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INTEGRAL INEQUALITIES FOR POLYNOMIALS HAVING A ZERO OF ORDER m AT THE ORIGIN

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Abstract. For a polynomial $p(z)$ of degree n, it is known that

$$\left(\int_0^{2\pi}\left|p'\left(e^{i\theta}\right)\right|^s d\theta\right)^{1/s} \leq n\left(\int_0^{2\pi}\left|p\left(e^{i\theta}\right)\right|^s d\theta\right)^{1/s}, \ s \geq 1.$$

We have obtained inequalities in the reverse direction for the polynomials having a zero of order m at the origin.

1. Introduction and statement of results

Let $p(z)$ be a polynomial of degree n. Zygmund [3] has shown that for $s \geq 1$

$$\left(\int_0^{2\pi}\left|p'\left(e^{i\theta}\right)\right|^s d\theta\right)^{1/s} \leq n\left(\int_0^{2\pi}\left|p\left(e^{i\theta}\right)\right|^s d\theta\right)^{1/s}. \tag{1.1}$$

In this paper, we have obtained similar type of integral inequalities, but in the reverse direction, for polynomials having a zero of order m at the origin. More precisely, we prove

Theorem 1.1. *Let $p(z)$ be a polynomial of degree n, having a zero of order m at $z=0$. Then for $s \geq 1$*

$$\left(\int_0^{2\pi}\left|p'\left(e^{i\theta}\right)\right|^s d\theta\right)^{1/s} \geq m\left(\int_0^{2\pi}\left|p\left(e^{i\theta}\right)\right|^s d\theta\right)^{1/s}. \tag{1.2}$$

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By letting $s \to \infty$ in (1.2), we obtain

COROLLARY 1.2. *Let $p(z)$ be a polynomial of degree n, having a zero of order m at $z = 0$. Then*

$$\max_{|z|=1} |p'(z)| \geq m \max_{|z|=1} |p(z)|.$$

THEOREM 1.3. *Let $p(z)$ be a polynomial of degree n, having all its zeros in $|z| \leq k$, $k \leq 1$, with a zero of order m at $z = 0$. Then for β with $|\beta| < k^{n-m}$ and $s \geq 1$*

$$\left(\int_0^{2\pi} \left| p'\left(e^{i\theta}\right) + \frac{mm'}{k^n} \overline{\beta} e^{i(m-1)\theta} \right|^s d\theta \right)^{1/s} \geq \tag{1.3}$$

$$\geq \left\{ n - (n-m) C_s^{(k)} \right\} \left(\int_0^{2\pi} \left| p\left(e^{i\theta}\right) + \frac{m'}{k^n} \overline{\beta} e^{im\theta} \right|^s d\theta \right)^{1/s},$$

where

$$m' = \min_{|z|=k} |p(z)|, \tag{1.4}$$

$$C_s^{(k)} = k \Bigg/ \left(\frac{1}{2\pi} \int_0^{2\pi} \left| 1 + k e^{i\alpha} \right|^s d\alpha \right)^{1/s}. \tag{1.5}$$

By taking $k = 1$ and $\beta = 0$ in Theorem 1.3, we obtain

COROLLARY 1.4. *If $p(z)$ is a polynomial of degree n, having all its zeros in $|z| \leq 1$, with a zero of order m at $z = 0$, then for $s \geq 1$*

$$\left(\int_0^{2\pi} \left| p'\left(e^{i\theta}\right) \right|^s d\theta \right)^{1/s} \geq \{ n - (n-m) D_s \} \left(\int_0^{2\pi} \left| p\left(e^{i\theta}\right) \right|^s d\theta \right)^{1/s}, \tag{1.6}$$

where

$$D_s = 1 \Bigg/ \left(\frac{1}{2\pi} \int_0^{2\pi} \left| 1 + e^{i\alpha} \right|^s d\alpha \right)^{1/s}. \tag{1.7}$$

Inequality (1.6), with $m = 0$, is also true for self–inversive polynomials. In other words we have

