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Modelling of the Pipeline as a Lumped Parameter System

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Original scientific paper

The paper deals with the simplification of pipeline models. A nonlinear distributed parameters model is linearised and its transfer function given. The pipeline is represented as a two-port system. Two causal representations – the hybrid ones which are used in practice – are studied further. They involve three different transcendent transfer functions which are then approximated by rational transfer functions using a Taylor series expansion. The derived models are valid for low frequencies and are used to discuss how to obtain better approximation. They equalise the high frequency gain of the transcendent and rational transfer functions and employ a Padé approximation. Due to the approximation of the high frequency gain, the derived models are only valid for a class of models – namely – well damped pipelines. The derived models which describe the pipeline as a lumped parameter system were verified on a real pipeline using experiment data.

Key words: modelling, model approximation, distributed-parameter system, lumped-parameter system, pipelines

1 INTRODUCTION

Models of pipelines are used for different purposes, such as controller design, leakage monitoring etc. Observer-based leak detection and localisation schemes for example require a pipeline model to compute the states of a pipeline without leak [1, 2]. The first industrial applications demonstrate the performance of observer-based methods [3].

In these schemes, the observer is derived using a mathematical model of the pipeline. The one-dimensional compressible fluid flow through pipelines is governed by nonlinear partial differential equations [4]. Pipelines are therefore distributed parameter systems. To date, there is no general closed-form solution. Numerical approaches like the Method of Characteristics are used instead [5] yielding the computational background for the observer algorithms.

However, sometimes simple models of the pipeline in the form of a lumped parameter system are use-ful: the classical system theory for Multi-Input Multi-Output (MIMO) systems can be used e.g. for controller design and system identification. The re-sulting algorithms are less time-consuming and hen-ce better suited for critical real time applications. Additionally, the analysis of fluid transients caused by leaks is much easier.

The paper is organised as follows: in Section 2 the nonlinear distributed parameter model of the pipeline is linearised and represented as a two-port system. Two representations of the pipeline's model are given. They include three different transcendent transfer functions which are approximated by rational transfer functions in Section 3. In Section 5, the verification of the simplified models by real experiment data is presented.

2 MATHEMATICAL MODELS OF THE PIPELINE

The analytical solution for unsteady flow problems is obtained by using the equations for continuity, momentum and energy. These equations correspond to the physical principles of mass conservation, Newton's second law $F=ma$ and energy conservation. Applying these equations leads to a coupled non-linear set of partial differential equations. Further problems arise in the case of turbulent flow, which introduces stochastic flow behaviour. Therefore, the mathematical derivation for the flow through a pipeline is a mixture of both theoretical and empirical approaches.

The following assumptions for the derivation of a mathematical model of the flow through pipelines are made: Fluid is compressible, viscous, adiabatic and homogenous. The resulting nonlinear distributed parameters model is described by the following two partial differential equations:

$$\frac{A}{a^2}\cdot\frac{\partial p}{\partial t}=-\frac{\partial q}{\partial x}, \tag{1}$$

$$\frac{1}{A}\cdot\frac{\partial q}{\partial t}+\bar{\rho}g\sin\alpha+\frac{\lambda(q)}{2DA^2\bar{\rho}}q^2=-\frac{\partial p}{\partial x}, \tag{2}$$

where A, D and α are the profile, the diameter and the inclination of the pipeline respectively, p the pressure, q the mass flow rate, λ the friction coefficient, ρ the density of the fluid, g the gravity constant and a the velocity of sound.

Nonlinear equations (2, 1) are linearised and written in a form using notations common in the analysis of electrical transmission lines. Also, the gravity effect is neglected – included into the working point. The corresponding system of linear partial differential equations is

$$L\frac{\partial q}{\partial t}+Rq=-\frac{\partial p}{\partial x} \tag{3}$$

$$C\frac{\partial p}{\partial t}=-\frac{\partial q}{\partial x} \tag{4}$$

where $L=\frac{1}{A}$, $R=\frac{\lambda(\bar{q})\bar{q}}{A^2\bar{\rho}D}$ ($\bar{q}$ is the flow at the working point) and $C=\frac{A}{a^2}$ are the inertance (inductivity), resistance and capacitance per unit length, respectively.

The next step in the derivation of a simple model of the pipeline is the analytical solution of linear quations (3, 4). The Laplace transformation of them yields the linear non-causal model

$$P_L(s)=Z_K Q_0(s)\sinh(nL_p)+P_0(s)\cosh(nL_p) \tag{5}$$

$$Q_L(s)=Q_0(s)\cosh(nL_p)-\frac{1}{Z_K}P_0(s)\sinh(nLp) \tag{6}$$

where L_p is the length of the pipeline and

$$n=\sqrt{(Ls+R)\cdot Cs} \tag{7}$$

$$Z_K=\frac{Ls+R}{n}=\sqrt{\frac{Ls+R}{Cs}}\,. \tag{8}$$

Finally, the linearised model of the pipeline can be written in one of the following four forms which differ from each other with respect to the model inputs (independent quantities) and outputs (dependent quantities)

1. *Hybrid representation: Inputs Q_0, P_L, outputs Q_L, P_0:*

$$\begin{bmatrix}P_0\\Q_L\end{bmatrix}=\begin{bmatrix}\frac{1}{\cosh(nL_p)} & Z_K\tanh(nL_p)\\ -\frac{1}{Z_K}\tanh(nL_p) & \frac{1}{\cosh(nL_p)}\end{bmatrix}\begin{bmatrix}P_L\\Q_0\end{bmatrix} \tag{9}$$

2. *Hybrid representation: Inputs Q_L, P_0, outputs Q_0, P_L:*

$$\begin{bmatrix}P_L\\Q_0\end{bmatrix}=\begin{bmatrix}\frac{1}{\cosh(nL_p)} & -Z_K\tanh(nL_p)\\ \frac{1}{Z_K}\tanh(nL_p) & \frac{1}{\cosh(nL_p)}\end{bmatrix}\begin{bmatrix}P_0\\Q_L\end{bmatrix} \tag{10}$$

