GLASNIK MATEMATIČKI
Vol. 36(56)(2001), 39 – 48

STARLIKE MAPPINGS OF ORDER α ON THE UNIT BALL IN COMPLEX BANACH SPACES

Hidetaka Hamada, Gabriela Kohr and Piotr Liczberski

Kyushu Kyoritsu University, Japan
Babeş-Bolyai University, Romania
Technical University of Lódź, Poland

Abstract. In this paper, we will give the growth theorem of starlike mappings of order α on the unit ball B in complex Banach spaces. We also give an analytic sufficient condition for a locally biholomorphic mapping on B to be a starlike mapping of order α.

1. Introduction

It is well known that the classical growth theorem of normalized biholomorphic mappings on the unit disc Δ in $\mathbf{C}$ cannot be generalized to normalized biholomorphic mappings on the Euclidean unit ball in $\mathbf{C}^n$. Barnard, FitzGerald and Gong [1] and Chuaqui [3] extended the classical growth theorem to normalized starlike mappings on the Euclidean unit ball in $\mathbf{C}^n$. Dong and Zhang [4] generalized the above result to normalized starlike mappings on the unit ball in complex Banach spaces. The first and second authors [7] generalized the above result to spirallike mappings of type α on the unit ball B in an arbitrary complex Banach space. The second author [12], [13] gave a growth theorem of normalized starlike mappings of order α on the Euclidean unit ball in $\mathbf{C}^n$.

On the other hand, Becker [2] showed that if a holomorphic function f on Δ satisfies

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{1}{1-|z|^2},$$

2000 *Mathematics Subject Classification.* 32A30, 32H02, 30C45.

Key words and phrases. Banach space, biholomorphic mappings, growth theorem, starlike mapping of order α.

then f is univalent on Δ. Pfaltzgraff [18] generalized the above result for normalized locally biholomorphic mappings on the Euclidean unit ball $\mathbf{B}^n$ in $\mathbf{C}^n$. He showed that if a normalized locally biholomorphic mapping f on $\mathbf{B}^n$ satisfies

$$\|(Df(z))^{-1}D^2f(z)(z,\cdot)\| \leq \frac{1}{1-\|z\|^2},$$

then f is univalent on $\mathbf{B}^n$ and

$$\frac{\|z\|}{(1+\|z\|)^2} \leq \|f(z)\| \leq \frac{\|z\|}{(1-\|z\|)^2}.$$

The third author [16] showed that if a locally biholomorphic mapping f on $\mathbf{B}^n$ satisfies

$$\|(Df(z))^{-1}D^2f(z)(z,\cdot)\| < \frac{1}{1+\|z\|},$$

then f is a starlike mapping on $\mathbf{B}^n$.

In this paper, we will give the growth theorem of normalized starlike mappings of order α on the unit ball B in complex Banach spaces. As a generalization of the result in [16], we also give a sufficient condition for locally biholomorphic mappings on the unit ball B to be starlike of order α.

2. Preliminaries

Let X be a complex Banach space with norm $\|\cdot\|$. The open ball $\{x \in X : \|x\| < r\}$ is denoted by B_r and the unit ball is abbreviated by $B_1 = B$. Let $\mathcal{L}(X,X)$ be the space of all continuous linear operators from X into X with the standard operator norm. By I we denote the identity in $\mathcal{L}(X,X)$. Let G be a domain in X and let $f : G \to X$. f is said to be holomorphic on G, if for any $z \in G$, there exists a $Df(z) \in \mathcal{L}(X,X)$ such that

$$\lim_{h\to 0} \frac{\|f(z+h) - f(z) - Df(z)h\|}{\|h\|} = 0.$$

A holomorphic mapping $f : G \to X$ is said to be locally biholomorphic on G if its Fréchet derivative $Df(z)$ is nonsingular at each $z \in G$. A holomorphic mapping $f : G \to X$ is biholomorphic if the inverse f^{-1} exists, is holomorphic on an open set $V \subset X$ and $f^{-1}(V) = G$.

