : T(A(x), r) \geq r\}.$$

Theorem 7. *Let G be a group and A a t-fuzzy subgroup of G, then the t-level subset A_r^T, for $r \in [0,1], r \leq T(A(e), r)$, is a subgroup of G, where e is the identity of G.*

Proof. $A_r^T = \{x \in G : T(A(x), r) \geq r\}$ is clearly nonempty. Let $x, y \in A_r^T$, then $T(A(x), r) \geq r$ and $T(A(y), r) \geq r$. Since A is a t-fuzzy subgroup of G, $A(xy) \geq T(A(x), A(y))$ is satisfied. This means

$$T(A(xy), r) \geq T(T(A(x), A(y)), r) = T(A(x), T(A(y), r)) \geq T(A(x), r) \geq r.$$

Hence $xy \in A_r^T$. Again $x \in A_r^T$ implies $T(A(x), r) \geq r$. Since A is a t-fuzzy subgroup, $A(x^{-1}) = A(x)$ and hence $T(A(x^{-1}), r) = T(A(x), r) \geq r$. This means that $x^{-1} \in A_r^T$. Therefore A_r^T is a subgroup of G. □

Theorem 8. *Let G be a group and A a fuzzy subgroup of G, then the t-level subset A_r^T, for $r \in [0,1], r \leq T(A(e), r)$, is a subgroup of G, where e is the identity of G.*

Proof. $A_r^T = \{x \in G : T(A(x), r) \geq r\}$ is clearly nonempty. Let $x, y \in A_r^T$, then $T(A(x), r) \geq r$ and $T(A(y), r) \geq r$. Since A is a subgroup of G, $A(xy) \geq min(A(x), A(y))$ is satisfied. This means that $T(A(xy), r) \geq T(min(A(x), A(y), r)$, where there are two cases: $min(A(x), A(y)) = A(x)$ or $min(A(x), A(y)) = A(y)$. Since $x, y \in A_r^T$, also in two cases $T(min(A(x), A(y)), r) \geq r$. Therefore $T(A(xy), r) \geq r$. Thus we get $xy \in A_r^T$. It is easily seen that, as above, $x^{-1} \in A_r^T$. Hence A_r^T is a subgroup of G. □

Theorem 9. *Let G be a group and A be a fuzzy subset of G such that A_r^T is a subgroup of G for all $r \in [0,1]$, $r \leq T(A(x), r)$, then A is a t-fuzzy subgroup of G.*

Proof. Let $x, y \in G$ and let $T(A(x), r_1) = r_1$ and $T(A(y), r_2) = r_2$. Then $x \in A_{r_1}^T$, $y \in A_{r_2}^T$. Let us assume $r_1 < r_2$. Then there follows $T(A(x), r_1) < T(A(y), r_2)$ and $A_{r_2}^T \subseteq A_{r_1}^T$. So $y \in A_{r_1}^T$. Thus $x, y \in A_{r_1}^T$ and since $A_{r_1}^T$ is a subgroup of G, by hypothesis, $xy \in A_{r_1}^T$. Therefore

$$T(A(xy), r_1) \geq r_1 = T(A(x), r_1) \geq T(A(x), T(A(y), r_1)) = T(T(A(x), A(y)), r_1).$$

Thus we get $T(A(xy), r_1) \geq T(T(A(x), A(y)), r_1)$. As a T-norm is monotone with respect to each variable and symmetric, we have $A(xy) \geq T(A(x), A(y))$. Next, let $x \in G$ and $T(A(x), r) = r$. Then $x \in A_r^T$. Since A_r^T is a subgroup, $x^{-1} \in A_r^T$. Therefore $T(A(x^{-1}), r) \geq r$ and hence $T(A(x^{-1}), r) \geq T(A(x), r)$. So we have $A(x^{-1}) \geq A(x)$. Thus A is a t-fuzzy subgroup of G. □

Theorem 10. *Let A and B be t-level subsets of the sets G and H, respectively, and let $r \in [0,1]$. Then $A \times B$ is also a t-level subset of $G \times H$.*

Proof. Since any t-norm T is associative, using *Definition 4* and *Definition 9*, we can write the following statements

$$T(A \times B)(a, b), r) = T(T(A(a), B(b)), r) = T(A(a), T(B(b), r)) \geq T(A(a), r) \geq r.$$

This completes the proof. □

Now, we introduce the following definition.

Definition 10. *Let G be a group and A a t-fuzzy subgroup of G. The subgroups A_r^T, $r \in [0,1]$ and $r \leq T(A(e), r)$ are called t-level subgroup of A.*

Theorem 11. *Let G and H be two groups, A and B a t-fuzzy subgroup of G and H, respectively. Then the t-level subset $(A \times B)_r^T$, for $r \in [0,1]$, is a subgroup of $G \times H$, where e_G and e_H are identities of G and H, respectively.*

Proof. $(A \times B)_r^T = \{(x,y) \in G \times H : T((A \times B)(x,y), r) \geq r\}$. Since

$$\begin{aligned} T((A \times B)(e_G, e_H), r) &= T(T(A(e_G), B(e_H), r)) = T(A(e_G), T(B(e_H), r)) \\ &\geq T(A(e_G), r) \geq r, \end{aligned}$$

