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A FAMILY OF SQUARE INTEGRABLE REPRESENTATIONS OF CLASSICAL p-ADIC GROUPS IN THE CASE OF GENERAL HALF-INTEGRAL REDUCIBILITIES

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Abstract. The main aim of this paper is a presentation of a construction of a large family of non-cuspidal irreducible square integrable representations $\delta(\Delta_1, \ldots, \Delta_k, \sigma)_\tau$ of symplectic and odd-orthogonal p-adic groups, starting from the cuspidal representations of the Levi subgroups. The only information that we need about these irreducible cuspidal representations are the generalized rank one reducibilities. We also get a number of interesting facts about these square integrable representations. In C. Mœglin and M. Tadić's paper "Construction of discrete series for classical p-adic groups" (J. Amer. Math. Soc. 15 (2002), 715-786) is given a general construction of all the irreducible square integrable representations of the classical p-adic groups modulo cuspidal data (under a natural assumption). The construction of the family that we present in this paper preceded the general construction. Although the general construction gives a construction of all the square integrable representations of classical p-adic groups, it is interesting to have also available this former construction. Namely, the construction that we present here is much more direct than the general construction, and it gives a number of explicit information about representations. These facts may be useful in further study of the representations of the family that we construct in this paper. It is for expecting that we shall deal a lot in the future with the representations of this family, since this family includes all the generic irreducible square integrable representations (for example). From the Shahidi's conjecture on existence of a generic representation in each L^2 L-packet, would follow that each L^2 L-packet contains some of the representation from the family whose construction we present in this paper.

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Introduction

The main aim of this paper is a presentation of a construction of a large family of non-cuspidal irreducible square integrable representations $\delta(\Delta_1, \ldots, \Delta_k, \sigma)_\tau$ of symplectic and odd-orthogonal p-adic groups, starting from the cuspidal representations of the Levi subgroups. The only information that we need about these irreducible cuspidal representations are the generalized rank one reducibilities. We also get a number of interesting facts about these square integrable representations. Some of the constructions and analysis of representations in the paper may be of independent interest.

Classifying of irreducible square integrable representations is one of the most basic steps in the development of the representation theory of reductive groups. In [MgT] are constructed all the irreducible square integrable representations of the classical p-adic groups modulo cuspidal data (under a natural assumption, which is proved in some cases and which is expected to hold in general). Let us note that the construction of the family that we present in this paper[1] preceded the construction in [MgT] (by the way, the construction that we present here played an important role in the development of ideas of the construction in [MgT]; therefore this paper may be helpful in understanding of [MgT]).

Although [MgT] gives a construction of all the square integrable representations of classical p-adic groups, it may be interesting to have also available this former construction. Namely, the construction that we present here is much more direct than in [MgT], and it gives a number of explicit information about representations, which are not present in [MgT]. These facts may be useful in further study of the representations of the family that we construct in this paper. It is for expecting that we shall deal a lot in the future with the representations of this family, since this family includes all the generic irreducible square integrable representations (this fact is proved by G. Muić in [Mi2]). Let us recall that the generic representations were in the last few decades intensively studied for the purpose of the Langlands' program. From the Muić's result and the Shahidi's conjecture on existence of a generic representation in each L^2 L-packet, would follow that each L^2 L-packet contains some of the representation from the family whose construction we present in this paper. This brings an additional interest for the family that we study in this paper.

To describe our results, we shall first introduce some notation. Let F be a local non-archimedean field. We shall assume $\mathrm{char}\,(F) \neq 2$. The modulus character of F is denoted by $|\ \ |_F$. Set $\nu = |\det|_F$. Based on the fact that Levi factor of a maximal parabolic subgroup of a general linear group is a product

[1] This paper, written in 1998, is essentially a revised version of the second part of [T7]. The ideas used in [T7] and here, are the same. The only difference is that formulations of results and the proofs are written in a way which holds also for non-generic reducibilities.

of two smaller general linear groups, using parabolic induction Bernstein and Zelevinsky defined multiplication $\times$ among representations of general linear groups (see 4.1 of [BZ], or the first section). Let $\mathcal{C}$ be the set of all equivalence classes of irreducible cuspidal representations of all $GL(p, F)$, $p \geq 1$. For $\rho \in \mathcal{C}$ and $n \geq 0$, the set $[\rho, \nu^n\rho] = \{\rho, \nu\rho, \ldots, \nu^n\rho\}$ is called a segment in $\mathcal{C}$. The set of all such segments is denoted by $\mathcal{S}(\mathcal{C})$. For $\Delta = [\rho, \nu^n\rho] \in \mathcal{S}(\mathcal{C})$, the representation $\nu^n\rho \times \nu^{n-1}\rho \times \cdots \times \nu\rho \times \rho$ contains a unique irreducible square integrable subquotient, which we denote by $\delta(\Delta)$.