3. *Impedance representation: Inputs Q_0, Q_L, outputs P_0, P_L:*

$$\begin{bmatrix}P_0\\P_L\end{bmatrix}=\begin{bmatrix}Z_K\coth(nL_p) & -Z_K\frac{1}{\sinh(nL_p)}\\ -Z_K\frac{1}{\sinh(nL_p)} & -Z_K\coth(nL_p)\end{bmatrix}\begin{bmatrix}Q_L\\Q_0\end{bmatrix} \tag{11}$$

4. *Admittance representation: Inputs P_0, P_L, outputs Q_0, Q_L:*

$$\begin{bmatrix}Q_0\\Q_L\end{bmatrix}=\begin{bmatrix}\frac{1}{Z_K}\coth(nL_p) & -\frac{1}{Z_K}\cdot\frac{1}{\sinh(nL_p)}\\ \frac{1}{Z_K}\cdot\frac{1}{\sinh(nL_p)} & -\frac{1}{Z_K}\coth(nL_p)\end{bmatrix}\begin{bmatrix}P_0\\P_L\end{bmatrix} \tag{12}$$

It should be noted that the forms 1 to 4 represent causal models, whilst Equations (5, 6) represent a noncausal model with inputs P_0, Q_0 and outputs P_L, Q_L. This completes the derivation of the transfer functions of the linearised pipeline. The resulting transfer functions are transcendent. In the next section, their approximations by rational transfer functions will be given.

3 LINEAR PIPELINE MODEL WITH LUMPED PARAMETERS

In this section, the rational transfer functions of the pipeline will be derived. There are seven different transcendent transfer functions in the four pipeline forms of the previous section:

$$\frac{1}{\cosh(nL_p)},\quad \frac{1}{Z_K}\tanh(nL_p),\quad Z_K\tanh(nL_p),$$

$$Z_k\coth(nL_p),\quad Z_K\cdot\frac{1}{\sinh(nL_p)},$$

$$\frac{1}{Z_K}\cdot\coth(nL_p)\ \textit{and}\ \frac{1}{Z_K}\cdot\frac{1}{\sinh(nL_p)}\,.$$

First, simple models will be derived. These models are only valid for low frequencies. On the basis of a discussion of simple models, better models, which approximate the transcendent transfer functions in a broad frequency range, will be derived in subsection 4.

3.1 Models for low frequencies

Simple models are obtained by expanding the transcendent transfer function into a Taylor series.

They are as follows:

1. *Hybrid representation*: $\dfrac{1}{\cosh\left(nL_p\right)} =$

Reverse pressure gain at upstream dead end:

$$\frac{P_0}{P_L}\Big|_{Q_0=0}$$

Flow gain at downstream reservoir:

$$\frac{Q_L}{Q_0}\Big|_{P_L=0}$$

Pressure gain at downstream dead end:

$$\frac{P_L}{P_0}\Big|_{Q_L=0}$$

Reverse flow gain at upstream reservoir:

$$\frac{Q_0}{Q_L}\Big|_{P_0=0}$$

$$\frac{1}{\cosh\left(nL_p\right)} = \frac{1}{\cosh\left(\sqrt{(Ls+R)\cdot Cs}\; L_p\right)} \approx$$

$$\approx \frac{1}{\left(\frac{1}{2}L_p^2 LC + \frac{1}{24}L_p^4 R^2 C^2\right)s^2 + \frac{1}{2}L_p^2 RCs + 1}. \quad (13)$$

This transfer function describes a change of one quantity (flow, pressure) at one end of the pipeline, if the same quantity is changing at the other end, whilst the other quantity remains constant, e.g. the change of the flow at the pipeline's output, if the flow changes at the input of the pipeline whilst the pressure at the output of the pipeline remains constant. The transfer function (13) can be interpreted as a second order system with the eigen-frequency

$$\omega_0' = \frac{1}{\sqrt{\frac{1}{2}L_p^2 LC + \frac{1}{24}L_p^4 R^2 C^2}} \approx \sqrt{\frac{2}{LC}} \cdot \frac{1}{L_p} \quad (14)$$

and damping

$$\zeta \approx \sqrt{\frac{C}{2L}} \cdot R \cdot L_p \quad (15)$$

where the second term in the square root of equation (14) was neglected for small R. The eigen-frequency can be interpreted as follows: with hybrid boundary conditions (pressure at one end and flow at the other end of the pipeline are kept constant) the shock wave originating at one end of the pipeline returns after reflection at the other end of the pipeline with the opposite phase. The half-period of the oscillations is consequently equal to the time needed by the shock wave to travel along the pipeline and back. This gives the radial eigen-frequency

$$\omega_0 = \frac{2\text{p}}{4\sqrt{LC}L_p} = \frac{\text{p}}{2\sqrt{LC}} \cdot \frac{1}{L_p} \quad (16)$$

which is only 10 % different from ω_0'. The static gain of the treated transfer function is 1, meaning that the step change of the flow at one end of the pipeline results in the same steady state change of the flow at the other end (if the pressure at the first end remains unchanged). Similarly, the step change of the pressure at one end results in the same steady state change of the pressure at the other end (if the steady state flow remains unchanged).

2. *Hybrid representation*: $\dfrac{1}{Z_K}\tanh\left(nL_p\right) =$

Negative output admittance at upstream dead end: $\dfrac{Q_L}{P_L}\Big|_{Q_0=0}$

Input admittance at downstream dead end:

$$-\frac{Q_0}{P_0}\Big|_{Q_L=0}$$

$$\frac{1}{Z_K}\tanh\left(nL_p\right) = \sqrt{\frac{Cs}{Ls+R}} \cdot \tanh\left(\sqrt{(Ls+R)Cs}L_p\right) \approx$$

$$\approx \frac{\frac{1}{6}L_p^3 RC^2 s^2 + L_p Cs}{\left(\frac{1}{2}L_p^2 LC + \frac{1}{24}L_p^4 R^2 C^2\right)s^2 + \frac{1}{2}L_p^2 RCs + 1}. \quad (17)$$

This transfer function describes a change of the flow at one end of the pipeline, if the pressure is changing at the same end whilst the flow at the other end of the pipeline remains constant. It has a zero in the origin of the s plane which causes its static gain to be 0. This is obvious since the flow at both ends of the pipeline must be the same in the steady state.

