

AN IMPROVED INEQUALITY FOR  $k$ -TH DERIVATIVE OF  
A POLYNOMIAL

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ABSTRACT. For a polynomial  $p(z)$  of degree  $n$ , we have obtained

$$|p^{(k)}(\beta)| \leq \frac{n(n-1)(n-2)\dots(n-k+1)}{|\beta|^k} \left[ \frac{1}{2^k} \{ |p(\beta)| + \max_{1 \leq i \leq n} |p(\beta z_i)| \} + \right. \\ \left. \left( 1 - \frac{1}{2^{k-1}} \right) \max_{1 \leq i \leq 2n} |p(\beta b_i)| \right], \beta \neq 0 \text{ \& } k \geq 1,$$

a refinement of the well known Bernstein's inequality

$$\max_{|z|=1} |p^{(k)}(z)| \leq n(n-1)(n-2)\dots(n-k+1) \max_{|z|=1} |p(z)|,$$

$z_1, z_2, \dots, z_n$  being the zeros of  $z^n + 1$  and  $b_1, b_2, \dots, b_{2n}$  the zeros of  $z^{2n} - 1$ .  
The inequality is sharp.

## 1. INTRODUCTION AND STATEMENT OF RESULTS

Let  $p(z)$  be a polynomial of degree  $n$ . It easily follows from the well known Bernstein's theorem [3] that

$$(1.1) \quad \max_{|z|=R} |p'(z)| \leq \frac{n}{R} \max_{|z|=R} |p(z)|, R > 0,$$

with equality in (1.1) for  $p(z) = \alpha z^n$ . On applying inequality (1.1) again and again, we get

$$(1.2) \quad \max_{|z|=R} |p^{(k)}(z)| \leq \frac{n(n-1)\dots(n-k+1)}{R^k} \max_{|z|=R} |p(z)|, \quad R > 0 \text{ \& } k \geq 1,$$

with equality in (1.2) for  $p(z) = \alpha z^n$ .

We have been able to improve the inequality (1.2) and obtain a new inequality, which is sharp. More precisely, we prove

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THEOREM 1.1. *Let  $p(z)$  be a polynomial of degree  $n$ . Then*

$$(1.3) \quad |p^{(k)}(\beta)| \leq \frac{n(n-1)\dots(n-k+1)}{|\beta|^k} \left[ \frac{1}{2^k} \{ |p(\beta)| + \max_{1 \leq i \leq n} |p(\beta z_i)| \} \right. \\ \left. + \left( 1 - \frac{1}{2^{k-1}} \right) \max_{1 \leq t \leq 2n} |p(\beta b_t)| \right], \beta \neq 0, \text{ \& } k \geq 1,$$

where  $z_1, z_2, \dots, z_n$  are the zeros of  $z^n + 1$  and  $b_1, b_2, \dots, b_{2n}$  are the zeros of  $z^{2n} - 1$ . The inequality is sharp and the extremal polynomial is  $p(z) = \alpha z^n$ .

## 2. LEMMAS

For the proof of Theorem 1.1, we require the following lemmas.

LEMMA 2.1. *Let  $p(z)$  be a polynomial of degree  $n$  and  $z_1'', z_2'', \dots, z_n''$  be the zeros of  $z^n + a$ ,  $a \neq 0$ . Then for any complex number  $\beta$  such that  $\beta^n + a \neq 0$ , we have*

$$(2.4) \quad p'(\beta) = \frac{n\beta^{n-1}}{a + \beta^n} p(\beta) + \frac{a + \beta^n}{na} \sum_{i=1}^n p(z_i'') \frac{z_i''}{(z_i'' - \beta)^2}$$

$$(2.5) \quad \frac{1}{na} \sum_{i=1}^n \frac{z_i'' \beta}{(z_i'' - \beta)^2} = -\frac{n\beta^n}{(\beta^n + a)^2}.$$

This lemma is due to Aziz [1].

LEMMA 2.2. *Let  $p(z)$  be a polynomial of degree  $n$ . Then*

$$(2.6) \quad \max_{|z|=1} |p'(z)| \leq n \max_{1 \leq t \leq 2n} |p(b_t)|,$$

where  $b_1, b_2, \dots, b_{2n}$  are as in Theorem 1.1.

This lemma is due to Frappier, Rahman and Ruscheweyh [2].