A holomorphic mapping $f : B \to X$ is said to be normalized if $f(0) = 0$ and $Df(0) = I$. Let X^* be the dual space of X. For each $z \in X \setminus \{0\}$, we define

$$T(z) = \{z^* \in X^* : \|z^*\| = 1, z^*(z) = \|z\|\}.$$

By the Hahn-Banach theorem, $T(z)$ is nonempty.

Definition 2.1. *A holomorphic mapping $f : B \to X$ is said to be starlike if f is biholomorphic, $f(0) = 0$ and $e^{-t}f(B) \subset f(B)$ for all $t \geq 0$.*

The following theorem is proved in Gurganus [6] (cf. [20]).

THEOREM 2.1. *Let $f : B \to X$ be a locally biholomorphic mapping with $f(0) = 0$. If f is a starlike mapping, then*

$$\mathrm{Re} z^*([Df(z)]^{-1}f(z)) > 0 \tag{2.1}$$

for $z \in B\backslash\{0\}$, $z^* \in T(z)$. Moreover, if $\|[Df(z)]^{-1}f(z)\|$ is bounded on B_r for each r with $0 < r < 1$ and (2.1) holds, then f is a starlike mapping.

REMARK. In Gurganus [6], he claimed that if $f : B \to X$ is a locally biholomorphic mapping with $f(0) = 0$ and (2.1) holds, then f is starlike. For the proof, he uses Theorem 2.1 of Pfaltzgraff [18]. However, to apply Theorem 2.1 of [18], $\|[Df(z)]^{-1}f(z)\|$ should be bounded on B_r for each r with $0 < r < 1$.

Now, we will define a subclass of starlike mappings.

DEFINITION 2.2. *Let $f : B \to X$ be a starlike mapping. Let $\alpha \in \mathbf{R}$ with $0 < \alpha < 1$. We say that f is a starlike mapping of order α if*

$$\left|\frac{1}{\|z\|}z^*\left([Df(z)]^{-1}f(z)\right) - \frac{1}{2\alpha}\right| < \frac{1}{2\alpha}$$

for $z \in B\backslash\{0\}$, $z^* \in T(z)$.

This definition generalizes the definition of starlike mappings of order α on the unit disc and on the Euclidean unit ball in $\mathbf{C}^n$ [11].

Let Δ denote the unit disc in $\mathbf{C}$. The following lemma is proved in [9], [17].

LEMMA 2.3. *Let $k \geq 1$ and let $g : \Delta \to \mathbf{C}$ be a holomorphic function with $g(0) = g'(0) = \cdots = g^{(k-1)}(0) = 0$. If there exists a $z_0 \in \Delta \setminus \{0\}$ such that*

$$|g(z_0)| = \max\{|g(z)| : |z| \leq |z_0|\} > 0,$$

then there exists a real number $m \geq k$ such that

$$z_0 g'(z_0) = m g(z_0).$$

3. GROWTH THEOREM OF NORMALIZED STARLIKE MAPPINGS OF ORDER α

In this section, we will prove the following theorem (cf. [12], [13]).

THEOREM 3.1. *Let $\alpha \in \mathbf{R}$ with $0 < \alpha < 1$. Let f be a normalized starlike mapping of order α from B to X. Then*

$$\frac{\|z\|}{(1 + \|z\|)^{2(1-\alpha)}} \leq \|f(z)\| \leq \frac{\|z\|}{(1 - \|z\|)^{2(1-\alpha)}}.$$

PROOF. Let $w(z) = [Df(z)]^{-1}f(z)$. Let $z \in B \setminus \{0\}$, $z^* \in T(z)$ be fixed and let

$$g(\zeta) = \frac{1}{\zeta}z^*\left(w\left(\zeta\frac{z}{\|z\|}\right)\right), \ \zeta \in \Delta \setminus \{0\}$$

and $g(0) = 1$. Then g is a holomorphic function on Δ and

$$\left| g(\zeta) - \frac{1}{2\alpha} \right| < \frac{1}{2\alpha},\ \zeta \in \Delta.$$

Hence $\mathrm{Re}(1/g(\zeta)) > \alpha,\ \zeta \in \Delta$, which is equivalent to

$$\mathrm{Re}\frac{\frac{1}{g(\zeta)} - \alpha}{1-\alpha} > 0, \quad \zeta \in \Delta.$$