A higher order Taylor extension of the transcendent transfer function would result in:

$$\frac{1}{Z_K}\tanh\left(nL_p\right) = \sqrt{\frac{Cs}{Ls+R}} \cdot \tanh\left(\sqrt{(Ls+R)Cs}L_p\right) \approx$$

$$\approx \frac{\frac{1}{60}C^L p^5 RLs^4 + \left(\frac{1}{6}C^2 Lp^3 L + \frac{1}{120}C^3 Lp^5 R^2\right)s^3 + \frac{1}{6}L_p^3 RC^2 s^2 + L_p Cs}{\frac{1}{24}L^2 C^2 Lp^4 s^4 + \frac{1}{12}RLC^2 Lp^4 s^3 + \left(\frac{1}{2}L_p^2 LC + \frac{1}{24}L_p^4 R^2 C^2\right)s^2 + \frac{1}{2}L_p^2 RCs + 1}. \quad (18)$$

For $R \to 0$ this transfer function has two complex conjugate zeros at the radial frequency

$$\omega = \sqrt{\frac{6}{LC}} \cdot \frac{1}{L_p} \tag{19}$$

which is close to the twice eigen-frequency of the pipeline with hybrid boundary conditions.

3. *Hybrid representation*: $Z_K \tanh(nL_p) =$

Input impedance at downstream reservoir:

$$\frac{P_0}{Q_0}\Big|_{P_L=0}$$

Negative output impedance at upstream reservoir: $-\frac{P_L}{Q_L}\Big|_{P_0=0}$

$$Z_K \tanh(nL_p) = \sqrt{\frac{Ls+R}{Cs}} \tanh\left(\sqrt{(Ls+R)Cs}\,L_p\right) \approx$$
$$\approx \frac{\frac{1}{3}L_p^3 PLCs^2 + \left(L_p L + \frac{1}{6}L_p^3 R^2 C\right)s + L_p R}{\left(\frac{1}{2}L_p^2 LC + \frac{1}{24}L_p^4 R^2 C^2\right)s^2 + \frac{1}{2}L_p^2 RCs + 1}. \tag{20}$$

This transfer function describes a change of the pressure at one end of the pipeline, if the flow is changing at the same end whilst the pressure at the other end remains constant. Its static gain is $L_p R$ which corresponds to the static change of the pressure due to changing flow.

A higher order Taylor extension of the transcendent transfer function would result in:

$$Z_K \tanh(nL_p) = \sqrt{\frac{Ls+R}{Cs}} \cdot \tanh\left(\sqrt{(Ls+R)Cs}\,L_p\right) \approx$$
$$\approx \frac{\frac{1}{40}RL^2 Lp^5 C^2 s^4 + \left(\frac{1}{6}L^2 Lp^3 C + \frac{1}{40}R^2 L Lp^5 C^2\right)s^3 + \frac{1}{3}L_p^3 RLCs^2 + \left(L_p L + \frac{1}{6}L_p^3 R^2 C\right)s + L_p Cs}{\frac{1}{24}L^2 C^2 Lp^4 s^4 + \frac{1}{12}RLC^2 Lp^4 s^3 + \left(\frac{1}{2}L_p^2 LC + \frac{1}{24}L_p^4 R^2 C^2\right)s^2 + \frac{1}{2}L_p^2 RCs + 1}. \tag{21}$$

For $R \to 0$ this transfer function has two complex conjugate zeros at the radial frequency

$$\omega = \sqrt{\frac{6}{LC}} \cdot \frac{1}{L_p} \tag{22}$$

which is close to the twice eigen-frequency of the pipeline with hybrid boundary conditions.

4. *Impedance representation*: $Z_K \coth(nL_p) =$

Input impedance at downstream dead end:

$$\frac{P_0}{Q_0}\Big|_{Q_L=0}$$

Negative output impedance at upstream dead end: $-\frac{P_L}{Q_L}\Big|_{Q_0=0}$

$$Z_K \coth(nL_p) = \sqrt{\frac{Ls+R}{Cs}} \coth\left(\sqrt{(Ls+R)Cs}\,L_p\right) \approx$$
$$\approx \frac{\left(\frac{1}{2}L_p^2 LC + \frac{1}{24}L_p^4 R^2 C^2\right)s^2 + \frac{1}{2}L_p^2 RCs + 1}{\left(\frac{1}{6}C^2 Lp^3 L + \frac{1}{120}C^3 Lp^5 R^2\right)s^3 + \frac{1}{6}L_p^3 RC^2 s^2 + L_p Cs}. \tag{23}$$

This transfer function describes the change of the pressure at one end of the pipeline if the flow is changing at the same end while the flow at the other end remains constant. It has a pole in the origin of the s plane and has on infinite static gain. This is obvious since different steady state flows on both ends of the pipeline cause a constant increase of the pressure. The remaining dynamics of the transfer function for small R is a second order system with the eigen-frequency

$$\omega = \sqrt{\frac{6}{LC}} \cdot \frac{1}{L_p} \tag{24}$$

which is close to the twice eigen-frequency of the pipeline with hybrid boundary conditions and the damping

$$\zeta \approx \sqrt{\frac{3C}{2L}} \cdot R \cdot L_p. \tag{25}$$

This is the inverse of the case 2.