THEOREM 1.5. *If $p(z)$ is a polynomial of degree n such that*

$$p(z) = z^n \overline{p(1/\overline{z})}, \tag{1.8}$$

then for $s \geq 1$,

$$\left(\int_0^{2\pi} \left| p'\left(e^{i\theta}\right) \right|^s d\theta \right)^{1/s} \geq n(1 - D_s) \left(\int_0^{2\pi} \left| p\left(e^{i\theta}\right) \right|^s d\theta \right)^{1/s}, \tag{1.9}$$

where D_s *is as in Corollary 1.4.*

By letting $s \to \infty$ in Theorem 1.3, we obtain

Corollary 1.6. *Let* $p(z)$ *be a polynomial of degree* n*, having all its zeros in* $|z| \leq k$*,* $k \leq 1$*, with a zero of order* m *at* $z = 0$*. Then for* β *with* $|\beta| < k^{n-m}$

$$\max_{|z|=1} \left| p'(z) + \frac{mm'}{k^n} \overline{\beta} z^{m-1} \right| \geq \left(\frac{n + mk}{1 + k} \right) \max_{|z|=1} \left| p(z) + \frac{m'}{k^n} \overline{\beta} z^m \right|, \tag{1.10}$$

where m' *is, as in Theorem 1.3.*

By choosing argument of β suitably and letting $|\beta| \to k^{n-m}$ in Corollary 1.6, we obtain

Corollary 1.7. *If* $p(z)$ *is a polynomial of degree* n*, having all its zeros in* $|z| \leq k$*,* $k \leq 1$*, with a zero of order* m *at* $z = 0$*, then*

$$\max_{|z|=1} |p'(z)| \geq \left(\frac{n + mk}{1 + k} \right) \max_{|z|=1} |p(z)| + \left(\frac{n - m}{1 + k} \right) \frac{m'}{k^m}, \tag{1.11}$$

where m' *is as in Theorem 1.3.*

2. Lemmas

For the proofs of the theorems, we require the following lemmas.

Lemma 2.1. *If* $p(z)$ *is a polynomial of degree* n*, having no zeros in* $|z| < k$*,* $k \geq 1$*, then for* $s \geq 1$

$$\left(\int_0^{2\pi} \left| p'\left(e^{i\theta}\right) \right|^s d\theta \right)^{1/s} \leq n E_s^{(k)} \left(\int_0^{2\pi} \left| p\left(e^{i\theta}\right) \right|^s d\theta \right)^{1/s},$$

where

$$E_s^{(k)} = 1 \Big/ \left(\frac{1}{2\pi} \int_0^{2\pi} \left| k + e^{i\alpha} \right|^s d\alpha \right)^{1/s}.$$

This lemma is due to Govil and Rahman [2].

Lemma 2.2. *If* $p(z)$ *is a polynomial of degree* n *such that*

$$p(z) = z^n \overline{p(1/\overline{z})},$$

then for $s \geq 1$

$$\left(\int_0^{2\pi} \left|p'\left(e^{i\theta}\right)\right|^s d\theta\right)^{1/s} \leq nD_s \left(\int_0^{2\pi} \left|p\left(e^{i\theta}\right)\right|^s d\theta\right)^{1/s},$$

where D_s *is as in Corollary 1.4.*

This lemma is due to Dewan and Govil [1].

3. Proofs of the theorems

Proof of Theorem 1.1. We obviously have

$$p(z) = z^m \phi(z), \tag{3.1}$$

where $\phi(z)$ is a polynomial of degree $n - m$, with the property that

$$\phi(0) \neq 0.$$

Then

$$\begin{aligned} q(z) &= z^n \overline{p(1/\overline{z})} \\ &= z^{n-m} \overline{\phi(1/\overline{z})}, \end{aligned} \tag{3.2}$$

is also a polynomial of degree $n - m$. Hence we have for $s \geq 1$,

$$\left(\int_0^{2\pi} \left|q'\left(e^{i\theta}\right)\right|^s d\theta\right)^{1/s} \leq (n-m) \left(\int_0^{2\pi} \left|q\left(e^{i\theta}\right)\right|^s d\theta\right)^{1/s}. \tag{3.3}$$

But by (3.2), we have for $0 \leq \theta \leq 2\pi$

$$\begin{aligned} \left|q'(e^{i\theta})\right| &= \left|np(e^{i\theta}) - e^{i\theta} p'(e^{i\theta})\right| \\ \left|q(e^{i\theta})\right| &= \left|p(e^{i\theta})\right|, \end{aligned}$$

which, by (3.3), imply that for $s \geq 1$,

$$\left(\int_0^{2\pi} \left|np\left(e^{i\theta}\right) - e^{i\theta} p'\left(e^{i\theta}\right)\right|^s d\theta\right)^{1/s} \leq (n-m) \left(\int_0^{2\pi} \left|p\left(e^{i\theta}\right)\right|^s d\theta\right)^{1/s}. \tag{3.4}$$