It is easy to see that the above inequality implies the following relation (see, for example [5], [19]):

$$\frac{1+|\zeta|}{1+(2\alpha-1)|\zeta|} \geq \mathrm{Re}g(\zeta) \geq \frac{1-|\zeta|}{1-(2\alpha-1)|\zeta|}, \quad \zeta \in \Delta.$$

Letting $\zeta = \|z\|$ in the above inequality, we obtain

$$\|z\|\frac{1+\|z\|}{1+(2\alpha-1)\|z\|} \geq \mathrm{Re}z^*(w(z)) \geq \|z\|\frac{1-\|z\|}{1-(2\alpha-1)\|z\|}. \tag{3.1}$$

Since z was arbitrarily chosen, we deduce that the inequality (3.1) holds for all $z \in B \setminus \{0\}$.

Let $0 < r_1 < r_2 < 1$. Let z_2 be a point such that $\|z_2\| = r_2$. Since f is starlike, the curve $c(t) = \exp(-t)f(z_2)$ is contained in $f(B)$ for all $t \geq 0$. Also $c(t) \to 0$ as $t \to \infty$. Since f is biholomorphic, the curve $f^{-1}(c(t))$ is well-defined and intersects the sphere $\|z\| = r_1$ at some point $z_1 = f^{-1}(c(t_1))$. For a C^1 curve $\gamma : [a,b] \to X$, let

$$s = \int_a^b \left\| \frac{d\gamma}{dt}(t) \right\| dt$$

be the arc length of γ. We will parameterize the curve $f^{-1}(c(t))$ $(0 \leq t \leq t_1)$ by the arc length from z_1 and write it as $z(s)$. Then $f(z(s)) = \exp(u(s))f(z_1)$, where $u(0) = 0$ and $u' > 0$. Differentiating $z(s) = f^{-1}(\exp(u(s))f(z_1))$, we have

$$\frac{dz}{ds} = [Df(z(s))]^{-1}u'(s)f(z(s)) = u'(s)w(z(s)).$$

Since $z(s)$ is parameterized by the arc length, we have

$$\|u'(s)w(z(s))\| = 1.$$

Therefore,

$$u'(s) = \frac{1}{\|w(z(s))\|}.$$

Then

$$\frac{dz}{ds} = \frac{1}{\|w(z(s))\|}w(z(s)) \tag{3.2}$$

and

$$\frac{df(z(s))}{ds} = u'(s)f(z(s)) = \frac{1}{\|w(z(s))\|}f(z(s)).$$

Let $g(s) = \|f(z(s))\|$. Since $\|f(z(s))\| = \exp(u(s))\|f(z_1)\|$, we have

$$\frac{dg}{ds} = \frac{1}{\|w(z(s))\|}g$$

on $(0, s_1)$, where $z(s_1) = z_2$. Let $v(t) = f^{-1}(c(t))$. Then

$$\frac{dv}{dt} = -[Df(v(t))]^{-1}f(v(t)).$$

Then $v(t)$ satisfies the following integral equation:

$$v(t) = z_2 - \int_0^t [Df(v(\tau))]^{-1}f(v(\tau))d\tau.$$

For any $0 \le s < s' \le s_1$, let $z(s) = v(t_1 - t)$ and $z(s') = v(t_1 - t')$. Then

$$\begin{aligned}
\Big|\|z(s)\| - \|z(s')\|\Big| &\le \|z(s) - z(s')\| \\
&= \|v(t_1 - t) - v(t_1 - t')\| \\
&= \left\|\int_{t_1-t}^{t_1-t'} \frac{dv(\tau)}{d\tau}d\tau\right\| \\
&\le \int_{t_1-t'}^{t_1-t} \left\|\frac{dv(\tau)}{d\tau}\right\| d\tau \\
&= \int_s^{s'} \left\|\frac{dz(s)}{ds}\right\| ds \\
&= \int_s^{s'} 1ds \\
&= |s - s'|.
\end{aligned}$$

