REAL RAMIFICATION POINTS AND REAL WEIERSTRASS POINTS OF REAL PROJECTIVE CURVES

E. Ballico

University of Trento, Italy

ABSTRACT. Here we construct real smooth projective curves with prescribed genus, gonality and topological type or with a real Weierstrass point with prescribed first positive non-gap.

1. INTRODUCTION

For any smooth and connected projective curve X of genus $g \geq 0$ defined over $\mathbb{R}$ let $X(\mathbb{R})$ denote its set of real points and $n(X)$ the number of the connected components of $X(\mathbb{R})$. Hence $X(\mathbb{R})$ is the disjoint union of $n(X)$ circles. Set $a(X) = 1$ if $X(\mathbb{C})\backslash X(\mathbb{R})$ is connected and $a(X) = 0$ if $X(\mathbb{C})\backslash X(\mathbb{R})$ is not connected, i.e. if $X(\mathbb{C})\backslash X(\mathbb{R})$ has two connected components. The topological pair $(X(\mathbb{C}), X(\mathbb{R}))$ is uniquely determined by the triple of integers $(g, n(X), a(X))$ and such a triple of integers (g, n, a) is associated to some smooth real genus g curve if and only if either $a = 0$, $n \equiv g + 1 \pmod{2}$ and $1 \leq n \leq g + 1$, or $a = 1$ and $0 \leq n \leq g$ ([5, Proposition 3.1]). Let X be a smooth and connected projective curve of genus $g \geq 2$. A point $P \in X$ is a Weierstrass point if and only if $h^0(X, \mathcal{O}_X(gP)) \geq 2$, i.e. (Riemann-Roch) if and only if $h^1(X, \mathcal{O}_X(gP)) > 0$. The first integer $t \geq 2$ such that $h^0(X, \mathcal{O}_X(tP)) = 2$ is called the first non-gap of P. The gonality $\mathrm{gon}(X)$ of X is the minimal integer $b > 0$ such that there is a degree b morphism $X \to \mathbf{P}^1$. It is known that $2 \leq \mathrm{gon}(X) \leq \lfloor (g + 3)/2 \rfloor$, that the general genus g curve has gonality $\lfloor (g+3)/2 \rfloor$, and that for every integer k such that $2 \leq k < \lfloor (g + 3)/2 \rfloor$ the set of all k-gonal curves of genus g is parametrized one-to-one by an irreducible algebraic variety of dimension $2g + 2k - 5$ ([1,

p. 346]). Here we prove the following results, the case $a = 1$ of Theorem 1.1 being previously known ([2, Theorem 0.1]).

THEOREM 1.1. *Fix integers g, k, a, n such that $g \geq 2k + 3 \geq 9$, $a \in \{0, 1\}$ and either $a = 1$ and $0 \leq n \leq g$, or $a = 0$, $1 \leq n \leq g + 1$ and $n \equiv g + 1$ (mod 2). If $n = 0$ assume k is even. Then there exists a smooth projective k-gonal curve X of genus g defined over $\mathbb{R}$ with $n(X) = n$, $a(X) = a$ and such that there is a degree k pencil on X is defined over $\mathbb{R}$.*

REMARK 1.2. The case $g \geq 5$ and $k = \lfloor (g + 3)/2 \rfloor$, i.e. the case of curves with general Brill-Noether theory for pencils, was proved by S. Chaudary ([3]).

THEOREM 1.3. *Fix integers g, k, a, n such that $g \geq 2k + 3 \geq 9$, $a \in \{0, 1\}$ and either $a = 1$ and $1 \leq n \leq g$ or $a = 0$ and $n = g + 1$. Then there exist a smooth projective k-gonal curve X of genus g defined over $\mathbb{R}$ with $n(X) = n$, $a(X) = a$, and $P \in X(\mathbb{R})$ such that k is the first positive non-gap of P and $|kP|$ is a degree k pencil on X.*

Fix a smooth genus g curve such that there is a degree k morphism $f : X \to \mathbf{P}^1$ with a total ramification point, P. Hence $\mathcal{O}_X(kP) \geq 2$. Hence if $k < g$, then P is a Weierstrass point of X. This observation explains why Theorems 1.1 and 1.3 are also existence theorems for real Weierstrass points.

