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ADDITIVE SELECTIONS OF (α, β)-SUBADDITIVE SET VALUED MAPS

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ABSTRACT. It is proved that an (α, β)-subadditive set valued map with close, convex and equibounded values in a Banach space has exactly one additive selection if α, β are positive numbers and $\alpha + \beta \neq 1$.

The study of subadditive set valued maps (s.v. maps) is related to the classical Hyers-Ulam stability problem for the Cauchy functional equation. This problem is cf.[1] and [4] the next one:

If $f : \mathbb{R} \to \mathbb{R}$ is a measurable solution of the inequality

$$|f(x+y) - f(x) - f(y)| < \varepsilon \tag{0.1}$$

where $\varepsilon > 0$ then there exists a linear function $a(x) = mx$, $m \in \mathbb{R}$, such that $|f(x) - a(x)| < \varepsilon$ for every $x \in \mathbb{R}$.

The inequality (1) can be written on the form

$$f(x+y) - f(x) - f(y) \in B(0, \varepsilon)$$

where $B(0, \varepsilon)$ is the ball with centre 0 and radius ε.

Hence we have

$$f(x+y) + B(0, \varepsilon) \subset f(x) + B(0, \varepsilon) + f(y) + B(0, \varepsilon)$$

and denoting by $F(x) = f(x) + B(0, \varepsilon)$, $x \in \mathbb{R}$, we get

$$F(x+y) \subseteq F(x) + F(y), \quad x, y \in \mathbb{R} \tag{0.2}$$

and

$$a(x) \in F(x), \quad x \in \mathbb{R},$$

which means that the s.v. map F has the additive selection a.

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Now one may ask in what conditions a subadditive s.v. map (i.e. a s.v. map that satisfies relation (2) admits an additive selection. An answer at this question is given in [1]. It is also proved that there exists subadditive s.v. maps that have not additive selection (see [1]).

In the sequel we shall generalize the notion of subadditive s.v. map and we shall give for such s.v. maps an existence and uniqueness theorem of an additive selection.

Let X be a real vector space. We shall denote by $\mathcal{P}(X)$ and $\mathcal{P}_0(X)$ the family of all subsets, respectively nonempty subsets of X. If Y is a real normed space the family of all close and convex subsets of Y will be denoted by $ccl(Y)$.

If A, B are nonempty subsets of the real vector space X and λ, μ are real numbers we define

$$\begin{aligned} A + B &= \{x \mid x = a + b,\ a \in A,\ b \in B\} \\ \lambda A &= \{x \mid x = \lambda a,\ a \in A\}. \end{aligned} \tag{0.3}$$

The next properties will be often used in what follows

$$\begin{aligned} \lambda(A + B) &= \lambda A + \lambda B \\ (\lambda + \mu)A &\subseteq \lambda A + \mu A. \end{aligned} \tag{0.4}$$

If A is a convex set and $\lambda\mu \geq 0$ we have (cf.[3])

$$(\lambda + \mu)A = \lambda A + \mu A. \tag{0.5}$$

A set $A \subseteq X$ is said to be a cone if $K + K \subseteq K$ and $\lambda K \subseteq K$ for all $\lambda > 0$. If the zero vector from X belongs to K we say that K is a zero cone.

Definition 1. *Let X, Y be real vector spaces, $K \subseteq X$ a cone and $\alpha > 0$, $\beta > 0$. A s.v. map $F : K \to \mathcal{P}_0(Y)$ is called (α, β)-subadditive if*

$$F(\alpha x + \beta y) \subseteq \alpha F(x) + \beta F(y) \tag{0.6}$$

for all $x, y \in X$.

For $\alpha = \beta = 1$, F is called subadditive s.v. map.

Let us remark that there exists (α, β)-subadditive s.v. map that are not subadditive.

The s.v. map $F : [0, +\infty) \to \mathcal{P}_0(\mathbb{R})$ given by

$$F(x) = \left[\sqrt{x}, +\infty\right)$$

is $\left(\frac{1}{2}, \frac{1}{2}\right)$-subadditive but is not subadditive.

For all $x, y \geq 0$ we have

$$F\left(\frac{x+y}{2}\right) = \left[\sqrt{\frac{x+y}{2}}, +\infty\right)$$

and
$$\frac{1}{2}F(x)+\frac{1}{2}F(y)=\left[\frac{\sqrt{x}+\sqrt{y}}{2},+\infty\right).$$

Taking account of
$$\sqrt{\frac{x+y}{2}}\geq\frac{\sqrt{x}+\sqrt{y}}{2}$$
it results that
$$F\left(\frac{x+y}{2}\right)\subseteq\frac{F(x)+F(y)}{2}$$
that means F is $\left(\frac{1}{2},\frac{1}{2}\right)$-subadditive, but it is not subadditive because
$$F(x+y)=\left[\sqrt{x+y},+\infty\right)\nsubseteq\left[\sqrt{x}+\sqrt{y},\infty\right)=F(x)+F(y)$$
for $x>0$, $y>0$.