This implies that $\|z(s)\|$ is an absolutely continuous function on $[0, s_1]$. Thus, $d\|z(s)\|/ds$ exists a.e., integrable on $[0, s_1]$ and

$$\frac{d\|z(s)\|}{ds} = \mathrm{Re}z(s)^*\left(\frac{dz}{ds}\right)$$

for $z(s)^* \in T(z(s))$ a.e. on $[0, s_1]$ by Lemma 1.3 of Kato [10]. Then

$$\|w(z(s))\|\frac{d\|z(s)\|}{ds} = \mathrm{Re}z(s)^*(w(z(s))) \tag{3.3}$$

by (3.2). By (3.1) and (3.3), we have

$$\begin{aligned}
\frac{1 + (2\alpha - 1)\|z(s)\|}{\|z(s)\|(1 + \|z(s)\|)}\frac{d\|z(s)\|}{ds} &\le \frac{1}{g}\frac{dg}{ds} = \frac{1}{\|w(z(s))\|} \\
&\le \frac{1 - (2\alpha - 1)\|z(s)\|}{\|z(s)\|(1 - \|z(s)\|)}\frac{d\|z(s)\|}{ds}.
\end{aligned}$$

Since $\|z(s)\|$ is strictly increasing on $[0, s_1]$ by (3.1) and (3.3), we have

$$\begin{aligned}\log g(s) - \log g(0) &\leq \int_0^s \frac{1-(2\alpha-1)\|z(s)\|}{\|z(s)\|(1-\|z(s)\|)}\frac{d\|z(s)\|}{ds}ds \\ &= \int_{\|z(0)\|}^{\|z(s)\|} \frac{1-(2\alpha-1)x}{x(1-x)}dx \\ &= \log\|z(s)\| - 2(1-\alpha)\log(1-\|z(s)\|) \\ &\quad -\{\log\|z(0)\| - 2(1-\alpha)\log(1-\|z(0)\|)\}\end{aligned}$$

and

$$\begin{aligned}\log g(s) - \log g(0) &\geq \log\|z(s)\| - 2(1-\alpha)\log(1+\|z(s)\|) \\ &\quad -\{\log\|z(0)\| - 2(1-\alpha)\log(1+\|z(0)\|)\}.\end{aligned}$$

Then

$$\begin{aligned}\frac{(1-\|z(s)\|)^{2(1-\alpha)}}{\|z(s)\|(1-\|z(0)\|)^{2(1-\alpha)}}\|f(z(s))\| &\leq \frac{\|f(z(0))\|}{\|z(0)\|} \\ &\leq \frac{(1+\|z(s)\|)^{2(1-\alpha)}}{\|z(s)\|(1+\|z(0)\|)^{2(1-\alpha)}}\|f(z(s))\|.\end{aligned}$$

If we put $s = s_1$, we have

$$\begin{aligned}\frac{(1-\|z_2\|)^{2(1-\alpha)}}{\|z_2\|(1-\|z(0)\|)^{2(1-\alpha)}}\|f(z_2)\| &\leq \frac{\|f(z(0))\|}{\|z(0)\|} \\ &\leq \frac{(1+\|z_2\|)^{2(1-\alpha)}}{\|z_2\|(1+\|z(0)\|)^{2(1-\alpha)}}\|f(z_2)\|.\end{aligned}$$

Letting $r_1 \to 0$, we obtain that

$$\frac{(1-\|z_2\|)^{2(1-\alpha)}}{\|z_2\|}\|f(z_2)\| \leq 1 \leq \frac{(1+\|z_2\|)^{2(1-\alpha)}}{\|z_2\|}\|f(z_2)\|,$$

since

$$\lim_{z\to 0}\frac{\|f(z)\|}{\|z\|} = \lim_{z\to 0}\frac{\|Df(0)z\|}{\|z\|} = 1.$$

This completes the proof. ∎

EXAMPLE 3.1. When

$$X = \ell_p = \{z = (z_1, z_2, \ldots) : \|z\|^p = \sum_{n=1}^{\infty}|z_n|^p < \infty\},$$

where $p \geq 1$, the estimates in Theorem 3.1 are sharp. We will show that the holomorphic mapping

$$f(z) = (f_1(z_1), f_2(z_2), \ldots)',$$

where

$$f_j(z_j) = \frac{z_j}{(1-z_j)^{2(1-\alpha)}},$$

is a normalized starlike mapping of order α which attains the equalities in Theorem 3.1. Since