LEMMA 2.3. *Let  $p(z)$  be a polynomial of degree  $n$ . Then for  $s \geq 1$  and  $|\beta| = 1$ ,*

$$(2.7) \quad |p^{(s)}(\beta)| \leq \frac{n-s+1}{2} \{ |p^{(s-1)}(\beta)| + \max_{1 \leq m \leq n-s+1} |p^{(s-1)}(u_m')| \},$$

where  $u_1', u_2', \dots, u_{n-s+1}'$  are the roots of

$$(2.8) \quad z^{n-s+1} + a = 0$$

$$(2.9) \quad a = e^{i\gamma(n-s+1)}$$

and

$$\gamma = \arg \beta$$

PROOF OF LEMMA 2.3. As

$$\beta^{n-s+1} + a \neq 0,$$

we have on applying Lemma 2.1 to the polynomial  $p^{(s-1)}(z)$ ,

$$\begin{aligned} p^{(s)}(\beta) &= \frac{(n-s+1)\beta^{n-s}}{\beta^{n-s+1} + a} p^{(s-1)}(\beta) \\ &+ \frac{\beta^{n-s+1} + a}{(n-s+1)a} \sum_{m=1}^{n-s+1} p^{(s-1)}(u'_m) \frac{u'_m}{(u'_m - \beta)^2}, \end{aligned}$$

which implies

$$\begin{aligned} (2.10) \quad |p^{(s)}(\beta)| &\leq \frac{n-s+1}{2} |p^{(s-1)}(\beta)| + \frac{2}{n-s+1} \left\{ \sum_{m=1}^{n-s+1} \left| \frac{u'_m}{(u'_m - \beta)^2} \right| \right\} \\ &\max_{1 \leq m \leq n-s+1} |p^{(s-1)}(u'_m)| = \frac{n-s+1}{2} |p^{(s-1)}(\beta)| \\ &+ \frac{2}{n-s+1} \left\{ \sum_{m=1}^{n-s+1} \left| \frac{u'_m \beta}{(u'_m - \beta)^2} \right| \right\} \max_{1 \leq m \leq n-s+1} |p^{(s-1)}(u'_m)| \\ &= \frac{n-s+1}{2} |p^{(s-1)}(\beta)| + \frac{2}{n-s+1} \left\{ - \sum_{m=1}^{n-s+1} \frac{u'_m \beta}{(u'_m - \beta)^2} \right\} \\ &\max_{1 \leq m \leq n-s+1} |p^{(s-1)}(u'_m)|, \end{aligned}$$

as  $\frac{u'_m \beta}{(u'_m - \beta)^2}$  is a negative real number for  $m = 1, 2, \dots, n-s+1$ . Now by the second part of Lemma 2.1, (2.10) can be written as

$$\begin{aligned} |p^{(s)}(\beta)| &\leq \frac{n-s+1}{2} |p^{(s-1)}(\beta)| + \frac{2}{n-s+1} \left\{ \frac{(n-s+1)^2 \beta^{n-s+1} a}{(\beta^{n-s+1} + a)^2} \right\} \\ &\max_{1 \leq m \leq n-s+1} |p^{(s-1)}(u'_m)| = \frac{n-s+1}{2} |p^{(s-1)}(\beta)| \\ &+ \frac{2}{n-s+1} (n-s+1)^2 \frac{1}{4} \max_{1 \leq m \leq n-s+1} |p^{(s-1)}(u'_m)| \end{aligned}$$

and this completes the proof of Lemma 2.3.

### 3. PROOF OF THEOREM 1.1

Let

$$(3.11) \quad T(z) = p(\beta z).$$

Then

$$(3.12) \quad |p^{(k)}(\beta)| = \frac{1}{|\beta|^k} |T^{(k)}(1)|$$

Now by Lemma 2.3, we have for  $k \geq 2$ .

$$(3.13) \quad |T^{(k)}(1)| \leq \frac{n-k+2}{2} \{ |T^{(k-1)}(1)| + \max_{1 \leq m \leq n-k+1} |T^{(k-1)}(u_m)| \}$$

where  $u_1, u_2, \dots, u_{n-k+1}$  are the roots of  $z^{n-k+1} + 1 = 0$ . Again by Lemma 2.3, we have

$$(3.14) \quad |T^{(k-1)}(1)| \leq \frac{n-k+2}{2} \{ |T^{(k-2)}(1)| + \max_{1 \leq j \leq n-k+2} |T^{(k-2)}(w_j)| \},$$

where  $w_1, w_2, \dots, w_{n-k+2}$  are the roots of  $z^{n-k+2} + 1 = 0$ . Combining (3.13) and (3.14), we obtain

$$\begin{aligned} |T^{(k)}(1)| &\leq \frac{n-k+1}{2} \frac{n-k+2}{2} |T^{(k-2)}(1)| + \\ &\quad \frac{n-k+1}{2} \frac{n-k+2}{2} \max_{1 \leq j \leq n-k+2} |T^{(k-2)}(w_j)| + \\ &\quad \frac{n-k+1}{2} \max_{1 \leq m \leq n-k+1} |T^{(k-1)}(u_m)|. \end{aligned}$$

