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ON CERTAIN CHARACTERIZATIONS AND INTEGRAL REPRESENTATIONS OF CHATTERJEA'S GENERALIZED BESSEL POLYNOMIAL

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Abstract. The present paper deals with certain recurrence relations, integral representations, characterizations and a Rodrigue's type n-th derivative formula for the generalized Bessel polynomial of Chatterjea.

1. Introduction

In 1949 Krall and Frink [10] initiated serious study of what they called Bessel polynomials. In their terminology the simple Bessel polynomial is

$$y_n(x) = {}_2F_0\left[-n, 1+n; -; -\frac{x}{2}\right] \tag{1.1}$$

and the generalized one is

$$y_n(a,b,x) = {}_2F_0\left[-n, a-1+n; -; -\frac{x}{b}\right]. \tag{1.2}$$

Several other authors including Agarwal [1], Al-Salam [2], Brafman [3], Burchnall [4], Carlitz [5], Dickinson [8], Grosswald [9], Rainville [11] and Toscano [14] have contributed to the study of the Bessel polynomials.

In 1965, Chatterjea [7] generalized (1.2) and obtained certain generating functions for his generalized polynomial defined by

$${}_2F_0\left(-n, C+kn; -; x\right),$$

where k is a positive integer.

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In this paper we propose to study further the generalized Bessel polynomials due to Chatterjea [7]. In particular we shall find some recurrence relations, integral representations and certain characterizations of these polynomials.

We shall adopt in this paper a notation used by Al-Salam [2]. In the notation of hypergeometric series, the generalized Bessel polynomials due to Chatterjea [7] are given by

$$y_n^{(\alpha)}(x;k) = {}_2F_0\left[-n, kn+\alpha; -; -\frac{x}{2}\right], \tag{1.3}$$

where k is a positive integer and $n = 0, 1, 2, \ldots$.

Sometimes we shall find it convenient to consider the following polynomial

$$\theta_n^{(\alpha)} = x^n y_n^{(\alpha)}(x;k). \tag{1.4}$$

2. Recurrence relations

From (1.3) it is easy to find that

$$y_n^{(\alpha+1)}(x;k) - y_n^{(\alpha)}(x;k) = \frac{nx}{2} y_{n-1}^{(\alpha+k+1)}(x;k). \tag{2.1}$$

This suggests the difference formula

$$\Delta_\alpha y_n^{(\alpha)}(x;k) = \frac{nx}{2} y_{n-1}^{(\alpha+k+1)}(x;k) \tag{2.2}$$

where $\Delta_\alpha f(\alpha) = f(\alpha+1) - f(\alpha)$ and $\Delta_\alpha^{r+1} f(\alpha) = \Delta_\alpha \Delta_\alpha^r f(\alpha)$.

In particular

$$\Delta_\alpha^n y_n^{(\alpha)}(x;k) = \left(\frac{x}{2}\right)^n. \tag{2.3}$$

Now Newton's formula

$$f(\alpha+u) = \sum_r \binom{u}{r} \Delta^r f(\alpha)$$

and (2.3) imply

$$y_n^{(\alpha+u)}(x;k) = \sum_r \binom{u}{r} \frac{n!}{(n-r)!} \left(\frac{x}{2}\right)^r y_{n-r}^{(\alpha+r+rk)}(x;k) \tag{2.4}$$

or equivalently

$$\theta_n^{(\alpha+u)}(x;k) = \sum_r \binom{u}{r} \frac{n! 2^{-r}}{(n-r)!} \theta_{n-r}^{(\alpha+r+rk)}(x;k). \tag{2.5}$$

Also, from (1.3) we find that

$$\frac{d}{dx} y_n^{(\alpha)}(x;k) = \frac{1}{2} n(kn+\alpha+1) y_{n-1}^{(\alpha+r+rk)}(x;k). \tag{2.6}$$

From (2.6) and (2.2), we see that the polynomial given in (1.3) satisfy the mixed equation

$$\Delta_\alpha y_n^{(\alpha)}(x;k) = \frac{x}{kn+\alpha+1}\frac{d}{dx}y_n^{(\alpha)}(x;k). \tag{2.7}$$

The following recurrence relation can easily be verified:

$$y_{n+1}^{(\alpha-k+1)}(x;k) - y_n^{(\alpha)}(x;k) = \frac{1}{2}x(kn+n+\alpha+2)y_n^{(\alpha+1)}(x;k). \tag{2.8}$$