2. THE PROOFS

PROPOSITION 2.1. *Fix a topological type (g, n, a) for smooth real curves such that $n \neq 0$. Then there exist smooth curves A, B defined over $\mathbb{R}$ and of topological type (g, n, a), degree 3 morphisms $f : A \to \mathbf{P}^1$ and $h : B \to \mathbf{P}^1$ defined over $\mathbb{R}$ and $P \in A(\mathbb{R})$, $Q \in B(\mathbb{C}) \backslash B(\mathbb{R})$ such that P is a total ramification point of f and Q is a total ramification point of h. When $g = 1$ we may also find a real degree 3 morphism $\tau : A \to \mathbf{P}^1$ with one real total ramification point and two non-real complex conjugate total ramification points. In the case $g \geq 2$ and $n = g + 1$ (and hence $a = 0$) we may even find a real curve of that topological type with a degree 3 real morphism with three total real ramification points and another one with one real total ramification point and two complex conjugate total ramification points.*

PROOF. We divide the proof into five steps.

(i) First assume $g = 0$. Hence $n = 1$ and $a = 0$. Take $P, P' \in \mathbf{P}^1(\mathbb{R})$, $P \neq P'$, $Q \in \mathbf{P}^1(\mathbb{C}) \backslash \mathbf{P}^1(\mathbb{R})$. Take as f the morphism induced by the linear system spanned by the effective divisors $3P$ and $3P'$ and as h the morphism induced by the linear system spanned by the effective divisors $3Q$ and $3\bar{Q}$. In this way we even get two total ramification points.

(ii) Now assume $g = 1$. If $n = 1$, then $a = 1$. If $n = 2$, then $a = 0$. Fix any real smooth genus 1 curve Z such that $Z(\mathbb{R}) \neq \emptyset$ and any degree 3 real embedding of Z into a plane. Any smooth real degree 3 plane curve

has three real flexes and six non-real flexes (case $n = 1$) or five real flexes and four non-real flexes (case $n = 2$). Use the morphism induced by the linear system spanned by the triple of two of these flexes to obtain f and h. To obtain the morphism τ of the "Furthermore" part take a degree 3 cyclic covering of $\mathbf{P}^1$ either ramified (and hence totally ramified) over three real points, or ramified (and hence totally ramified) over a real point and two non-real complex conjugate points.

(iii) Now assume $g \geq 2$ and $n = g + 1$. Hence $a = 0$. We first construct A, P, P' and f. Let (A_i, P_i, P'_i, f_i), $i = 1, \ldots, g$, be solutions for the case $g = 1$ and $n = 2$. Let E be the genus g stable curve of compact type constructed in the following way. E has g irreducible components and we will call them $Z_1, \ldots, Z_g$ because they are isomorphic to the previously chosen smooth genus one curves. We will identify all the points P_i, P'_i with the corresponding points of E. E has $g - 1$ singular points $O_1, \ldots, O_{g-1}$. Notice that P_1 and P'_g are smooth points of E, while all other points P_i, P'_i are singular points of E. We have $Z_i \cap Z_j \neq \emptyset$ if and only if $|i - j| \leq 1$. E is obtained from these data just gluing the point P'_i of Z_i to the point P_{i+1} of Z_{i+1}, $1 \leq i \leq g-1$. Let $\mathbb{T}$ denote the genus 0 nodal connected curve with g irreducible components $\mathbb{E}_i \cong \mathbf{P}^1$, $1 \leq i \leq g$, and $g - 1$ singular points $\mathbb{O}_i$, $1 \leq i \leq g - 1$, with $\mathbb{O}_i = \mathbb{E}_i \cap \mathbb{E}_{i+1}$, $1 \leq i \leq g - 1$. We identify (over $\mathbb{R}$) $\mathbb{E}_i \cong \mathbf{P}^1$ with the target of f_i. In this way the set of all f_i's induces a degree 3 admissible cover $\phi : E \to \mathbb{E}$ ([6, §4]) with P_1 and P'_g as total ramification points contained in E_{reg}. We prescribe that each $\mathbb{O}_i$ is obtained gluing together a real point of $\mathbb{E}_i$ with a real point of $\mathbb{E}_{i+1}$. In this way $\mathbb{E}$ gets a real structure compatible with ϕ. We get f smoothing (E, ϕ), but keeping P_1 and P'_g as total ramification points (see [3, Chapter 2] for real smoothings of admissible coverings). To construct h we take the same construction, except that for Z_1 and Z_g we use $\tau_1 : A_1 \to \mathbf{P}^1$ and $\tau_g : A_g \to \mathbf{P}^1$ with P'_1 and P_g a real total ramification point, while P_1 and P'_g are non-real total ramification points. For the last assertion of Proposition 2.1 we take two of the curves Z_i with three real total ramification points.

