Fixed points of Ćirić quasi-contractive operators in normed spaces

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Abstract. *We establish a general theorem to approximate fixed points of Ćirić quasi-contractive operators on a normed space through the Mann iteration process with errors in the sense of Xu [10]. Our result generalizes and improves upon, among others, the corresponding results of [1,8].*

Key words: *iteration process, contractive condition, strong convergence*

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1. Introduction and preliminaries

Let C be a nonempty convex subset of a normed space E and $T : C \to C$ a mapping.

The Mann iteration process is defined by the sequence $\{x_n\}_{n=0}^{\infty}$ (see [8]):

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = (1 - b_n)\,x_n + b_n T x_n,\ n \geq 0 \end{cases} \tag{1.1}$$

where $\{b_n\}$ is a sequence in $[0, 1]$.

In 1998, Xu [11] introduced more satisfactory error terms in the sequence defined by:

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = a_n x_n + b_n T x_n + c_n u_n,\ n \geq 0. \end{cases} \tag{1.2}$$

where $\{a_n\}, \{b_n\}, \{c_n\}$ are sequences in $[0, 1]$ such that $a_n + b_n + c_n = 1$ and $\{u_n\}$ is a bounded sequence in C. Clearly, this iteration process contains the process (1.1) as its special case.

We recall the following definitions in a metric space (X, d). A mapping $T : X \to X$ is called an a-contraction if

$$d(Tx, Ty) \leq a d(x, y) \text{ for all } x, y \in X, \tag{1.3}$$

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where $a \in (0,1)$.

The map T is called a Kannan mapping [5] if there exists $b \in (0, \frac{1}{2})$ such that

$$d(Tx,Ty) \leq b[d(x,Tx) + d(y,Ty)] \text{ for all } x, y \in X. \tag{1.4}$$

A similar definition is due to Chatterjea [2]: there exists $c \in (0, \frac{1}{2})$ such that

$$d(Tx,Ty) \leq c[d(x,Ty) + d(y,Tx)] \text{ for all } x, y \in X. \tag{1.5}$$

Combining these three definitions, in 1972 Zamfirescu [12] proved the following result.

Theorem 1. *Let (X,d) be a complete metric space and $T : X \to X$ a mapping for which there exist real numbers a, b and c satisfying $a \in (0,1)$, $b, c \in (0, \frac{1}{2})$ such that for any pair x, $y \in X$, at least one of the following conditions holds:*

(z_1) $d(Tx,Ty) \leq ad(x,y)$,
(z_2) $d(Tx,Ty) \leq b[d(x,Tx) + d(y,Ty)]$,
(z_3) $d(Tx,Ty) \leq c[d(x,Ty) + d(y,Tx)]$.

Then T has a unique fixed point p and the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = Tx_n, \ n = 0, \ 2, ...$$

converges to p for any arbitrary but fixed $x_0 \in X$.

Remark 1. *The conditions $(z_1) - (z_3)$ can be written in the following equivalent form*

$$d(Tx,Ty) \leq h \max\{d(x,y), \ \frac{d(x,Tx) + d(y,Ty)}{2}, \ \frac{d(x,Ty) + d(y,Tx)}{2}\},$$

$\forall x$, $y \in X$; $0 < h < 1$. *Thus, a class of mappings satisfying the contractive conditions $(z_1) - (z_3)$ is a subclass of mappings satisfying the following condition*

$$d(Tx,Ty) \leq h \max\{d(x,y), \ d(x,Tx), d(y,Ty), \ \frac{d(x,Ty) + d(y,Tx)}{2}\}, \tag{C}$$

$0 < h < 1$.

The class of mappings satisfying (C) was introduced and investigated by Ćirić [3] in 1971.

Remark 2. *A mapping satisfying (C) is commonly called a Ćirić generalized contraction.*

In 2000, Berinde [1] introduced a new class of operators on a normed space E satisfying

$$\|Tx - Ty\| \leq \delta \|x - y\| + L \|Tx - x\|, \tag{1.6}$$

for any $x, y \in E$, $0 \leq \delta < 1$ and $L \geq 0$.