$(A\times B)_r^T$ is nonempty. Let $(x_1, y_1), (x_2, y_2) \in (A\times B)_r^T$, then $T(A\times B)(x_1, y_1), r) \geq r$ and $T(A \times B)(x_2, y_2), r) \geq r$. Since $A \times B$ is a t-fuzzy group of $G \times H$, we get $(A \times B)((x_1, y_1)(x_2, y_2)) = (A \times B)(x_1 x_2, y_1 y_2) = T(A(x_1 x_2), B(y_1 y_2))$. Using A and B are t-fuzzy subgroup, we get

$$\begin{aligned} T(A \times B)(x_1 x_2, y_1 y_2) &\geq T(T(A(x_1 x_2), B(y_1 y_2)), r) \\ &= T(A(x_1 x_2), T(B(y_1 y_2), r)) \\ &\geq T(A(x_1 x_2), r) \geq r. \end{aligned}$$

Hence $(x_1, y_{1)}), (x_2, y_{2)}) \in (A \times B)_r^T$. Again $(x,y) \in (A \times B)_r^T$ implies

$$\begin{aligned} T((A \times B)(x,y)^{-1}, r) &= T((A \times B)(x^{-1}, y^{-1}), r) \\ &= T(T(A(x^{-1}), B(y^{-1})), r) \\ &= T(A(x^{-1}), T(B(y^{-1})), r) \\ &\geq T(A(x^{-1}), r) \geq r. \end{aligned}$$

This means that $(x,y)^{-1} \in (A \times B)_r^T$. Therefore $(A \times B)_r^T$ is a subgroup of $G \times H$. □

Theorem 12. *Let G be a group and A_r^T a t-level subgroup of G. If A is a normal t-fuzzy subgroup, then A_r^T is a normal subgroup of G.*

Proof. By *Theorem 7* A_r^T is a t-level subgroup of G. Now let us show that A_r^T is normal. For all $a \in G$ and $x \in A_r^T$, $T(A(axa^{-1}), r) = T(A(a^{-1}ax), r) = T(A(x), r) \geq r$. Thus $axa^{-1} \in A_r^T$. Hence A_r^T is a normal subgroup. □

Theorem 13. *Let A, B be fuzzy subsets of the sets G and H, respectively, T be a t-norm and $r \in [0,1]$. Then $A_r^T \times B_r^T = (A \times B)_r^T$.*

Proof. Let (a,b) be an element of $A_r^T \times B_r^T$. Then $a \in A_r^T$ and $b \in B_r^T$. By *Definition 9*, we can write $T(A(a), r) \geq r$ and $T(B(b), r) \geq r$. Using *Definition 2* and *Definition 4* we get $T((A \times B)(a,b), r) = T(T(A(a), B(b)), r) = T(A(a), T(B(b), r)) \geq T(A(a), r) \geq r$. Thus we have $(a,b) \in (A \times B)_r^T$. Now, let $(a,b) \in (A \times B)_r^T$, This is required following statements $T((A \times B)(a,b), r) = T(T(A(a), B(b)), r) = T(A(a), T(B(b), r)) \geq r = T(1, r)$. Thus the inequalities $T(B(b), r) \geq r$ and $T(A(a), r) \geq r$ is satisfied. Hence (a,b) is in $A_r^T \times B_r^T$. This completes the proof. □

Theorem 14. *Let $A_1, A_2, \dots, A_n$ be fuzzy subgroups under a minimum operation in groups $G_1, G_2, \dots, G_n$, respectively, and let $r \in [0,1]$. Then*

$$(A_1 \times A_2 \times \cdots \times A_n)_r^T = A_{1r}^T \times A_{2r}^T \times \cdots A_{nr}^T.$$

Proof. Let $(a_1, a_2, \dots, a_n)$ be an element of $(A_1 \times A_2 \times \cdots \times A_n)_r^T$. Then using *Definition 6* and *Definition 9* we can write $T(min((A_1 \times A_2 \times \cdots \times A_n)(a_1, a_2, \dots, a_n), r)$ $= T(min(A_1(a_1), A_2(a_2), \dots, A_n(a_n)), r)$. For all $i = 1, 2, \dots, n$, $min(A_1(a_1), \dots, A_i(a_i), \dots, A_n(a_n)) = A_i(a_i)$. This gives us

$$T(min(A_1(a_1), \dots, A_i(a_i), \dots, A_n(a_n)), r) = T(A_i(a_i), r) \geq r.$$

Thus we have $a_i \in A_{ir}^T$. That is $(a_1, a_2, \dots, a_n) \in A_{1r}^T \times A_{2r}^T \times \cdots \times A_{nr}^T$. Similarly, let $(a_1, a_2, \dots, a_n)$ be an element of $A_{1r}^T \times A_{2r}^T \times \cdots A_{nr}^T$. Then for all $i = 1, 2, \dots, n$, $a_i \in A_{ir}^T$. That is, $T(A_i(a_i), r) \geq r$. Since $min(A_1(a_1), \dots, A_i(a_i), \dots, A_n(a_n)) = A_i(a_i)$ and $T(A_i(a_i), r) \geq r$, we get

$$\begin{aligned} T((A_1 \times A_2 \times & \cdots \times A_n)(a_1, a_2, \dots, a_n), r) \\ &= T(min(A_1(a_1), \dots, A_i(a_i), \dots, A_n(a_n)), r) = T(A_i(a_i), r) \geq r. \end{aligned}$$

Thus $(a_1, a_2, \dots, a_n) \in (A_1 \times A_2 \times \cdots \times A_n)_r^T$. Finally, this completes the proof. □

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