5. *Impedance representation*: $Z_K \frac{1}{\sinh(nL_p)} =$

Negative reverse transimpedance at upstream dead end: $-\frac{P_0}{Q_L}\Big|_{Q_0=0}$

Negative transimpedance at downstream dead end: $-\frac{P_L}{Q_0}\Big|_{Q_L=0}$

$$Z_K \frac{1}{\sinh(nL_p)} = \sqrt{\frac{Ls+R}{Cs}} \cdot \frac{1}{\sinh\left(\sqrt{(Ls+R)Cs}L_p\right)} \approx$$
$$\approx \frac{1}{\left(\frac{1}{6}C^2Lp^3L + \frac{1}{120}C^3Lp^5R^2\right)s^3 + \frac{1}{6}L_p^3RC^2s^2 + L_pCs}. \quad (26)$$

This transfer function describes the change of the flow at one end of the pipeline if the pressure is changing at the same end while the pressure at the other end remains constant. This is an inverse of the case 3, so a higher order Taylor extension of the transcendent transfer function would result in:

$$\frac{1}{Z_K}\coth(nL_p) = \sqrt{\frac{Cs}{Ls+R}}\coth\left(\sqrt{(Ls+R)Cs}L_p\right) \approx$$
$$\approx \frac{\frac{1}{24}L^2C^2Lp^4s^4 + \frac{1}{12}RLC^2Lp^4s^3 + \left(\frac{1}{2}L_p^2LC + \frac{1}{24}L_p^4R^2C^2\right)s^2 + \frac{1}{2}L_p^2RCs + 1}{\frac{1}{40}RL^2Lp^5C^2s^4 + \left(\frac{1}{6}L^2Lp^3C + \frac{1}{40}R^2LLp^5C^2\right)s^3 + \frac{1}{3}L_p^3RLCs^2 + \left(L_pL + \frac{1}{6}L_p^3R^2C\right)s + L_pR}. \quad (28)$$

This transfer function describes the change of the pressure at one end of the pipeline, if the flow is changing at the other end while the flow at the same end remains constant. It has a pole at the origin of the s plane, causing its integral character. The remaining dynamics of the transfer function for small R is again a second order system with the same eigen-frequency and damping as in the previous case.

6. *Admittance representation*: $\frac{1}{Z_K}\coth(nL_p) =$

Input admittance at downstream reservoir:

$$\frac{Q_0}{P_0}\Big|_{P_L=0}$$

Negative output admittance at upstream reservoir:

$$-\frac{Q_L}{P_L}\Big|_{P_0=0}$$

$$\frac{1}{Z_K}\coth(nL_p) = \sqrt{\frac{Cs}{Ls+R}}\coth\left(\sqrt{(Ls+R)Cs}L_p\right) \approx$$
$$\approx \frac{\left(\frac{1}{2}L_p^2LC + \frac{1}{24}L_p^4R^2C^2\right)s^2 + \frac{1}{2}L_p^2RCs + 1}{\frac{1}{3}L_p^3RLCs^2 + \left(L_pL + \frac{1}{6}L_p^3R^2C\right)s + L_pR}. \quad (27)$$

For $R \approx 0$ this transfer function has a pole at the origin of the s plane, i.e. the integral character. The remaining dynamics of the transfer function is a second order system with the eigen-frequency

$$\omega = \sqrt{\frac{6}{LC}} \cdot \frac{1}{L_p} \quad (29)$$

which is close to the twice eigen-frequency of the pipeline with hybrid boundary conditions and the damping

$$\zeta \approx \sqrt{\frac{3C}{2L}} \cdot R \cdot L_p. \quad (30)$$

7. *Admittance representation*: $\frac{1}{Z_K \sinh(nL_p)} =$

Negative reverse transadmittance at upstream reservoir:

$$-\frac{Q_0}{P_L}\Big|_{P_0=0}$$

Transadmittance at downstream reservoir:

$$\frac{Q_L}{P_0}\Big|_{P_L=0}$$

$$\frac{1}{Z_K \sinh(nL_p)} = \sqrt{\frac{Cs}{Ls+R}} \cdot \frac{1}{\sinh\left(\sqrt{(Ls+R)Cs}L_p\right)} \approx$$
$$\approx \frac{1}{\frac{1}{40}RL^2Lp^5C^2s^4 + \left(\frac{1}{6}L^2Lp^3C + \frac{1}{40}R^2LLp^5C^2\right)s^3 + \frac{1}{3}L_p^3RLCs^2 + \left(L_pL + \frac{1}{6}L_p^3R^2C\right)s + L_pR}. \quad (31)$$

This transfer function describes the change of the flow at one end of the pipeline if the pressure is changing at the other end while the pressure at the same end remains constant. It has a static gain $\frac{1}{L_p R}$ which corresponds to the static change of the flow due to changing pressure. For $R \approx 0$ this transfer function has a pole at the origin of the s plane, i.e. the integral character. The remaining dynamics of the transfer function is a second order system with the eigen-frequency and damping as in the previous case.

4 MODELS FOR WELL-DAMPED PIPELINES

In this section, the pipeline will be presented as a second order lumped parameter system in the form

$$G(s) = \frac{b_2 s^2 + b_1 s + b_0}{a_2 s^2 + a_1 s + a_0} e^{-sT_d} \tag{32}$$

where T_d is the dead time. The transcendent transfer functions will be approximated by a rational transfer function with dead time however only for a class of pipelines, as shown later. The procedure of the approximation is as follows: as the simple models are valid for low frequencies, their static gain is exact and will be applied to the derived models. Also, the eigen-frequency as discussed in the earlier subsection (Equation 16) will be used. Next, the high frequency gain will be approximated from transcendent function and known dead time will be applied. In this way, four coefficients of the transfer function (32) are determined. The remaining two are obtained using a Padé approximation of the transcendent transfer function. Since the high frequency gain approximation is valid only under certain conditions, the derived models are only valid for one class of pipelines. The derived models are as follows

1. *Hybrid representation: Reverse pressure gain at upstream dead end, Flow gain at downstream reservoir, Pressure gain at downstream dead end and Reverse flow gain at upstream reservoir*

$$G_0(s) = \frac{1}{\cosh(nL_p)} = \frac{1}{\cosh(nL_p)\, e^{-sT_d}} e^{-sT_d}. \tag{33}$$

Since this transfer function connects quantities at different ends of the pipeline, the dead time for (33) is known – it is the time needed for the shock wave to travel along the pipeline.