3. Integral representations

It is easy to derive the following integral representations for the Chatterjea's generalized Bessel polynomials (1.3):

$$\begin{aligned}(3.1)\qquad &\int_0^1 t^{\beta-1}(1-t)^{\gamma-1}y_n^{(\alpha)}(xt;k) = \\ &= \frac{\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\beta+\gamma)}{}_3F_1\left[\begin{matrix}-n,\beta,kn+\alpha+1; & \\ & -\frac{x}{2}\\ \beta+\gamma; & \end{matrix}\right].\end{aligned}$$

$$\begin{aligned}(3.2)\qquad &\int_0^1 t^{\beta-1}(1-t)^{\gamma-1}y_n^{(\alpha)}(x(1-t);k) = \\ &= \frac{\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\beta+\gamma)}{}_3F_1\left[\begin{matrix}-n,\gamma,kn+\alpha+1; & \\ & -\frac{x}{2}\\ \beta+\gamma; & \end{matrix}\right].\end{aligned}$$

$$\int_0^\infty e^{st}t^{kn+\alpha}\left(1+\frac{sxt}{2}\right)^n dt = \frac{\Gamma(kn+\alpha+1)}{s^{kn+\alpha+1}}y_n^{(\alpha)}(x;k). \tag{3.3}$$

$$\begin{aligned}&\int_0^1 y_m^{(\alpha)}\left(\frac{t}{x};k\right)y_n^{(\beta)}\left(\frac{t}{1-x};k\right)x^{-km-\alpha-1}(1-x)^{-kn-\beta-1}dx\\ (3.4)\qquad &= -\frac{\pi\sin\pi(\alpha+\beta)\Gamma(km+kn+\alpha+\beta+1)}{\sin\pi\alpha\sin\pi\beta\Gamma(km+\alpha+1)\Gamma(kn+\beta+1)}y_{m+n}^{(\alpha+\beta)}(t;k).\end{aligned}$$

Some interesting particular cases of (3.1), (3.2), (3.3) and (3.4) are as follows:

(i) Taking $\beta = kn+1$, $\gamma=\alpha$ in (3.1), we get

$$\int_0^1 t^{kn}(1-t)^{\alpha-1}y_n^{(\alpha)}(xt;k)dt = \frac{\Gamma(kn+1)\Gamma(\alpha)}{\Gamma(kn+\alpha+1)}y_n^{(0)}(x;k). \tag{3.5}$$

(ii) Taking $\beta = n+\alpha+1$, $\gamma = kn-n$ in (3.1), we obtain

$$\int_0^1 t^{n+\alpha}(1-t)^{kn-n-1}y_n^{(\alpha)}(xt;k)dt = \frac{\Gamma(kn-n)\Gamma(n+\alpha+1)}{\Gamma(kn+\alpha+1)}y_n^{(\alpha)}(x;k). \tag{3.6}$$

(iii) Replacing β by $kn+\alpha+\beta+1$ and taking $\gamma = 1-\beta$ in (3.1), we get

$$\begin{aligned}\int_0^1 & t^{kn+\alpha+\beta}(1-t)^{\beta}y_n^{(\alpha)}(xt;k)dt = \\ & = -\frac{\pi\Gamma(kn+\alpha+\beta+1)}{\sin\pi\beta\Gamma(1+\beta)\Gamma(kn+\alpha+1)}y_n^{(\alpha+\beta)}(x;k).\end{aligned} \tag{3.7}$$

(iv) Replacing β by $kn+\alpha-\beta+1$ and taking $\gamma = 1+\beta$ in (3.1), we have

$$\int_0^1 t^{kn+\alpha-\beta}(1-t)^{\beta}y_n^{(\alpha)}(xt;k)dt = \frac{\Gamma(kn+\alpha-\beta+1)\Gamma(\beta)}{\Gamma(kn+\alpha+1)}y_n^{(\alpha-\beta)}(x;k). \tag{3.8}$$

Results similar to (3.5), (3.6), (3.7) and (3.8) will hold by suitable selection of β and γ in (3.2) also.