$$Df(z)x = \left(\frac{(1-2\alpha)z_1 + 1}{(1-z_1)^{3-2\alpha}} x_1, \frac{(1-2\alpha)z_2 + 1}{(1-z_2)^{3-2\alpha}} x_2, \dots \right)',$$

f is a normalized locally biholomorphic mapping. Moreover,

$$2\alpha[Df(z)]^{-1}f(z) - z = \left(\frac{z_1(2\alpha - 1 - z_1)}{(1-2\alpha)z_1 + 1}, \frac{z_2(2\alpha - 1 - z_2)}{(1-2\alpha)z_2 + 1}, \dots \right)'. \qquad (3.4)$$

When $1 < p < \infty$, $T(z)$ $(z \neq 0)$ consists of one element

$$z^*(y) = \sum_{j=1}^{\infty} \frac{|z_j|^p}{z_j \|z\|^{p-1}} y_j.$$

Then

$$\begin{aligned} |z^*(2\alpha[Df(z)]^{-1}f(z) - z)| &= \left| \sum_{j=1}^{\infty} \frac{|z_j|^p}{\|z\|^{p-1}} \frac{2\alpha - 1 - z_j}{(1-2\alpha)z_j + 1} \right| \\ &\leq \frac{1}{\|z\|^{p-1}} \sum_{j=1}^{\infty} |z_j|^p \left| \frac{2\alpha - 1 - z_j}{(1-2\alpha)z_j + 1} \right| \\ &< \frac{1}{\|z\|^{p-1}} \sum_{j=1}^{\infty} |z_j|^p \\ &= \|z\|. \end{aligned}$$

When $p = 1$, $T(z)$ $(z \neq 0)$ consists of those functionals z^* given by

$$z^*(y) = \sum_{z_j \neq 0} \frac{|z_j|}{z_j} y_j + \sum_{z_j = 0} \alpha_j y_j,$$

where $|\alpha_j| \leq 1$. Then we can show that $|z^*(2\alpha[Df(z)]^{-1}f(z) - z)| < \|z\|$ as above. Since $\|[Df(z)]^{-1}f(z)\|$ is bounded on B_r for each r with $0 < r < 1$ by (3.4), f is a starlike mapping of order α. For $z = (r, 0, 0, \dots) \in B$, we have $\|f(z)\| = \|z\|/(1 - \|z\|)^{2(1-\alpha)}$, and for $z = (-r, 0, 0, \dots) \in B$, we have $\|f(z)\| = \|z\|/(1 + \|z\|)^{2(1-\alpha)}$.

Remark. Let $f : B \to X$ be a normalized convex mapping. That is, f is a biholomorphic mapping from B onto a convex domain with $f(0) = 0$, $Df(0) = I$. Then we can show that f is a starlike mapping of order 1/2. Then we obtain the following growth theorem from the above theorem.

$$\frac{\|z\|}{1 + \|z\|} \leq \|f(z)\| \leq \frac{\|z\|}{1 - \|z\|}.$$

For details, see Theorem 2.1 of [8] (cf. [11], [12]).

4. A sufficient condition to be starlike of order α

In this section, we will give a sufficient condition for locally biholomorphic mappings on the unit ball in complex Banach spaces to be starlike of order α.

First, we will generalize Lemma 2.3 to complex Banach spaces (cf. [14], [15]).

THEOREM 4.1. *Let B be the unit ball in a complex Banach space X. Let $f : B \to X$ be a holomorphic mapping with $f(0) = 0$. Suppose that there exists an $a \in B \setminus \{0\}$ such that*

$$\|f(a)\| = \max\{\|f(\zeta a)\| : |\zeta| \leq 1\} > 0.$$

Then there exists a real number $s \geq 1$ such that

$$\|Df(a)(a)\| = s\|f(a)\|.$$

Moreover, if $Df(0) = 0$, then $s \geq 2$.