It may be noted that (1.6) is equivalent to

$$\|Tx - Ty\| \leq \delta \|x - y\| + L \min\{\|Tx - x\|, \|Ty - y\|\}, \tag{1.7}$$

for any $x, y \in E$, $0 \leq \delta < 1$ and $L \geq 0$.

Berinde [1] proved that this class is wider than the class of Zamfiresu operators and used the Mann iteration process to approximate fixed points of this class of operators in a normed space given in the form of the following theorem:

Theorem 2. *Let C be a nonempty closed convex subset of a normed space E. Let $T : C \to C$ be an operator satisfying* (1.6). *Let $\{x_n\}_{n=0}^{\infty}$ be defined through the iterative process* (1.1). *If $F(T) \neq \emptyset$ and $\sum_{n=1}^{\infty} b_n = \infty$, then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique fixed point of T.*

In this paper, a convergence theorem of Rhoades [8] regarding the approximation of fixed points of some quasi contractive operators in uniformly convex Banach spaces using the Mann iteration process, is extended to arbitrary normed spaces using the Mann iteration process with errors in the sense of Xu [10]. The conditions on the parameters $\{b_n\}$ that define the Mann iteration are also weakened.

2. Main results

In this paper we shall consider a class of mappings satisfying the following condition

$$d(Tx, Ty) \leq h \max\{d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2},\ d(x, Ty), d(y, Tx)\},$$

$0 < h < 1$. This class of mappings is a subclass of mappings satisfying the following condition

$$d(Tx, Ty) \leq h \max\{d(x, y), d(x, Tx), d(y, Ty), ,\ d(x, Ty), d(y, Tx)\}, \tag{QC}$$

$0 < h < 1$. The class of mappings satisfying (QC) is introduced and investigated by Ćirić [4] in 1974 and a mapping satisfying is commonly called a Ćirić quasi contraction.

The following lemma is now well known.

Lemma 1. *Let $\{r_n\}, \{s_n\}, \{t_n\}$ and $\{k_n\}$ be sequences of nonnegative numbers satisfying*

$$r_{n+1} \leq (1 - s_n) r_n + s_n t_n + k_n \quad \textit{for all } n \geq 1.$$

If $\sum_{n=1}^{\infty} s_n = \infty$, $\lim_{n\to\infty} t_n = 0$ and $\sum_{n=1}^{\infty} k_n < \infty$ hold, then $\lim_{n\to\infty} r_n = 0$.

Theorem 3. *Let C be a nonempty closed convex subset of a normed space E. Let $T : C \to C$ be an operator satisfying the condition*

$$\|Tx - Ty\| \leq h \max\{\|x - y\|, \frac{\|x - Tx\| + \|y - Ty\|}{2}, \|x - Ty\|, \|y - Tx\|\}, \tag{CR}$$

$\forall x,\ y \in C$; $0 < h < 1$. Let $\{x_n\}_{n=0}^{\infty}$ be defined by the iterative process (1.2). *If $\sum_{n=1}^{\infty} b_n = \infty$ and $c_n = o(b_n)$, then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique fixed point of T.*

Proof. By Ćirić [4], we know that T has a unique fixed point in C, say w. Consider $x, y \in C$. Without loss of generality, we may suppose that

$$\min\{\|Tx - x\|, \|Ty - y\|\} = \|Tx - x\|.$$

Since T is a Ćirić operator, then, if from (CR) we have

$$\|Tx - Ty\| \leq \frac{h}{2}\left[\|x - Tx\| + \|y - Ty\|\right],$$

then

$$\begin{aligned}\|Tx - Ty\| &\leq \frac{h}{2}\left[\|x - Tx\| + \|y - Ty\|\right] \\ &\leq \frac{h}{2}\left[\|x - Tx\| + \|y - x\| + \|x - Tx\| + \|Tx - Ty\|\right].\end{aligned}$$

Hence

$$(1 - \frac{h}{2})\|Tx - Ty\| \leq \frac{h}{2}\|x - y\| + h\|x - Tx\|,$$

which yields (using the fact that $0 < h < 1$)

$$\|Tx - Ty\| \leq \frac{\frac{h}{2}}{1 - \frac{h}{2}}\|x - y\| + \frac{h}{1 - \frac{h}{2}}\|x - Tx\|. \tag{2.1}$$