(v) Taking $\gamma = 1$ in (3.3), it becomes

$$\int_0^\infty e^{-t}t^{kn+\alpha}\left(1+\frac{xt}{2}\right)^n dt = \Gamma(kn+\alpha+1)y_n^{(\alpha)}(x;k). \tag{3.9}$$

(vi) Taking $\gamma = 1$ in (3.1), it reduces to

$$\int_0^1 t^{\beta-1}y_n^{(\alpha)}(xt;k)dt = \frac{1}{\beta}{}_3F_1\left[\begin{matrix} -n,\beta,kn+\alpha+1; \\ \beta+1; \end{matrix} \; -\tfrac{x}{2}\right]. \tag{3.10}$$

(vii) Taking $\beta = 1$ in (3.2), it reduces to

$$\int_0^1 (1-t)^{\gamma-1}y_n^{(\alpha)}(x(1-t);k)dt = \frac{1}{\gamma}{}_3F_1\left[\begin{matrix} -n,\gamma,kn+\alpha+1; \\ \gamma+1; \end{matrix} \; -\tfrac{x}{2}\right]. \tag{3.11}$$

(viii) Taking $\beta = kn+\alpha$, $\gamma = 1$ in (3.1), it reduces to

$$\int_0^1 t^{kn+\alpha-1}y_n^{(\alpha)}(xt;k)dt = \frac{1}{kn+\alpha}y_n^{(\alpha+1)}(x;k). \tag{3.12}$$

(ix) Taking $\beta = 1$ and $\gamma = kn+\alpha$ in (3.2), it becomes

$$\int_0^1 (1-t)^{kn+\alpha-1}y_n^{(\alpha)}(x(1-t);k)dt = \frac{1}{kn+\alpha}y_n^{(\alpha+1)}(x;k). \tag{3.13}$$

(x) For $n = 0$, (3.4) becomes

$$\int_0^1 y_m^{(\alpha)}\left(\frac{t}{x};k\right) x^{-km-\alpha-1}(1-x)^{-\beta-1}dx$$

$$= -\frac{\pi \sin \pi(\alpha+\beta)\Gamma(km+\alpha+\beta+1)}{\sin \pi\alpha \sin \pi\beta \Gamma(km+\alpha+1)\Gamma(1+\beta)} y_m^{(\alpha+\beta)}(t;k). \tag{3.14}$$

(xi) For $m = 0$, (3.4) becomes

$$\int_0^1 y_n^{(\beta)}\left(\frac{t}{1-x};k\right) x^{-\alpha-1}(1-x)^{-kn-\beta-1}dx$$

$$= -\frac{\pi \sin \pi(\alpha+\beta)\Gamma(kn+\alpha+\beta+1)}{\sin \pi\alpha \sin \pi\beta \Gamma(\alpha+1)\Gamma(kn+\beta+1)} y_n^{(\alpha+\beta)}(t;k). \tag{3.15}$$

We now give a two dimensional version of (3.4). It is given by

$$\iint_{u+v\le 1} y_m^{(\alpha)}\left(\frac{t}{u};k\right) y_n^{(\beta)}\left(\frac{t}{v};k\right) y_p^{(\gamma)}\left(\frac{t}{1-u-v};k\right)$$

$$u^{-km-\alpha-1}v^{-kn-\beta-1}(1-u-v)^{-kp-\gamma-1}dudv$$

$$= \frac{\pi^2 \sin \pi(\alpha+\beta+\gamma)}{\sin \pi\alpha \sin \pi\beta \sin \pi\gamma}$$

$$\frac{\Gamma(km+kn+kp+\alpha+\beta+\gamma+1)}{\Gamma(km+\alpha+1)\Gamma(kn+\beta+1)\Gamma(kp+\gamma+1)} y_{m+n+p}^{(\alpha+\beta+\gamma)}(t;k) \tag{3.16}$$

where the integration is over the interior of the triangle bounded by the u and v axes and the line $u + v = 1$. The extension of (3.16) to higher dimensions is immediate.

4. Some characterizations

In this section we obtain some characterizations of Chatterjea's generalized Bessel polynomials (1.3) similar to those obtained by Al-Salam [2] for Bessel polynomial.

THEOREM 4.1. *Given a sequence* $\{f_n^{(\alpha)}(x;k)\}$ *of polynomials in* x *where* $\deg f_n^{(\alpha)}(x;k) = n$, *and* α *is a parameter, such that*

$$\frac{d}{dx} f_n^{(\alpha)}(x;k) = \frac{1}{2} n(kn+\alpha+1) f_{n-1}^{(\alpha+k+1)}(x;k) \tag{4.1}$$

and $f_n^{(\alpha)}(0;k) = 1$. *Then* $f_n^{(\alpha)}(x;k) = y_n^{(\alpha)}(x;k)$.

