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A NECESSARY AND SUFFICIENT CONDITION FOR A SPACE TO BE INFRABARRELLED OR POLYNOMIALLY INFRABARRELLED

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ABSTRACT. A locally convex space E is infrabarrelled (resp. polynomially infrabarrelled) if and only if, for every Banach space F (resp. for every positive integer m and for every Banach space F), the space of all continuous linear mappings from E into F (resp. the space of all continuous m-homogeneous polynomials from E into F) is quasi-complete for the topology of bounded convergence.

Pfister [3] has shown that a locally convex space E is barrelled if and only if, for every Banach space F , the space of all continuous linear mappings from E into F is quasi-complete for the topology of pointwise convergence. The corresponding characterization of polynomially barrelled spaces has been obtained in [1]. In this note we prove that analogous results are valid in the infrabarrelled case, if one replaces the topology of pointwise convergence by the topology of bounded convergence.

In what follows, all vector spaces under consideration are vector spaces over a field **K** which is either **R** or **C**. Given a positive integer m and two locally convex spaces E and F, $P(^mE;F)$ denotes the vector space of all continuous m-homogeneous polynomials from E into F and τ_b denotes the locally convex topology of bounded convergence on $P(^mE;F)$. When $m = 1$, $P(^mE;F)$ is the vector space of all continuous linear mappings from E into F, represented by $L(E;F)$.

A locally convex space E is :(i) infrabarrelled if every strongly bounded subset of the topological dual of E is equicontinuous; (ii) polynomially infrabarrelled ([4], Proposition 3.10 and [5], Lemma 2) if, for every positive

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integer m, every τ_b-bounded subset of $P(^mE;\mathbf{K})$ is equicontinuous. Every polynomially infrabarrelled space is obviously infrabarrelled, but the converse is not true in general ; see Example 4.8 of [**4**].

In Theorems 1 and 2 below, E represents a locally convex space.

THEOREM 1. *For E to be an infrabarrelled space it is necessary and sufficient that, for every Banach space F, the space $(L(E;F),\tau_b)$ be quasi-complete.*

PROOF. The necessity of the condition follows immediately from (5), p. 144 of [**2**]. The sufficiency of the condition is established as in the proof of Theorem 2 below, with the obvious modifications. □

THEOREM 2. *For E to be a polynomially infrabarrelled space it is necessary and sufficient that, for every positive integer m and for every Banach space F, the space $(P(^mE;F),\tau_b)$ be quasi-complete.*

PROOF. The necessity of the condition follows immediately from Proposition 3.14 of [**4**].

In order to establish the sufficiency of the condition, let m be a positive integer and let X be a τ_b-bounded subset of $P(^mE;\mathbf{K})$. We shall prove that X is equicontinuous. For this purpose, let $B(X)$ be the space of all bounded mappings from X in $\mathbf{K}$, endowed with the supremum norm: if $h \in B(X)$, $||h|| = \sup\{|h(f)|; f \in X\}$. Then $(B(X),||.||)$ is a Banach space. Define $P: E \to B(X)$ by $P(x)(f) = f(x)$ for $x \in E$, $f \in X$ (note that $P(x) \in B(X)$ because X is pointwise bounded). It is easily seen that P is an m-homogeneous polynomial from E into $B(X)$.

Let Ω be the set of all non-empty bounded subsets of E. Arguing as in the proof of the theorem obtained in [**1**], we construct a subset $\{P_{B,\epsilon}; B \in \Omega, \epsilon > 0\}$ of $P(^mE;B(X))$ satisfying the following properties:

(1) For every $B \in \Omega$ and for every $\epsilon > 0$, $||P_{B,\epsilon}(x) - P(x)|| \leq \epsilon$ if $x \in B$;
(2) $\{P_{B,\epsilon}; B \in \Omega, \epsilon > 0\}$ is τ_b-bounded in $P(^mE;B(X))$.

For $B, B' \in \Omega$ and $\epsilon, \epsilon' > 0$, put $(B,\epsilon) \leq (B',\epsilon')$ if and only if $B \subset B'$ and $\epsilon' \leq \epsilon$. In this way, $\Omega \times]0,\infty[$ becames a directed set, and thus the set $\{P_{B,\epsilon}; B \in \Omega, \epsilon > 0\}$ may be regarded as a net in $P(^mE;B(X))$. By (1), $(P_{B,\epsilon})$ is a Cauchy net in $(P(^mE;B(X)),\tau_b)$. Therefore, by (2) and the hypothesis, there exists a $P' \in P(^mE;B(X))$ such that $(P_{B,\epsilon})$ converges to P' according to τ_b. By (1), $P = P'$, proving that $P \in P(^mE;B(X))$. Consequently, X is uniformly bounded on a neighborhood of 0 in E, which implies the equicontinuity of X by Theorem 1 of [**5**].

This completes the proof of the theorem. □

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