$$T_d = \sqrt{LC}\, L_p . \tag{34}$$

The static gain of (33) is 1, so the coefficients

$$\begin{aligned} a_0 &= 1 \\ b_0 &= 1 \end{aligned} \tag{35}$$

The eigen frequency ω_0 (Equation 16) determines the coefficient

$$a_2 = \frac{1}{\omega_0^2} = \frac{4}{\pi^2} L_p^2 LC . \tag{36}$$

The high frequency gain is determined by

$$\lim_{\omega\to\infty} |G_0(j\omega)| = \lim_{\omega\to\infty} \left| \frac{1}{\cosh\left(\sqrt{(Lj\omega + R)Cj\omega L_p}\right)} \right| \tag{37}$$

which does not exist. However,

$$\lim_{\omega\to\infty} \left| e^{\pm\left(\sqrt{(Lj\omega+R)C_j\omega L_p}\right)} \right| = e^{\pm\sqrt{\frac{C}{L}}\cdot\frac{RL_p}{2}} \tag{38}$$

does exist and in the case

$$\sqrt{\frac{C}{L}} \cdot \frac{RL_p}{2} >> 1 \tag{39}$$

the high frequency gain (37) can be approximated by

$$\lim_{\omega\to\infty} G(s) = \frac{b_2}{a_2} \approx 2e^{-\sqrt{\frac{C}{L}}\cdot\frac{RL_p}{2}} \tag{40}$$

yielding

$$b_2 = \frac{8}{\pi^2} LCL_p^2 e^{-\sqrt{\frac{C}{L}}\cdot\frac{RL_p}{2}} . \tag{41}$$

The condition (39) can also be written as

$$\zeta >> \frac{1}{\sqrt{2}},$$

so the derived models are valid for well damped pipelines only.

The remaining coefficients b_1 and a_1 are determined using a Padé approximation of (33) i.e. by comparing the corresponding terms of the following two series expansions:

$$\frac{a_2 s^2 + a_1 s + 1}{b_2 s^2 + b_1 s + 1} =$$
$$= 1 + (a_1 - b_1)s + (a_2 - b_2 - b(a_1 - b_1))s^2 + \ldots \tag{42}$$

and

$$\cosh\left(\sqrt{(Ls+R)Cs}\, L_p\right) e^{-\sqrt{LCs} L_p} =$$
$$= 1 + \left(\frac{1}{2} L_p^2 RC - L_p \sqrt{LC}\right) s +$$
$$+ \left(L_p^2 LC + \frac{1}{24} L_p^4 R^2 C^2 - \frac{1}{2} L_p^3 RC\sqrt{LC}\right) s^2 + \ldots$$

The result is (taking into account Equations 36 and 41)

$$b_1 = \frac{L_p C\left(96L - 192Le^{-\sqrt{\frac{C}{L}}\cdot\frac{RL_p}{2}} - 24\mathsf{p}^2 L - \mathsf{p}^2 L_p^2 R^2 C + 12\mathsf{p}^2 L_p R\sqrt{LC}\right)}{12\mathsf{p}^2\left(L_p RC - 2\sqrt{LC}\right)} \qquad (43)$$

$$a_1 = \frac{L_p C\left(96L - 192Le^{-\sqrt{\frac{C}{L}}\cdot\frac{RL_p}{2}} + 5\mathsf{p}^2 L_p^2 R^2 C - 12\mathsf{p}^2 L_p R\sqrt{LC}\right)}{12\mathsf{p}^2\left(L_p RC - 2\sqrt{LC}\right)}. \qquad (44)$$

2. *Hybrid representation: Negative output admittance at upstream dead end and Input admittance at downstream dead end*

$$G_0(s) = \frac{1}{Z_K}\tanh\left(nL_p\right). \qquad (45)$$

The dead time of the treated transfer function is zero since it connects a change of the flow at one end of the pipeline if the pressure changes at the same end:

$$T_d = 0. \qquad (46)$$

The static gain of $G_0(s)$ is zero, so the coefficients

$$\begin{aligned} a_0 &= 1 \\ b_0 &= 0. \end{aligned} \qquad (47)$$

The eigen frequency ω_0 is the same as in the previous case:

$$a_2 = \frac{1}{\omega_0^2} = \frac{4}{\mathsf{p}^2} L_p^2 LC. \qquad (48)$$

The high frequency gain is determined by

$$\frac{b_2}{a_2} = \lim_{\omega\to\infty}\left|\frac{1}{Z_K}\tanh\left(nL_p\right)\right| =$$

$$= \lim_{\omega\to\infty}\left|\sqrt{\frac{Cj\omega}{Lj\omega + R}}\tanh\left(\sqrt{-LC\omega^2 + RCj\omega}L_p\right)\right| =$$

$$= \sqrt{\frac{C}{L}} \qquad (49)$$

yielding

$$b_2 = \frac{4}{\mathsf{p}^2} L_p^2 C\sqrt{LC}. \qquad (50)$$

The remaining coefficients b_1 and a_1 are determined by the same procedure as in the previous case. A comparison of

$$\frac{b_2 s^2 + b_1 s}{a_2 s^2 + a_1 s + 1} = b_1 s + (b_2 - b_1 a_1)s^2 + \ldots \qquad (51)$$

and

$$\sqrt{\frac{Cs}{Ls + R}}\tanh\left(\sqrt{(Ls + R)Cs}\,L_p\right) =$$

$$= L_p Cs - \frac{1}{3}L_p^3 RC^2 + \ldots \qquad (52)$$

yields

$$b_1 = L_p \cdot C \qquad (53)$$

$$a_1 = \frac{1}{3\mathsf{p}^2} L_p\left(12\sqrt{LC} + \mathsf{p}^2 L_p RC\right). \qquad (54)$$