In the following theorem we shall give a sufficient condition for an (α,β)-subadditive multifunction to admit an additive selection. Let us recall that if $F:X\to\mathcal{P}(Y)$ is a s.v. map a function $f:X\to Y$ with the property $f(x)\in F(x)$ for all $x\in X$ is said to be a selection of F.

Theorem 2. *Let X be a real vector space, $K\subseteq X$ a zero cone, Y a Banach space and $\alpha>0$, $\beta>0$.*

If $F:K\to ccl(Y)$ is (α,β)-subadditive s.v. map which satisfies the conditions:

1) $\alpha+\beta\neq 1$

2) $\sup\{diam\,F(x):\ x\in X\}<+\infty$

then F admits exactly one additive selection.

Proof. Using (α,β)-subadditivity of F and convexity of its values we have for all $x\in K$
$$F((\alpha+\beta)x)\subseteq\alpha F(x)+\beta F(y)=(\alpha+\beta)F(x) \tag{0.7}$$
and replacing x by $(\alpha+\beta)^n x$, $n\in\mathbb{N}$, in (7) we obtain
$$F((\alpha+\beta)^{n+1}x)\subseteq(\alpha+\beta)F((\alpha+\beta)^n x)$$
and
$$\frac{F((\alpha+\beta)^{n+1}x)}{(\alpha+\beta)^{n+1}}\subseteq\frac{F((\alpha+\beta)^n x)}{(\alpha+\beta)^n}.$$

I. Let $\alpha+\beta>1$.

Denoting by $F_n(x)=\dfrac{F((\alpha+\beta)^n x)}{(\alpha+\beta)^n}$, $x\in K$, $n\in\mathbb{N}$, it results that $(F_n(x))_{n\geq 0}$ is a decreasing sequence of closed subsets of the Banach space Y. We have also
$$diam\,F_n(x)=\frac{1}{(\alpha+\beta)^n}diam\,F((\alpha+\beta)^n x)$$

and taking account of the condition 2) we obtain

$$\lim_{n\to\infty} diam\, F_n(x) = 0. \tag{0.8}$$

Using the Cantor theorem for the sequence $(F_n(x))_{n\geq 0}$ the intersection $\bigcap\limits_{n\geq 0} F_n(x)$ is a singleton and we denote this intersection by $a(x)$ for all $x \in K$. Thus we obtain a function $a : K \to Y$ which is a selection of F because $a(x) \in F_0(x) = F(x)$ for all $x \in K$.

We shall show that a is additive. We have

$$F_n(\alpha x + \beta y) = \frac{F((\alpha+\beta)^n(\alpha x + \beta y))}{(\alpha+\beta)^n} = \frac{F(\alpha(\alpha+\beta)^n x + \beta(\alpha+\beta)^n y)}{(\alpha+\beta)^n} \subseteq$$

$$\subseteq \frac{\alpha F((\alpha+\beta)^n x) + \beta F((\alpha+\beta)^n y)}{(\alpha+\beta)^n} = \alpha\frac{F((\alpha+\beta)^n x)}{(\alpha+\beta)^n} + \beta\frac{F((\alpha+\beta)^n y)}{(\alpha+\beta)^n} =$$
$$= \alpha F_n(x) + \beta F_n(y)$$

for all $x, y \in K$ and $n \in \mathbb{N}$.

From the last relation it results that

$$a(\alpha x + \beta y) = \bigcap_{n\geq 0} F_n(\alpha x + \beta y) \subseteq \bigcap_{n\geq 0} (\alpha F_n(x) + \beta F_n(y))$$

and

$$\|a(\alpha x + \beta y) - (\alpha a(x) + \beta a(y))\| \leq diam(\alpha F_n(x) + \beta F_n(y)) \leq$$
$$\leq \alpha diam\, F_n(x) + \beta diam\, F_n(y) \to 0,$$

when $n \to \infty$, cf.(8).

So

$$a(\alpha x + \beta y) = \alpha a(x) + \beta a(y), \tag{0.9}$$

for all $x, y \in K$.