(iv) Now assume $g \geq 2$, $a = 0$ and $n < g + 1$. Hence $q := (g + 1 - n)/2$ is a positive integer. To get h we make the following construction. Take a solution (B, Q', h') for the topological type $(n - 1, n, 0)$ and the non-real total ramification point Q'. Let U be any smooth complex genus q curve equipped with a degree 3 morphism $u : U \to \mathbf{P}^1$ with two distinct total ramification points W, W'. Let $\bar{U}$ denote the complex conjugate curve of U; if U is seen as a Riemann surface with local charts, then $\bar{U}$ is the same topological manifold with, as gluing data for the same atlas, the complex conjugation of the gluing data of U; if U is seen as the zero-locus of some homogeneous complex polynomials in some projective space, then $\bar{U}$ is the zero-locus in the same projective space of the complex conjugate polynomials, i.e. of the polynomials obtained taking the complex conjugate of all coefficients; if U is seen as an abstract algebraic scheme in the sense of Groethendieck or as a variety in the

very old set-up of Weil, then $\bar{U}$ is obtained from U applying the action of the Galois group of the field extension $\mathbb{C}/\mathbb{R}$. Any of these equivalent descriptions gives the existence of a unique $\bar{W} \in \bar{U}(\mathbb{C})$ corresponding to W and a unique $\bar{W}' \in \bar{U}(\mathbb{C})$ corresponding to W'. Furthermore, u induces a degree 3 morphism $\bar{u} : \bar{U} \to \mathbf{P}^1$ such that $\bar{W}$ and $\bar{W}'$ are total ramification points of $\bar{u}$. Let Γ be the genus g nodal curve of compact type constructed in the following way. Γ has three irreducible components, respectively isomorphic to B (over $\mathbb{R}$), to U and to $\bar{U}$, and two singular points $O, \bar{O}$, with O obtained gluing together the point W' of U with the point Q' of B and $\bar{O}$ obtained gluing together the point $\bar{Q}'$ of B with the point $\bar{W}$ of $\bar{U}$. With these gluings the curve Γ has a real structure such that U and $\bar{U}$ are exchanged by the complex conjugation and hence $\Gamma(\mathbb{R}) = B(\mathbb{R})$. Thus $\Gamma(\mathbb{R})$ is the disjoint union of n circles. It is easy to check that $\Gamma(\mathbb{C})\backslash\Gamma(\mathbb{R})$ has two connected components. Taking as target a genus 0 nodal curves with three irreducible components and using $h', u, \bar{u}$ we get an admissible cover whose general smoothing gives a solution for h for the topological type $(g, n, 0)$: the non-real total ramification point comes from $W \in \Gamma$. Now we show the existence of f, A, P. We start with a real smooth curve A' of type $(n-1, n, 0)$ equipped with a degree 3 real morphism $f' : A' \to \mathbf{P}^1$ with one real total ramification point and two complex conjugate complex ramification points. We use the two complex conjugate ramification points to add two genus q tails as in the proof of the existence of h.