3. *Hybrid representation: Input impedance at downstream reservoir and Output impedance at upstream reservoir*

$$G_0(s) = Z_k \tanh\left(nL_p\right). \qquad (55)$$

The dead time of the treated transfer function is zero, since it connects a change of the pressure at one end of the pipeline if the flow changes at the same end, so

$$T_d = 0. \qquad (56)$$

The static gain of $G_0(s)$ is $L_p R$, yielding

$$\begin{aligned} a_0 &= 1 \\ b_0 &= L_p R. \end{aligned} \qquad (57)$$

The eigen-frequency is the same as in the previous two cases:

$$a_2 = \frac{1}{\omega_0^2} = \frac{4}{\mathsf{p}^2} L_p^2 LC. \qquad (58)$$

The high frequency gain is determined by

$$\frac{b_2}{a_2} = \lim_{\omega\to\infty}\left|Z_k \tanh\left(nL_p\right)\right| =$$

$$= \lim_{\omega\to\infty}\left|\sqrt{\frac{Lj\omega + R}{Cj\omega}}\tanh\left(\sqrt{-LC\omega^2 + RCj\omega}L_p\right)\right| =$$

$$= \sqrt{\frac{L}{C}} \qquad (59)$$

yielding

$$b_2 = \frac{4}{\mathsf{p}^2} L_p^2 L\sqrt{LC}. \qquad (60)$$

The remaining coefficients b_1 and a_1 are determined by the same procedure as in the two previous cases. A comparison of

$$\frac{b_2 s^2 + b_1 s + b_0}{a_2 s^2 + a_1 s + 1} =$$
$$= b_0 + (b_1 - b_0 a_1)s + (b_2 - b_0 a_2 + (-b_1 + b_0 a_1)a_1)s^2 + \dots \tag{61}$$

and

$$\sqrt{\frac{Ls+R}{Cs}}\tanh\left(\sqrt{(Ls+R)Cs}\,L_p\right) = L_p R + \left(L_p L - \frac{1}{3}L_p^3 R^2 C\right)s +$$
$$+\left[\frac{1}{3}L_p^3 RLC - L_p R\left(\frac{1}{2}L_p^2 LC + \frac{1}{24}L_p^4 R^2 C^2\right) + \frac{1}{2}L_p^2 RC\left(\frac{1}{3}L_p^3 R^2 C - L_p L\right)\right]s^2 + \dots \tag{62}$$

yields

$$b_1 = \frac{L_p\left(288 L_p^2 R^2 LC + \mathrm{p}^2 L_p^4 R^4 C^2 - 288 L_p RL\sqrt{LC} - 72\mathrm{p}^2 L^2\right)}{24\mathrm{p}^2\left(L_p^2 R^2 C - 3L\right)} \tag{63}$$

and

$$a_1 = \frac{L_p\left(96 L_p RLC - 96L\sqrt{LC} - 16\mathrm{p}^2 L_p RLC + 3\mathrm{p}^2 L_p^3 R^3 C^2\right)}{8\mathrm{p}^2\left(L_p^2 R^2 C - 3L\right)}. \tag{64}$$

The resulting transfer function has an integral character. Since the models are valid for well damped pipelines only, a second order transfer function is sufficient and the eigen-frequency of the pipeline with impedance boundary conditions which is close to the twice eigen-frequency of the pipeline with hybrid boundary conditions can not be recognised.

4. *Impedance representation: Input impedance at downstream dead end and negative output impedance at upstream dead end*

$$G_0(s) = Z_K \coth\left(nL_p\right) \tag{65}$$

The dead time of the treated transfer function is zero, since it connects a change of the pressure at one end of the pipeline if the flow changes at the same end, so

$$T_d = 0 \tag{66}$$

Transfer function (65) is the inverse of the transfer function (45) so the coefficients are:

$$\begin{aligned}
a_0 &= 0\\
a_1 &= L_p \cdot C\\
a_2 &= \frac{4}{\mathrm{p}^2} L_p^2 C\sqrt{LC}\\
b_0 &= 1\\
b_1 &= \frac{1}{3\mathrm{p}^2} L_p\left(12\sqrt{LC} + \mathrm{p}^2 L_p RC\right)\\
b_2 &= \frac{4}{\mathrm{p}^2} L_p^2 LC.
\end{aligned} \tag{67}$$

5. *Impedance representation: Negative reverse transimpedance at upstream dead end and negative transimpedance at downstream dead end*

$$G_0(s) = Z_K \frac{1}{\sinh\left(nL_p\right)}. \tag{68}$$

Since this transfer function connects change of the pressure at one end of the pipeline if the flow changes at the other end of the pipeline, the dead time for (68) is known – it is the time needed for the shock wave to travel along the pipeline.

$$T_d = \sqrt{LC}\,L_p. \tag{69}$$

If the pipeline is approximated with the second order system, what is sufficient for well damped pipelines, the known eigen-frequency can not be used. A determination of all coefficients of the second order transfer function by the Padé approximation results in a unsolvable system of equations. So an other procedure should be used. By this procedure it is supposed that the dynamics (i.e. the denominator of the corresponding transfer function) of all hybrid representations of the pipeline is the same.

Equation (68) can be written as:

$$Z_K \frac{1}{\sinh\left(nL_p\right)} = Z_K \coth\left(nL_p\right)\frac{1}{\cosh\left(nL_p\right)} \tag{70}$$

and is the product of transfer functions (65) and (33). The dead time is a part of the transfer function (33). The denominator of the Padé approximation of this transfer function cancels out the numerator of the Padé approximation of the transfer function (65), so the coefficients of the resulting transfer functions are:

$$\begin{aligned}
a_0 &= 0\\
a_1 &= L_p \cdot C\\
a_2 &= \frac{4}{\mathrm{p}^2} L_p^2 C\sqrt{LC}\\
b_0 &= 1\\
b_1 &= \frac{L_p C\left(96L - 192Le^{-\sqrt{\frac{C}{L}}\cdot\frac{RL_p}{2}} - 24\mathrm{p}^2 L - \mathrm{p}^2 L_p^2 R^2 C + 12\mathrm{p}^2 L_p R\sqrt{LC}\right)}{12\mathrm{p}^2\left(L_p RC - 2\sqrt{LC}\right)}\\
b_2 &= \frac{8}{\mathrm{p}^2} LCL_p^2 e^{-\sqrt{\frac{C}{L}}\cdot\frac{RL_p}{2}}
\end{aligned} \tag{71}$$