PROOF. Let $b = f(a)$ and let $F(\zeta) = b^*(f(\zeta a/\|a\|))$, where $b^* \in T(b)$. Then F is a holomorphic function on Δ and $F(0) = 0$. Since $F(\|a\|) = \|f(a)\|$ and $|F(\zeta)| \leq \|f(a)\|$ for $|\zeta| \leq \|a\|$, there exists a real number $m \geq 1$ such that $\|a\|F'(\|a\|) = mF(\|a\|)$ by Lemma 2.3. This implies that $b^*(Df(a)(a)) = m\|f(a)\|$. Since $\|b^*\| = 1$, we can find a real number s with $s \geq m \geq 1$ such that $\|Df(a)(a)\| = s\|f(a)\|$.

If $Df(0) = 0$, then $F'(0) = 0$. Then by Lemma 2.3, $s \geq m \geq 2$. This completes the proof. □

The following theorem generalizes the result of third author's paper [16].

THEOREM 4.2. *Let B be the unit ball in a complex Banach space X. Let $f : B \to X$ be a locally biholomorphic mapping with $f(0) = 0$. Assume that f satisfies one of the following two conditions:*

(i) $1/2 < \alpha < 1$ *and*

$$\|(Df(z))^{-1}D^2f(z)(z,\cdot)\| < \frac{1-(2\alpha-1)\|z\|}{1+\|z\|};$$

(ii) $\alpha = 1/2$ *and*

$$\|(Df(z))^{-1}D^2f(z)(z,\cdot)\| < \frac{2}{1+\|z\|}.$$

Then f is starlike of order α. Moreover, if f is normalized, then

$$\frac{\|z\|}{(1+\|z\|)^{2(1-\alpha)}} \leq \|f(z)\| \leq \frac{\|z\|}{(1-\|z\|)^{2(1-\alpha)}}.$$

PROOF. Let $p(z) = 2\alpha[Df(z)]^{-1}f(z) - z$. First we show that

$$\|p(z)\| < 1, z \in B. \tag{4.1}$$

If the inequality (4.1) does not hold, then there exists a point $a \in B \setminus \{0\}$ such that

$$\|p(a)\| = \max\{\|p(\zeta a)\| : |\zeta| \leq 1\} = 1.$$

By Theorem 4.1, there exists a real number $s \geq 1$ such that

$$\|Dp(a)(a)\| = s\|p(a)\| = s \geq 1.$$

When $\alpha = 1/2$, $Dp(0) = 0$ and therefore, $s \geq 2$. Since

$$[Df(z)]^{-1}D^2f(z)(p(z) + z, \cdot) + Dp(z) = (2\alpha - 1)I,$$

we have

$$\begin{aligned} s \leq \|Dp(a)(a)\| &= \|(2\alpha - 1)a - [Df(a)]^{-1}D^2f(a)(p(a) + a, a)\| \\ &\leq (2\alpha - 1)\|a\| + \|[Df(a)]^{-1}D^2f(a)(a, \cdot)\|\|p(a) + a\| \\ &\leq (2\alpha - 1)\|a\| + \|[Df(a)]^{-1}D^2f(a)(a, \cdot)\|(1 + \|a\|). \end{aligned}$$

Then

$$\frac{s - (2\alpha - 1)\|a\|}{1 + \|a\|} \leq \|[Df(a)]^{-1}D^2f(a)(a, \cdot)\|$$

This is a contradiction. So, $\|p(z)\| < 1$ on B. Since $p(0) = 0$, $\|p(z)\| \leq \|z\|$ for $z \in B$ by the Schwarz lemma. For fixed $z \in B \setminus \{0\}$, $z^* \in T(z)$, let $w = z/\|z\|$ and let

$$g(\zeta) = z^*\left(\frac{p(\zeta w)}{\zeta}\right).$$

Then g is a holomorphic function on Δ with $|g(\zeta)| \leq 1$. Since $g(0) = z^*(Dp(0)w) = 2\alpha - 1$, $|g(0)| < 1$. Then $|g(\zeta)| < 1$ by the maximum principle. This implies that

$$\frac{1}{2\alpha}|g(\|z\|)| = \left|\frac{1}{\|z\|}z^*\left([Df(z)]^{-1}f(z)\right) - \frac{1}{2\alpha}\right| < \frac{1}{2\alpha}$$

Since $\|[Df(z)]^{-1}f(z)\|$ is bounded on B from (4.1), f is a starlike mapping of order α. By Theorem 3.1, we obtain the growth theorem. This completes the proof. ◻

Acknowledgements.