We fix one of the families $Sp(m, F)$ $(m \geq 0)$ or $SO(2m+1, F)$ $(m \geq 0)$ of classical groups. The group of rank m from the fixed family will be denoted by S_m. The Levi factor of a maximal parabolic subgroup of S_m is isomorphic to $GL(k, F) \times S_{m-k}$, with $1 \leq k \leq m$. Now, as in the case of general linear groups, using parabolic induction, one can introduce multiplication $\rtimes$ between representations of general linear groups and representations of the groups S_m. The products are representations of the groups S_m (see the first section).

Let $\rho \in \mathcal{C}$ be unitarizable and σ an irreducible cuspidal representation of S_q. Suppose that $\nu^\alpha\rho \rtimes \sigma$ reduces for some real α. Look at the simplest case when the induced representation is a representation of $Sp(1, F) = SL(2, F)$ or $SO(3, F)$ (then ρ is a character of $F^\times$ and σ is trivial representation). Then there exits

$$\alpha_0 \in \{0, 1/2, 1\}$$

such that $\nu^{\alpha_0}\rho \rtimes \sigma$ reduces and $\nu^\alpha\rho \rtimes \sigma$ is irreducible for $\beta \in \mathbb{R}\backslash\{\pm\alpha_0\}$ (in general, we shall than say that (ρ, σ) satisfies $(\mathrm{C}\alpha_0)$). F. Shahidi has shown that this is the case in general, if σ is generic and $\mathrm{char}\,(F) = 0$. We shall say that ρ and σ have generic (or non-exceptional) reducibility if they satisfy the above condition on reducibility (σ does not need to be generic). Otherwise, we shall say that ρ and σ have exceptional (or non-generic) reducibility. It is expected that in general for any reducibility α_0 we have

$$\alpha_0 \in (1/2)\,\mathbb{Z}$$

(this would follow from a F. Shahidi's conjecture on existence of a generic representation in each L^2 L-packet).

The first exceptional reducibilities seems to be proved to exist in 1996, by M. Reeder ([Re]) and by C. Mœglin ([Mg2]). It is a hard problem to determine the reducibility point α_0 for a general (ρ, σ). F. Shahidi has determined this in a number of cases ([Sd2], Theorem 3.3, Propositions 3.5 and 3.10). Earlier, J.-L. Waldspurger settled one such case ([W], Proposition 5.1). C. Mœglin has computed in some cases these reducibilities, and she has also formulated a conjecture about the generalized rank one reducibilities (see Remarks 3.2 for more comments).

Construction of square integrable representations is closely related to the reducibility of the generalized principal series representations. Particularly important for us is the reducibility of $\delta(\Delta) \rtimes \sigma$ ($\Delta \in \mathcal{S}(\mathcal{C})$ and σ is irreducible cuspidal). In [T5] we have proved that the reducibility of $\delta(\Delta)\rtimes\sigma$ is equivalent to

(RCS) $\rho \rtimes \sigma$ reduces for some $\rho \in \Delta$,

if char $(F) = 0$ and σ is generic (the proof in [T5] covers the case of general half-integral reducibility, which could be the general case according to the Shahidi's conjecture that we have mentioned above).

The following theorem is one of the main results of the paper.

THEOREM 1.1. *Let* $\Delta_i = [\nu^{-n_i}\rho_i, \nu^{m_i}\rho_i] \in \mathcal{S}(\mathcal{C})$, $i = 1, \ldots, k$. *Suppose that* ρ_i *are unitarizable,* $n_i, m_i \in (1/2)\mathbb{Z}$ *and* $n_i < m_i$. *Let* σ *be an irreducible cuspidal representation of* S_q. *Suppose*

1. $\Delta_i \cap \tilde{\Delta}_i$ *and* σ *satisfy (RCS), or* $\Delta_i \cap \tilde{\Delta}_i = \emptyset$ *and* $\nu^{-n_i}\rho_i \rtimes \sigma$ *reduces.*
2. *If* $\Delta_i \cap \Delta_j \neq \emptyset$, *for some* $1 \leq i < j \leq k$, *then either* $\Delta_i \cup \tilde{\Delta}_i \subsetneq \Delta_j \cap \tilde{\Delta}_j$, *or* $\Delta_j \cup \tilde{\Delta}_j \subsetneq \Delta_i \cap \tilde{\Delta}_i$.