6. *Admittance representation: Input admittance at downstream reservoir and output admittance at upstream reservoir*

$$G_0(s) = \frac{1}{Z_K}\coth\left(nL_p\right) \tag{72}$$

The dead time of the treated transfer function is zero, since it connects a change of the flow at one end of the pipeline if the pressure changes at the same end, so

$$T_d = 0 \tag{73}$$

Transfer function (72) is the inverse of the transfer function (55) so the coefficients are:

$$\begin{aligned}
a_0 &= L_p R\\
a_1 &= \frac{L_p\left(288L_p^2 R^2 LC + \mathrm{p}^2 L_p^4 R^4 C^2 - 288L_p RL\sqrt{LC} - 72\mathrm{p}^2 L^2\right)}{24\mathrm{p}^2\left(L_p^2 R^2 C - 3L\right)}\\
a_2 &= \frac{4}{\mathrm{p}^2} L_p^2 L\sqrt{LC}\\
b_0 &= 1\\
b_1 &= \frac{L_p\left(96L_p RLC - 96L\sqrt{LC} - 16\mathrm{p}^2 L_p RLC + 3\mathrm{p}^2 L_p^3 R^3 C^2\right)}{8\mathrm{p}^2\left(L_p^2 R^2 C - 3L\right)}\\
b_2 &= \frac{1}{\omega_0^2} = \frac{4}{\mathrm{p}^2} L_p^2 LC.
\end{aligned}$$

7. *Admittance representation: Reverse transadmittance at upstream reservoir and admittance at downstream reservoir*

$$G_0(s) = \frac{1}{Z_K \sinh\left(nL_p\right)} = \frac{1}{Z_K \sinh\left(nL_p\right)e^{-sT_d}} e^{-sT_d}. \tag{74}$$

Since this transfer function connects change of the flow at one end of the pipeline if the pressure changes at the other end of the pipeline, the dead time for (74) is known – it is the time needed for the shock wave to travel along the pipeline.

$$T_d = \sqrt{LC}\, L_p. \tag{75}$$

If the pipeline is approximated with the second order system, what is sufficient for well damped pipelines, the known eigen-frequency can not be used. A determination of all coefficients of the second order transfer function by the Padé approximation results in a unsolvable system of equations. So again the procedure will be used which supposes that the dynamics of all hybrid representations of the pipeline is the same.
Equation (74) can be written as:

$$\frac{1}{Z_K \sinh\left(nL_p\right)} = \frac{1}{Z_K} \coth\left(nL_p\right) \frac{1}{\cosh\left(nL_p\right)} \qquad (76)$$

and is the product of transfer functions (72) and (33). The dead time is a part of the transfer function (33). The denominator of the Padé approximation of this transfer function cancels out the numerator of the Padé approximation of the transfer function (72), so the coefficients of the resulting transfer functions are:

$$\begin{aligned}
a_0 &= L_p R \\
a_1 &= \frac{L_p\left(288 L_p^2 R^2 LC + \mathrm{p}^2 L_p^4 R^4 C^2 - 288 L_p RL\sqrt{LC} - 72\mathrm{p}^2 L^2\right)}{24\mathrm{p}^2\left(L_p^2 R^2 C - 3L\right)} \\
a_2 &= \frac{4}{\mathrm{p}^2} L_p^2 L \sqrt{LC} \\
b_0 &= 1 \\
b_1 &= \frac{L_p C\left(96L - 192 L e^{-\sqrt{\frac{C}{L}} \cdot \frac{RL_p}{2}} - 24\mathrm{p}^2 L - \mathrm{p}^2 L_p^2 R^2 C + 12\mathrm{p}^2 L_p R\sqrt{LC}\right)}{12\mathrm{p}^2\left(L_p RC - 2\sqrt{LC}\right)} \\
b_2 &= \frac{8}{\mathrm{p}^2} LCL_p^2 e^{-\sqrt{\frac{C}{L}} \cdot \frac{RL_p}{2}}.
\end{aligned} \qquad (77)$$

5 VERIFICATION OF THE MODELS

The models were verified on a real pipeline with the following data: Length of the pipeline $L_p = 31\,146$ m, diameter $D = 0.143$ m, relative roughness $k_c = 0.0291$ mm, inclination $\alpha = 0$ and the fluid data: Density $\rho = 752$ kg/m³, kinematic viscosity $\nu = 7.18 \times 10^{-7}$ m²/s and velocity of sound $a = 1110$ m/s.

Fluid transients were generated for experimental verification by closing a shunt valve at the beginning of the pipeline. This leads to a quick drop in pressure causing fluid transients. There were no controllers for flow rate or pump pressure.

The excitation was quite high (more than 50 % of the steady state values of the signals). For this reason, the model was linearised at 75 % of the steady state flow (18 kg/s) yielding the parameters $L = 62.3$, $R = 10.6$ and $C = 1.3 \times 10^{-8}$. The damping factor for this case is $\zeta = 3.37$. The experimental data were first evaluated by the solving the nonlinear partial differential equations. The evaluation gave unsatisfactory results due to the dynamics of the transducers. Since all the data – both input and output – are measured, only the relative dynamics of the pressure and flow transducers is significant. A good coincidence was obtained by including the first order lag with the time constant of 15 s as the flow transducer dynamics.

Accordingly these dynamics were taken into account in the verification of the lumped parameter model. If the measured flow was used as the input to a model, the applied input was obtained by an approximate deconvolution (filtering by the $G_s^{-1} \approx \frac{15s+1}{\varepsilon s+1}$) where ε should be small, but was taken to be 3 so as not to magnify the measurement noise. The simulated output flow was first filtered by $G_s = \frac{1}{15s+1}$ and then compared with the measured flow.