Continuing similarly, we obtain for  $k \geq 2$

$$\begin{aligned} |T^{(k)}(1)| &\leq \frac{n-k+1}{2} \frac{n-k+2}{2} \dots \frac{n}{2} |T(1)| \\ &\quad + \frac{n-k+1}{2} \frac{n-k+2}{2} \dots \frac{n}{2} \max_{1 \leq i \leq n} |T(z_i)| \\ &\quad + \frac{n-k+1}{2} \frac{n-k+2}{2} \dots \frac{n-1}{2} \max_{1 \leq h \leq n-1} |T'(x_h)| \\ &\quad + \frac{n-k+1}{2} \frac{n-k+2}{2} \dots \frac{n-2}{2} \max_{1 \leq g \leq n-2} |T''(d_g)| + \dots \\ &\quad \dots + \frac{n-k+1}{2} \frac{n-k+2}{2} \max_{1 \leq j \leq n-k+2} |T^{(k-2)}(w_j)| \\ (3.15) \quad &\quad + \frac{n-k+1}{2} \max_{1 \leq m \leq n-k+1} |T^{(k-1)}(u_m)|, \end{aligned}$$

where  $z_1, z_2, \dots, z_n$  are the roots of  $z^n + 1 = 0$ ,  $x_1, x_2, \dots, x_{n-1}$  are the roots of  $z^{n-1} + 1 = 0$ ,  $d_1, d_2, \dots, d_{n-2}$  are the roots of  $z^{n-2} + 1 = 0$ , and so on.

Now, by Lemma 2.2 and Bernstein's theorem [3], we have from (3.15), for  $k \geq 2$

$$\begin{aligned} |T^{(k)}(1)| &\leq \frac{n-k+1}{2} \frac{n-k+2}{2} \dots \frac{n}{2} |T(1)| \\ &\quad + \frac{n-k+1}{2} \frac{n-k+2}{2} \dots \frac{n}{2} \max_{1 \leq i \leq n} |T(z_i)| \end{aligned}$$

$$\begin{aligned}
& + \frac{n-k+1}{2} \frac{n-k+2}{2} \cdots \frac{n-1}{2} n \max_{1 \leq t \leq 2n} |T(b_t)| \\
& + \frac{n-k+1}{2} \frac{n-k+2}{2} \cdots \frac{n-2}{2} (n-1)n \max_{1 \leq t \leq 2n} |T(b_t)| \\
& \cdots + \frac{n-k+1}{2} \frac{n-k+2}{2} (n-k+3) \cdots n \max_{1 \leq t \leq 2n} |T(b_t)| \\
& + \frac{n-k+1}{2} (n-k+2)(n-k+3) \cdots n \max_{1 \leq t \leq 2n} |T(b_t)| \\
& = (n-k+1)(n-k+2) \cdots (n-1)n \left[ \frac{1}{2^k} \{|T(1)| + \max_{1 \leq i \leq n} |T(z_i)|\} \right. \\
(3.16) \quad & \left. + \left(1 - \frac{1}{2^{k-1}}\right) \max_{1 \leq t \leq 2n} |T(b_t)| \right]
\end{aligned}$$

Further, for  $k = 1$ , we have by Lemma 2.3

$$(3.17) \quad |T'(1)| \leq \frac{n}{2} \{|T(1)| + \max_{1 \leq i \leq n} |T(z_i)|\},$$

On combining (3.16) and (3.17), we get for  $k \geq 1$

$$(3.18) \quad |T^{(k)}(1) \leq (n-k+1)(n-k+2) \cdots n \left[ \frac{1}{2^k} \{|T(1)| \right.$$

$$(3.19) \quad \left. + \max_{1 \leq i \leq n} |T(z_i)|\} + \left(1 - \frac{1}{2^{k-1}}\right) \max_{1 \leq t \leq 2n} |T(b_t)| \right]$$

which, by (3.11) & (3.12), implies

$$(3.20) \quad |p^{(k)}(\beta)| \leq \frac{1}{|\beta|^k} (n-k+1)(n-k+2) \cdots n \left[ \frac{1}{2^k} \{|p(\beta)| \right.$$

$$(3.21) \quad \left. + \max_{1 \leq i \leq n} |p(\beta z_i)|\} + \left(1 - \frac{1}{2^{k-1}}\right) \max_{1 \leq t \leq 2n} |p(\beta b_t)| \right]$$

thereby proving Theorem 1.1.

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