If from (CR) ,

$$\|Tx - Ty\| \leq h\|x - Ty\|,$$

then

$$\|Tx - Ty\| \leq \frac{h}{1 - h}\|x - Tx\|. \tag{2.2}$$

Also, if from (CR), $\|Tx - Ty\| \leq h\|y - Tx\|$, then we have

$$\|Tx - Ty\| \leq h\|y - Tx\| \leq h\|x - y\| + h\|x - Tx\|. \tag{2.3}$$

Denote

$$\delta = \max\left\{h, \frac{\frac{h}{2}}{1 - \frac{h}{2}}\right\} = h, \qquad L = \max\left\{h, \frac{h}{1 - \frac{h}{2}}, \frac{h}{1 - h}\right\} = \frac{h}{1 - h}.$$

Thus, in all cases,

$$\|Tx - Ty\| \leq \delta\|x - y\| + L\|x - Tx\| = h\|x - y\| + \frac{h}{1 - h}\|x - Tx\| \tag{2.4}$$

holds for all $x, y \in C$.

Also from (CR) with $y = w = Tw$ we have

$$\begin{aligned}\|Tx - w\| &\leq h\max\{\|x - w\|,\ \frac{\|x - Tx\|}{2},\ \|x - w\|, \|w - Tx\|\} \\ &= h\max\{\|x - w\|,\ \frac{\|x - Tx\|}{2},\ \|w - Tx\|\} \\ &\leq h\max\{\|x - w\|,\ \frac{\|x - w\| + \|w - Tx\|}{2},\ \|w - Tx\|\} \\ &\leq h\max\{\|x - w\|,\ \|w - Tx\|\}.\end{aligned}$$

If we suppose that $\max\{\|x-w\|, \|w-Tx\|\} = \|Tx-w\|$, then we have

$$\|Tx-w\| \leq h\,\|w-Tx\|,$$

which is impossible for $Tx \neq w$. Thus

$$\|Tx-w\| \leq h\,\|x-w\|. \tag{2.4a}$$

Assume that

$$M = \sup_{n\geq 0}\|u_n - w\|.$$

Using (1.2), we have

$$\begin{aligned}\|x_{n+1}-w\| &= \|a_n x_n + b_n T x_n + c_n u_n - (a_n+b_n+c_n)w\| \\ &= \|a_n(x_n-w) + b_n(Tx_n-w) + c_n(u_n-w\| \\ &\leq a_n\|x_n-w\| + b_n\|Tx_n-w\| + c_n\|u_n-w\| \\ &\leq (1-b_n)\|x_n-w\| + b_n\|Tx_n-w\| + Mc_n. \end{aligned} \tag{2.5}$$

Now (2.4) or (2.4a) gives

$$\|Tx_n-w\| \leq h\,\|x_n-w\|. \tag{2.6}$$

From (2.5)-(2.6), we obtain

$$\|x_{n+1}-w\| \leq [1-(1-h)b_n]\,\|x_n-w\| + Mc_n.$$

By *Lemma 1*, we get that $\lim\|x_n-w\| = 0$. Consequently $x_n \to w \in F$ and this completes the proof. □

Corollary 1. *Let C be a nonempty closed convex subset of a normed space E. Let $T : C \to C$ be an operator satisfying* (2.4). *Let $\{x_n\}_{n=0}^{\infty}$ be defined by the iterative process* (1.1). *If $\sum_{n=1}^{\infty} b_n = \infty$, then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique fixed point of T.*

Remark 3.

1. *The contractive condition* (1.3) *makes T a continuous function on X while this is not the case with the contractive conditions* $(1.4-1.5)$ *and* (2.4).
2. *The Chatterjea's and the Kannan's contractive conditions* (1.5) *and* (1.4) *are both included in the class of Ćirić operators and so their convergence theorems for the Mann iteration process with errors are obtained in* Theorem 3.
3. *Theorem 4 of Rhoades [8] in the context of Mann iteration on a uniformly convex Banach space has been extended in* Corollary 1 .
4. In Corollary 1, *Theorem 8 of Rhoades [8] is generalized to the setting of normed spaces.*

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