The first author is supported by Grant-in-Aid for Scientific Research (C) no.11640194 from Japan Society for the Promotion of Science, 1999.

References

[1] R.W. Barnard, C.H. FitzGerald and S. Gong, *The growth and* $1/4$-*theorems for starlike mappings in* $\mathbf{C}^n$, Pacific J. Math., **150** (1991), 13–22.

[2] J. Becker, *Löwnersche differentialgleichung und quasikonform fortsetzbare schlichte funktionen,* J. Reine Angew. Math., **255** (1972), 23–43.

[3] M. Chuaqui, *Application of subordination chains to starlike mappings in* $\mathbf{C}^n$, Pacific J. Math. **168** (1995), 33–48.

[4] D. Dong and W. Zhang, *Growth and* $1/4$-*theorem for starlike maps in the Banach space,* Chin. Ann. Math., Ser.A **13**, No.4 (1992), 417–423.

[5] A.W. Goodman, *Univalent Functions*, I-II, Mariner, Tampa Florida, 1983.
[6] K. R. Gurganus, *Φ-like holomorphic functions in $\mathbf{C}^n$ and Banach spaces*, Trans. Amer. Math. Soc., **205** (1975), 389–406.
[7] H. Hamada and G. Kohr, *Subordination chains and the growth theorem of spirallike mappings,* to appear in Mathematica(Cluj).
[8] H. Hamada and G. Kohr, *Growth and distortion results for convex mappings in infinite dimensional spaces,* submitted.
[9] I.S. Jack, *Functions starlike and convex of order α,* J. London Math. Soc. **3** (1971), 469–474.
[10] T. Kato, *Nonlinear semigroups and evoluation equations,* J. Math. Soc. Japan **19** (1967), 508–520.
[11] G. Kohr, *Certain partial differential inequalities and applications for holomorphic mappings defined on the unit ball of $\mathbf{C}^n$*, Ann. Univ. Mariae-Curie Sklodowska, **50** (1996), 87–94.
[12] G. Kohr, *On certain conditions of starlikeness of order α in $\mathbf{C}^n$,* Zeszyty Nauk. Politech. Rzeszowskiej, Mat., **20** (1996), 61–72.
[13] G. Kohr, *On starlikeness and strongly starlikeness of order alpha in $\mathbf{C}^n$,* Mathematica(Cluj), **40(63)** (1998), 95–109.
[14] G. Kohr and P. Liczberski, *On some sufficient conditions for univalence in $\mathbf{C}^n$,* Demonstratio Mathematica, **29** (1996), 407–412.
[15] P. Liczberski, *Jack's lemma for holomorphic mappings in $\mathbf{C}^n$,* Ann. Univ. Mariae Curie-Sklodowska, Sect.A **15** (1986), 131–139.
[16] P. Liczberski, *A starlikeness criterion for holomorphic mappings in $\mathbf{C}^n$,* Complex Variables **25** (1994), 193–195.
[17] S.S. Miller and P.T. Mocanu, *Second order differential inequalities in the complex plane,* J. Math. Analysis and Applications **65** (1978), 289–305.
[18] J.A. Pfaltzgraff, *Subordination chains and univalence of holomorphic mappings in $\mathbf{C}^n$,* Math. Ann., **210** (1974), 55–68.
[19] Ch. Pommerenke, *Univalent functions*, Vandenhoeck & Ruprecht, Göttingen, 1975.
[20] T.J. Suffridge, *Starlike and convex maps in Banach spaces*, Pacif. J. Math., **46** (1973), 575–589.

Faculty of Engineering
Kyushu Kyoritsu University
1-8 Jiyugaoka, Yahatanishi-ku
Kitakyushu 807-8585, Japan
E-mail: hamada@kyukyo-u.ac.jp

Faculty of Mathematics
Babeş-Bolyai University
1 M. Kogălniceanu Str.
3400 Cluj-Napoca, Romania
E-mail: gkohr@math.ubbcluj.ro

Institute of Mathematics
Technical University of Lódź
Av.Politechniki 11
90 - 924, Lódź Poland
E-mail: piliczb@ck-sg.p.lodz.pl

Received: 10.11.99.