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SOME SYMMETRIC $(71, 15, 3)$ DESIGNS WITH AN INVOLUTORY ELATION

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Abstract. Symmetric designs for $(71, 15, 3)$ with the semi-standard automorphism group $G \cong E_8 \cdot F_{21}$ have been investigated. There were constructed exactly three nonisomorphic designs, two of them with an involutory elation.

1. Introduction

In [1], we have proved that exactly six nonisomorphic triplanes for $(71, 15, 3)$ exists having the automorphism group $G \cong E_8 \cdot F_{21}$ acting non-semistandardly. The group G is a faithful extension of an elementary abelian group E_8 of order 8 by a Frobenius group of order 21. Here we shall confine our attention to the semi-standard automorphism group situation. We use the method from [2] and this group G to "construct" the triplanes for $(71, 15, 3)$ under this assumption. We have found that there exist triplanes with these parameters nonisomorphic to the triplanes in [1], admitting an elation of order 2 and also another class of triplanes without that property. Hence, the property of admitting an involutory elation yields a new approach to the construction and the classification of symmetric design for $(71, 15, 3)$.

2. Notation and basic definitions

Suppose G is an authomorphism group of a symmetric $D = (v, k, \lambda)$ design. Actually, G has orbits of lengths $a_1, a_2, \dots, a_c$ on points and orbits of lengths $b_1, b_2, \dots, b_c$ on blocks of D respectively.

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DEFINITION 1. *An automorphism group G is said to be semi-standard if, after possibly renumbering the orbits we have $a_i \leq a_j$ and $b_i \leq b_j$ for $i \leq j$ and if $a_i = b_i$ for all $i = 1, 2, \ldots, c$.*

We also say that a group G acts semistandardly. In the oposite case, we say G acts non-semistandardly.

DEFINITION 2. *Let $\alpha \neq 1$ be an automorphism on a symmetric $2-(v,k,\lambda)$ design. We say that α is an elation if there exists an α-fixed block b containing exactly k α-fixed points and if α fixes a set of k blocks having exactly one α-fixed point of b in common.*

Notation and other definitions can be found in [1].

3. Results of the investigation in the semi-standard group situation

Now, suppose the group $G \cong E_8 \cdot F_{21}$ is an automorphism group of a symmetric $2-(71,15,3)$ design. This group G is generated by the set of three elements of order two, three and seven respectively. (For details of the action of generators see [1].) It was proved in [1] that an involution had seven or fifteen points, an automorphism of order three fixed five points and an automorphism of order seven fixed exactly one point. In the case where G acts semistandardly it was proved that G had three orbits on the points(blocks) having lengths $1, 14, 56$. Here we have constructed symmetric designs D_5, D_6 and D_7, all of them nonisomorphic. (See the method in [2].) Without loss of generality the constructed designs have the following block orbit representatives:

The first block orbit representative of D_5, D_6 and D_7:

$$1_1\ 2_1\ 2_2\ 2_3\ 2_4\ 2_5\ 2_6\ 2_7\ 2_8\ 2_9\ 2_{10}\ 2_{11}\ 2_{12}\ 2_{13}\ 2_{14}$$

The second block orbit representative of D_5, D_6 and D_7:

$$1_1\ 2_1\ 2_2\ 3_8\ 3_9\ 3_{11}\ 3_{18}\ 3_{19}\ 3_{21}\ 3_{25}\ 3_{33}\ 3_{38}\ 3_{45}\ 3_{51}\ 3_{53}$$

The third block orbit representatives for

$$D_5:\ \ 2_5\ 2_7\ 2_8\ 3_8\ 3_9\ 3_{11}\ 3_{15}\ 3_{16}\ 3_{26}\ 3_{27}\ 3_{30}\ 3_{35}\ 3_{48}\ 3_{52}\ 3_{55}$$

$$D_6:\ \ 2_5\ 2_7\ 2_8\ 3_8\ 3_9\ 3_{11}\ 3_{15}\ 3_{27}\ 3_{28}\ 3_{32}\ 3_{37}\ 3_{42}\ 3_{46}\ 3_{49}\ 3_{52}$$

$$D_7:\ 2_5\ 2_7\ 2_8\ 3_{10}\ 3_{12}\ 3_{13}\ 3_{16}\ 3_{25}\ 3_{26}\ 3_{30}\ 3_{33}\ 3_{35}\ 3_{48}\ 3_{51}\ 3_{55}$$

Here the intersection numbers of all triples of blocks for D_5,D_6 and D_7 have been calculated but the constructed designs D_6 and D_7 do not differ in this characteristic. Hence we have used the method of spreads (see [1]) to prove the nonisomorphism of designs D_5, D_6 and D_7. We proved that all of them are mutually nonisomorphic as well as nonisomorphic to the designs constructed in [1]. Compare the results of investigation in the *Table* 1 with the results in [1].

Table 1:

D_i	b_*	(r_1, r_2, r_3)	(s_1, s_2, s_3)
D_5	1	(35, 63, 69)	(56, 52, 100)
D_6	1	(35, 51, 63)	(56, 160, 86)
D_7	1	(35, 51, 63)	(56, 160, 70)

Here, b_* is the number of special blocks of a design D_i, r_i means the number of blocks of a reduced structure I_b/R where b is an element of the i-th orbit of blocks of D_i and s_i means the number of spreads with respect to the block b in the i-th orbit of blocks of D_i.

It was interesting to investigate the acting of involution $f = \rho^*$ fixing fifteen points of the constructed designs D_5 ,D_6 and D_7. (See the permutation ρ^* in [1]). Here we have proved that f was not an automorphism on D_5 but f acted on D_6 and D_7 as an elation.

Now the first orbit of blocks of D_6 and D_7 contains just one G-fixed block containing exactly fifteen f-fixed points. The second orbit consists of fourteen blocks which all have the point 1_1 in common. We find that the blocks incident with the point 1_1 are all the f-fixed blocks. Here a pair of points of these blocks is either fixed or interchanged by f. The third orbit of blocks of D_6 and D_7 contains further fifty-six blocks of these designs. By verification it turns out that this permutation f breaks up this orbit of blocks in twenty-eight pairs of blocks interchanged by f for both D_6 and D_7. As one can see, the permutation f is an elation on the symmetric designs D_6 and D_7 respectively. This automorphism f is different from, and commutes with the automorphisms provided by the group G and it was found under this assumption. We have seen that the permutation f was not an automorphism on D_1, it acted as an automorphism on the symmetric designs D_2 and D_3 (see in [1]) but it was not an elation on designs D_2 and D_3.

We have investigated only the symmetric $(71, 15, 3)$ designs with the automorphism group $G \cong E_8 \cdot F_{21}$. We did not check whether it was possible to produce still more symmetric $(71, 15, 3)$ designs with an involutory elation in another case. Thus we have the following result:

THEOREM 3.1. *There exist at least two non-isomorphic symmetric* $(71, 15, 3)$ *designs admitting an involutory elation. Both of them are non-self-dual.*

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