Now, by Minkowski inequality, we have for $s \geq 1$

$$n\left(\int_0^{2\pi} \left|p\left(e^{i\theta}\right)\right|^s d\theta\right)^{1/s} \leq$$

$$\leq \left(\int_0^{2\pi} \left|np\left(e^{i\theta}\right) - e^{i\theta}p'\left(e^{i\theta}\right)\right|^s d\theta\right)^{1/s} + \left(\int_0^{2\pi} \left|e^{i\theta}p'\left(e^{i\theta}\right)\right|^s d\theta\right)^{1/s}$$

$$\leq (n-m)\left(\int_0^{2\pi} \left|p\left(e^{i\theta}\right)\right|^s d\theta\right)^{1/s} + \left(\int_0^{2\pi} \left|p'\left(e^{i\theta}\right)\right|^s d\theta\right)^{1/s}, \quad \text{(by (3.4))}$$

and Theorem 1.1 follows. □

Proof of Theorem 1.3. The polynomial $q(z)$, given by (3.2) will have no zeros in $|z| < \frac{1}{k}$. Now if

$$m_0 = \min_{|z|=\frac{1}{k}} |q(z)| = \min_{|z|=\frac{1}{k}} \left|z^n\overline{p(1/\overline{z})}\right| = \frac{m'}{k^n}, \quad \text{by (1.4)).} \tag{3.5}$$

then, by Rouché's theorem, the polynomial

$$q(z) + m_0\beta z^{n-m}, \quad |\beta| < k^{n-m},$$

of degree $n-m$, will also have no zeros in $|z| < \frac{1}{k}$, $\frac{1}{k} \geq 1$. Hence, by Lemma 2.1, we have for $s \geq 1$ and $|\beta| < k^{n-m}$

$$\left(\int_0^{2\pi} \left|q'\left(e^{i\theta}\right) + \frac{m'}{k^n}\beta e^{i(n-m-1)\theta}(n-m)\right|^s d\theta\right)^{1/s} \leq$$

$$(n-m)C_s^{(k)}\left(\int_0^{2\pi} \left|q\left(e^{i\theta}\right) + \frac{m'}{k^n}\beta e^{i(n-m)\theta}\right|^s d\theta\right)^{1/s},$$

i.e.,

$$\left(\int_0^{2\pi} \left|np\left(e^{i\theta}\right) - e^{i\theta}p'\left(e^{i\theta}\right) + \overline{\beta}\frac{m'}{k^n}(n-m)e^{im\theta}\right|^s d\theta\right)^{1/s} \leq$$

$$(n-m)C_s^{(k)}\left(\int_0^{2\pi} \left|p\left(e^{i\theta}\right) + \overline{\beta}\frac{m'}{k^n}e^{im\theta}\right|^s d\theta\right)^{1/s}, \quad \text{(by (3.2)).} \tag{3.6}$$

Now by Minkowski inequality, we have for $s \geq 1$ and $|\beta| < k^{n-m}$

$$n\left(\int_0^{2\pi}\left|p\left(e^{i\theta}\right)+\frac{m'}{k^n}\overline{\beta}e^{im\theta}\right|^s d\theta\right)^{1/s} \leq$$

$$\left(\int_0^{2\pi}\left|np\left(e^{i\theta}\right)+\frac{m'}{k^n}\overline{\beta}(n-m)e^{im\theta}-e^{i\theta}p'\left(e^{i\theta}\right)\right|^s d\theta\right)^{1/s} +$$

$$\left(\int_0^{2\pi}\left|e^{i\theta}p'\left(e^{i\theta}\right)+m\frac{m'}{k^n}\overline{\beta}e^{im\theta}\right|^s d\theta\right)^{1/s},$$

and Theorem 1.3 follows, by (3.6). □

Proof of Theorem 1.5. The polynomial

$$q(z) = z^n\overline{p(1/\overline{z})}$$

is a polynomial of degree n, with the property

$$q(z) = z^n\overline{q(1/\overline{z})}, \quad \text{(by (1.8))}.$$

Hence, by Lemma 2.2, we have for $s \geq 1$

$$\left(\int_0^{2\pi}\left|q'\left(e^{i\theta}\right)\right|^s d\theta\right)^{1/s} < nD_s\left(\int_0^{2\pi}\left|q\left(e^{i\theta}\right)\right|^s d\theta\right)^{1/s}.$$

Now Theorem 1.5 follows on lines, similar to those of Theorem 1.1. □

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