    (v) Now we assume $g \geq 2$ and $a = 1$. By assumption we have $1 \leq n \leq g$. We fix g smooth real genus one curves A_i, $1 \leq i \leq g$, such that for all i there is a degree 3 real morphism $f_i : A_i \to \mathbf{P}^1$ with two real total ramification points P_i, P_i'. If $n(A_i) = 2$, i.e. if $a(A_i) = 1$, we also assume that the points P_i and P_i' are in different connected components of $A_i(\mathbb{R})$. We assume $n(A_i) = 1$ for $1 \leq i \leq g + 1 - n$ and $n(A_i) = 2$ for all $i > g + 1 - n$ (if any). Let E be the unique curve of compact type defined over $\mathbb{R}$ with arithmetic genus g built in the following way. E has g irreducible components B_i, $1 \leq i \leq g$, each of them defined over $\mathbb{R}$. We assume the existence of a real isomorphism $u_i : A_i \to B_i$ and we set $Q_i := u_i(P_i)$, $Q_i' := u_i(P_i')$. We assume $B_i \cap B_j \neq \emptyset$ if and only if $|i - j| \leq 1$. For all $1 \leq i \leq g - 1$ the only point of $B_i \cap B_{i+1}$ is the point $Q_i' \in B_i$, which is identified (glued) to the point $Q_{i+1} \in B_{i+1}$. Since for all $1 \leq i \leq g - 1$ the point P_i' is a total ramification point of f_i and P_{i+1} is a total ramification point of f_{i+1}, the g degree 3 coverings $f_1, \ldots, f_g$ define a degree 3 admissible covering ϕ of E. Both E and ϕ are defined over $\mathbb{R}$. Notice that $E(\mathbb{R})$ has n connected components. Since $a(B_1) = 1$, the open set $E(\mathbb{C})\backslash E(\mathbb{R})$ is connected. Notice that Q_1 and Q_g' are total ramification points. To conclude the proof we take a real smoothing of the admissible covering ϕ. $\quad\square$

PROOF OF THEOREM 1.1. If $a(X) = 1$ the result is [2, Theorem 0.1], except for the uniqueness part. The uniqueness part is true because it is true outside a proper closed subset (for the Zariski topology) of the variety parametrizing all smooth complex k-gonal curves of genus g. Hence we will only check the case $a = 0$.