Model forms 1 and 2 as described in section 2 were evaluated with the following results:

1. *Hybrid representation: Inputs Q_0, P_L, outputs Q_L, P_0*

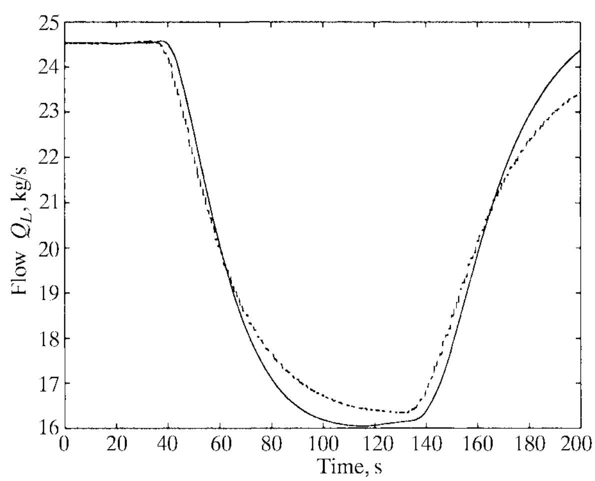

Fig. 1 The simulated (solid line) and the measured (dashed line) flows at the end of the pipeline

In Figure 1 comparison of the simulated (solid line) and measured (dashed line) flows at the end of the pipeline is shown.

Figure 2 depicts the simulated and the measured pressure at the beginning of the pipeline.

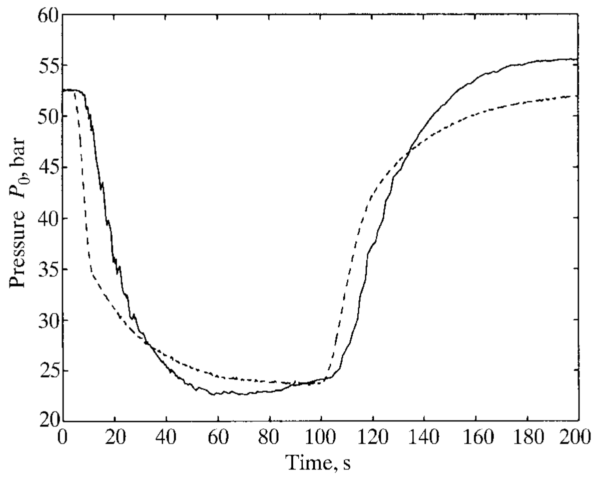

Fig. 2 The simulated (solid line) and the measured (dashed line) pressures at the beginning of the pipeline

In both figures an acceptable coincidence can be observed. The error between the outputs of the model and the real plant is caused by nonlinearities due to high excitation and, of course, by the approximation of the pipeline using a lumped parameter model. The »noise« in the simulated signals (especially in Figure 2) is due to the deconvolution of the model input signal (flow at the beginning of the pipeline) which increased the measurement noise.

2. *Hybrid representation: Inputs Q_L, P_0, outputs Q_0, P_L*

In Figure 3 the comparison of the simulated and measured flows at the beginning of the pipeline is shown.

Acceptable coincidence can again be observed. The verification of the fourth model (output pressure as the function of the output flow and input pressure) failed. The reason for that failure is as follows: according to Equation (10) the resulting output pressure is the difference of two terms (pressure changes due to the change of the output flow and input pressure, respectively).

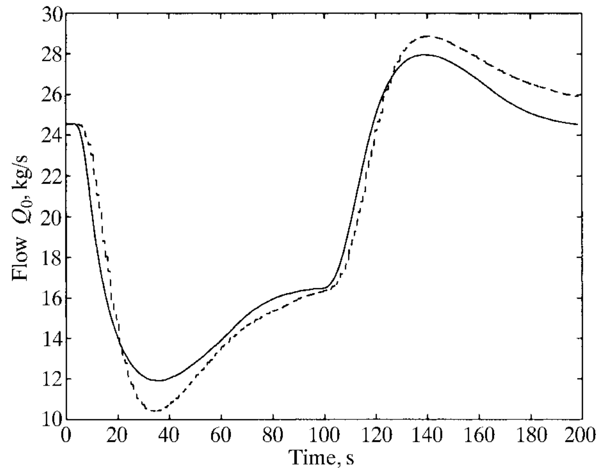

Fig. 3 The simulated (solid line) and the measured (dashed line) flows at the beginning of the pipeline

In the treated case, both terms have a magnitude of about 25 bars, while their difference should have a magnitude of 0.6 bars. The numerical problems arising due to nonlinearities and measurement noise magnified by the deconvolution caused the failure of the validation.

6 CONCLUSION

Three transcendent transfer functions which are parts of the linear model of the pipeline as a distributed parameter system are approximated by rational transfer functions with time delay. The approximations were verified on a real pipeline by using real experiment data. The approximations are only valid for one class of models – well damped pipelines. Simple models for pipelines which are able to oscillate are being investigated.

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Modeliranje cjevovoda kao sustava s usredotočenim parametrima. U članku se opisuje postupak pojednostavljenja matematičkih modela cjevovoda. Lineariziran je nelinearni matematički model s raspodijeljenim parametrima iz čega je dobivena prijenosna funkcija. Cjevovod je prikazan kao dvoulazni sustav. Analizirana su dva međusobno povezana prikaza triju različitih transcedentnih prijenosnih funkcija koje se potom nadomještaju racionalnim prijenosnim funkcijama uporabom Taylorovog razvoja u red. Izvedeni modeli valjani su za niske frekvencije i koriste se kako bi se došlo do boljih aproksimacija. Uspoređuje se visokofrekvencijsko pojačanje transcedentnih i racionalnih prijenosnih funkcija. Zbog aproksimacije visokofrekvencijskog pojačanja izvedeni su modeli valjani samo za dobro prigušene cjevovode. Izvedeni modeli, koji opisuju cjevovod kao sustav s usredotočenim parametrima, provjereni su na realnom cjevovodu.

Ključne riječi: modeliranje, aproksimacija modela, sustav s raspodijeljenim parametrima, sustav s usredotočenim parametrima, cjevovod

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