PROOF. Let

$$f_n^{(\alpha)}(x;k) = \sum_{r=0}^{n} C_r(\alpha,n,k)\left(-\frac{x}{2}\right)^r.$$

Then by (3.1), we have

$$C_r(\alpha, n, k) = -\frac{n(kn + \alpha + a)}{r} C_{r-1}(\alpha + k + 1, n - 1, k).$$

Since $C_0(\alpha, n, k) = 1$,

$$C_r(\alpha, n, k) = \frac{(-n)_r (kn + \alpha + 1)_r}{r!},$$

which proves the theorem. ◻

Another characterization is suggested by (2.2) as given in the following theorem:

THEOREM 4.2. *Given a sequence of functions* $\{f_n^{(\alpha)}(x; k)\}$ *such that*

$$\Delta_\alpha f_n^{(\alpha)}(x; k) = \frac{1}{2} n x f_{n-1}^{(\alpha+k+1)}(x; k) \tag{4.2}$$

$$f_n^{(\alpha)}(0; k) = 1, \quad f_0^{(\alpha)}(x; k) = 1. \tag{4.3}$$

Then $f_n^{(\alpha)}(x; k) = y_n^{(\alpha)}(x; k)$.

PROOF. We observe from (4.2) that $f_n^{(\alpha)}(x; k)$ is a polynomial in α of degree n. Hence we can write

$$f_n^{(\alpha)}(x; k) = \sum_{r=0}^{n} C_r(n, x) \frac{(kn + \alpha + 1)_r}{r!}.$$

Hence (4.2) gives

$$C_r(n, x) = \frac{nx}{2} C_{r-1}(n - 1, x).$$

From this recurrence and condition (4.3), we obtain

$$C_r(n, x) = (-n)_r \left(-\frac{x}{2}\right)^r.$$

This proves the theorem. ◻

Now equation (2.7) gives the following:

THEOREM 4.3. *Let the sequence* $\{f_n^{(\alpha)}(x; k)\}$*, where* $f_n^{(\alpha)}(x; k)$ *is a polynomial of degree* n *in* x*, and* α *is a parameter, satisfy*

$$\Delta_\alpha f_n^{(\alpha)}(x; k) = \frac{x}{kn + \alpha + 1} \frac{d}{dx} f_n^{(\alpha)}(x; k) \tag{4.4}$$

such that

$$f_n^{(0)}(x; k) = {}_2F_0 \left[-n, kn + 1; -; -\frac{x}{2}\right]. \tag{4.5}$$

Then $f_n^{(\alpha)}(x; k) = y_n^{(\alpha)}(x; k)$.

The proof is similar to that of Theorem 4.1 and 4.2.

Similarly (2.8) suggests yet another characterization of $y_n^{(\alpha)}(x;k)$ given in the form of the following theorem:

Theorem 4.4. *Given a sequence of functions* $f_n^{(\alpha)}(x;k)\}$ *such that*

$$f_{n+1}^{(\alpha-k+1)}(x;k) - f_n^{(\alpha)}(x;k) = \frac{1}{2}x(kn+n+\alpha+2)f_n^{(\alpha+1)}(x;k) \tag{4.6}$$

and

$$f_0^{(\alpha)}(x;k) = 1 \quad \textit{for all } x \textit{ and } \alpha.$$

Then $f_n^{(\alpha)}(x;k) = y_n^{(\alpha)}(x;k)$.

The proof of this theorem follows by induction on n.

5. Rodrigue's formula

Krall and Frink [10] gave the following Rodrigue's type formula for the Bessel polynomials $y_n(x,a,b)$:

$$y_n(x,a,b) = b^{-n}x^{2-a}e^{b/x}\frac{d^n}{dx^n}\left(x^{2n+a-2}e^{-b/x}\right). \tag{5.1}$$

It is not difficult to establish the following Rodrigue's type formula for the polynomial $y_n^{(\alpha)}(x;k)$:

$$y_n^{(\alpha)}(x;k) = \frac{1}{2^n x^{kn-n+\alpha}}e^{\frac{2}{x}}D^n\left[x^{kn+n+\alpha}e^{-\frac{2}{x}}\right], \quad D \equiv \frac{d}{dx}. \tag{5.2}$$

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