Putting $x = y = 0$ in (9) it results $a(0) = 0$ and for $y = 0$ and $x = 0$ in (9) we obtain

$$\begin{aligned} a(\alpha x) &= \alpha a(x) \\ a(\beta y) &= \beta a(y) \end{aligned} \tag{0.10}$$

for all $x, y \in K$. Using (10) we have

$$a(x+y) = a\left(\alpha\frac{x}{\alpha} + \beta\frac{y}{\beta}\right) = \alpha a\left(\frac{x}{\alpha}\right) + \beta a\left(\frac{y}{\beta}\right) = a(x) + a(y)$$

for all $x, y \in K$, thus a is additive.

II. For $\alpha + \beta < 1$, we replace x in (7) by $\dfrac{x}{(\alpha+\beta)^{n+1}}$, $n \in \mathbb{N}$, and we obtain multiplying this relation by $(\alpha+\beta)^n$

$$(\alpha+\beta)^n F\left(\frac{x}{(\alpha+\beta)^n}\right) \subseteq (\alpha+\beta)^{n+1} F\left(\frac{x}{(\alpha+\beta)^{n+1}}\right). \tag{0.11}$$

The sequence $(F_n'(x))_{n\geq 0}$,

$$F_n'(x) = (\alpha+\beta)^n F\left(\frac{x}{(\alpha+\beta)^n}\right),$$

is increasing and it follows that the sequence of positive numbers $(diam\, F_n'(x))_{n\geq 0}$ is increasing too. We have

$$diam\, F_n'(x) = (\alpha+\beta)^n diam\, F\left(\frac{x}{(\alpha+\beta)^n}\right)$$

and

$$\lim_{n\to\infty} diam\, F_n'(x) = 0,$$

hence $F_n'(x)$ is single valued for all $x \in K$. The s.v. map F is single valued

$$F(\alpha x + \beta y) = \alpha F(x) + \beta F(y)$$

for all $x, y \in K$, and using the same methods as in case I, we obtain the additivity of F. We conclude that F is its own selection.

We have proved the existence of an additive selection of F and let us prove the uniqueness.

Suppose that F has two additive selection $a_1, a_2 : K \to Y$. We have

$$na_i(x) = a_i(nx) \in F(nx)$$

for all $n \in \mathbb{N}$, $x \in K$, $1 \leq i \leq 2$. We have

$$n\|a_1(x) - a_2(x)\| = \|na_1(x) - na_2(x)\| = \|a_1(nx) - a_2(nx)\| \leq diam\, F(nx)$$

for all $x \in K$, $n \in \mathbb{N}$, and taking account of the condition 2) from Theorem 1, it results $a_1(x) = a_2(x)$ for all $x \in K$. □

Corollary 3. *([1]). Let X be a real vector space, $K \subset X$ a cone with zero and Y a Banach space. If $F : K \to ccl(Y)$ is a subadditive s.v. map with*

$$\sup\{diam\, F(x) : \; x \in K\} < +\infty$$

then F admits exactly one additive selection.

Proof. We take $\alpha = \beta = 1$ in Theorem 1. □

Corollary 4. *Let X be a real vector space, $K \subseteq X$ a cone with zero, Y a normed space. If $F : K \to \mathcal{P}_0(Y)$ is (α, β)-subadditive with $\alpha + \beta < 1$ and convex values and*

$$\sup\{diam\, F(x) : \; x \in K\} < +\infty,$$

then F is single valued and additive.

Proof. It results from the proof of Theorem 1, part II. □

REMARK 5. There exists (α, β)-subadditive s.v. maps with $\alpha + \beta = 1$, which satisfies the conditions of Theorem 1 and have not additive selection.

Concave s.v. map are (α, β)-subadditive s.v. maps with $\alpha + \beta = 1$. It is known that if $f, g : [0, \infty) \to \mathbb{R}$ are two functions such that f is concave and g is convex and $f(x) \leq g(x)$, for each $x \in \mathbb{R}$ then the s.v. map $F : \mathbb{R} \to \mathcal{P}_0(\mathbb{R})$, $F(x) = [f(x), g(x)]$ is concave (see [3]). Using this result it follows that

$$F(x) = \left[\frac{x}{x+1}, 1\right], \quad x \in [0, \infty)$$

is concave without additive selection, because if $a : \mathbb{R} \to \mathbb{R}$ is an additive selection of F we must have $a(x) = mx$ for each $x \in [0, +\infty) \cap \mathbb{Q}$ $(m \in \mathbb{R})$ and

$$\frac{x}{x+1} \leq mx \leq 1, \ \forall\, x \in [0, +\infty) \cap \mathbb{Q}.$$

The last inequality is impossible because taking the limit it results $\lim\limits_{x \to \infty} mx = 1$.

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