(a) Here we will check the case $n = g + 1$. Let Y be a smooth hyperelliptic curve of genus g defined over $\mathbb{R}$ with $n(Y) = g+1$ and $a(Y) = 0$; we also assume that the hyperelliptic involution is the only non-trivial holomorphic automorphism of Y. The curve Y obviously exists and it corresponds to the case in which all $2g + 2$ ramification points of the hyperelliptic pencil are real ([5, §6]) and not too special. The assumption on $\mathrm{Aut}(Y)$ implies that Y has a unique real structure. Let $\Delta(Y)$ be the local moduli space of Y. The germ $\Delta(Y)$ is a smooth $3g - 3$ dimensional germ of a complex space around the point $[Y]$ representing Y. The complex conjugation σ acts on $\Delta(Y)$. Since $Y(\mathbb{R}) \neq \emptyset$ and near $[Y]$ all complex curves have at most one complex structure, the set of real curves near Y is parametrized by the fix point set $\Delta(Y)^\sigma$ of σ. Since $\Delta(Y)$ is smooth, every connected component of $\Delta(Y)^\sigma$ is a real $3g - 3$ dimensional differential manifold. Since $[Y] \in \Delta(Y)^\sigma$, we see that the image of $\Delta(Y)^\sigma$ in the moduli space $\mathcal{M}_g$ is a Zariski dense subset of the moduli space $\mathcal{M}_g$ containing the element $[[Y]]$ representing Y. Let $\mathcal{M}_g^\circ$ denote the open subset of $\mathcal{M}_g$ parametrizing all smooth curves without non-trivial automorphism. The variety $\mathcal{M}_g^\circ$ is smooth. Since each $C \in \mathcal{M}_g^\circ$ has no holomorphic involution, each $C \in \mathcal{M}_g^\circ$ has at most one real structure. There is an anti-holomorphic involution σ on $\mathcal{M}_g^\circ$ whose fixed locus $\mathcal{M}_g^\circ(\mathbb{R})$ is the set of all $C \in \mathcal{M}_g^\circ$ with a real structure. Hence every connected component of $\mathcal{M}_g^\circ(\mathbb{R})$ is a $(3g - 3)$-dimensional differentiable manifold. Set $\mathcal{M}_{g;k}^\circ := \{C \in \mathcal{M}_g^\circ : C$ is k-gonal$\}$. $\mathcal{M}_{g;k}^\circ \neq \emptyset$ because $3 \leq k \leq (g - 3)/2$. Furthermore, $\mathcal{M}_{g;k}^\circ$ is smooth, irreducible and of dimension $2g + 2k - 5$. The closure $\Gamma(k)$ of $\mathcal{M}_{g;k}^\circ$ in $\mathcal{M}_g$ contains the hyperelliptic locus. Hence in $\Delta(Y)$ there is an irreducible σ-invariant $2g + 2k - 5$ complex space with $[Y]$ in its closure and parametrizing k-gonal curves. Hence $[Y]$ is in the closure in $\Delta(Y)^\sigma$ of a family of real k-gonal curves, each of them with the real topological type of Y, concluding the proof of the case $n = g + 1$.

(b) Here we will check the case $a = 0$, $1 \leq n \leq g - 1$ and $n \equiv g + 1$ (mod 2). Just use the previous proof starting from the case $k = 3$ (proved in Proposition 2.1) instead of the hyperelliptic case lifted from [5, §6]. $\square$

PROOF OF THEOREM 1.3. We modify the proof of Proposition 2.1 taking degree k morphisms (and their total ramification points) instead of degree 3 morphisms. In the case $g = 0$ we just write k instead of 3. First assume $a = 0$. If $k \geq 4$, there are real genus 1 smooth curves with two total real ramification points, but by Riemann-Hurwitz formula there is no genus 1 smooth curve with a degree k morphism with three or more total ramification points. The

proof of the case $n = g + 1$ of Proposition 2.1 works verbatim when $k \geq 4$, and this is the reason of our restriction in the case $a = 0$ of Theorem 1.3. If $a = 1$ part (v) of the proof of Proposition 2.1 works verbatim.   □

References

[1] E. Arbarello and M. Cornalba, *Footnotes to a paper of Beniamino Segre: The number of g_d^1's on a general d-gonal curve, and the unirationality of the Hurwitz spaces of 4-gonal and 5-gonal curves*, Math. Ann. **256** (1981), 341–362.

[2] E. Ballico, *Real algebraic curves and real spanned bundles*, Ricerche Mat. **50** (2001), 223-241.

[3] S. Chaudary, The Brill-Noether theorem for real algebraic curves, Ph. D. Thesis, Duke University, 1995.

[4] D. Eisenbud and J. Harris, *Limit linear series: basic theory*, Invent. Math. **85** (1986), 337-371.

[5] B. H. Gross and J. Harris, *Real algebraic curves*, Ann. Scient. École Norm. Sup. (4) **14** (1981), 157-182.

[6] J. Harris and D. Mumford, *On the Kodaira dimension of the moduli space of curves*, Invent. Math. **67** (1982), 23-88.

E. Ballico
Department of Mathematics
University of Trento
38050 Povo (TN)
Italy
E-mail: `ballico@science.unitn.it`