Let $l = card\{i; 1 \leq i \leq k$ *and* $\Delta_i \cap \tilde{\Delta}_i \neq \emptyset\}$. *Then:*

(i) $\left(\prod_{i=1}^k \delta(\Delta_i \cap \tilde{\Delta}_i)\right) \rtimes \sigma$ *decomposes into a sum* $\oplus_{j=1}^{2^l} \tau_j$ *of* 2^l *inequivalent irreducible (tempered) representations. Each representation* $\left(\prod_{i=1}^k \delta(\Delta_i \backslash \tilde{\Delta}_i)\right) \rtimes \tau_j$ *has a unique irreducible subrepresentation, which we denote by*

$$\delta(\Delta_1, \ldots, \Delta_k, \sigma)_{\tau_j}.$$

The representations $\delta(\Delta_1, \ldots, \Delta_k, \sigma)_{\tau_j}$ *are square integrable.*

(ii) Each irreducible subrepresentation of $\left(\prod_{i=1}^k \delta(\Delta_i)\right) \rtimes \sigma$ *has multiplicity one. There exist exactly* 2^l *irreducible subrepresentations of* $\left(\prod_{i=1}^k \delta(\Delta_i)\right) \rtimes \sigma$ *and they are*

$$\{\delta(\Delta_1, \ldots, \Delta_k, \sigma)_{\tau_j}; j = 1, \ldots, 2^l\}.$$

The approach in this paper is different from the one in [MgT]. The square integrable representations in [MgT] are introduced inductively, while the definition in the above theorem is much more direct. Further, in this paper we get explicit upper bounds for Jacquet modules. This is the reason why we expect that in the study of the generic representations, or for computing some invariants, the approach of this paper could be useful.

The above theorem directly implies that the standard modules of the groups S_m, induced by generic (essentially tempered) representations, do not have injective Whittaker models in general (this is different from the case of the general linear groups; see Proposition 3.2 of [JcSl]).

D. Vogan showed us in 1992 number of places where he expected square integrable representations for symplectic groups (having in mind the conjectural local Langlands' correspondence). This was one of the motivations to construct such representations using the techniques developed in [T4]. Another motivation for our work was getting a parameterization of the non-unitary dual (in particular, getting a parameterization which is convenient for the work on the unitarizability problem).

Let us say a few words about the methods that we use in the construction. In [T3] (Theorem 7.2), we have constructed the structure which enables us to obtain, in a simple way, composition series of Jacquet modules of parabolically induced representations. The fact that Levi factors of maximal parabolic subgroups of S_m are isomorphic to products of general linear groups and groups S_q enables us to use the full power of the well understood representation theory of general linear groups in the representation theory of $Sp(n,F)$ and $SO(2n+1,F)$. In our construction of the representations $\delta(\Delta_1,\dots,\Delta_k,\sigma)_\tau$ in this paper, the basis is understanding of the representations $\delta(\Delta_i,\sigma)_{\tau'}$, which were introduced in [T6].

Although our work in this paper deals with the representations of the groups $Sp(n,F)$ and $SO(2n+1,F)$, this work will be not hard to extend to other classical groups.

The first two sections of this paper introduce notation and recall some previous results that we use often in the paper. We recall of the representations $\delta(\Delta_i,\sigma)_{\tau'}$ in the third section. In the fourth section, we give the construction of the representations $\delta(\Delta_1,\dots,\Delta_k,\sigma)_{\tau_j}$. The last section presents another proof of a result of D. Goldberg. We include this proof, because it works also in the positive characteristic.

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1. Preliminaries

In this paper, we fix a local non-archimedean field F of characteristic different from two. At the beginning of this section, we shall recall the standard notation from the representation theory of $GL(n,F)$ (see [Z1] for complete definitions). The minimal parabolic subgroup of $GL(n,F)$ consisting of all upper triangular matrices in $GL(n,F)$ is fixed. Parabolic subgroups of $GL(n,F)$ which contain this minimal parabolic subgroup will be called standard parabolic subgroups of $GL(n,F)$.

Let π_i be an admissible representation of $GL(n_i,F)$, for $i=1,2$. Then $\pi_1\times\pi_2$ denotes the representation of $GL(n_1+n_2,F)$ which is parabolically induced from the representation $\pi_1 \otimes \pi_2$ of a suitable standard parabolic subgroup. Then, $\pi_1 \times (\pi_2 \times \pi_3) \cong (\pi_1 \times \pi_2) \times \pi_3$.

If G is a reductive group over F, then there is always a natural order on the Grothendieck group of the category of all admissible representations of G of finite length. We shall denote by $\tilde{G}$ the set of all equivalence classes of irreducible admissible representations of G. The set of unitarizable classes in $\tilde{G}$ is denoted by $\hat{G}$.

Let the Grothendieck group of the category of all admissible representations of $GL(n,F)$ of finite length be denoted by R_n. The canonical mapping from the objects of the category to R_n is denoted by s.s. (the image forms a cone of positive elements). Set $R = \oplus_{n\geq 0} R_n$. One lifts the above multiplication to a multiplication $\times$ on R. The induced mapping $R\otimes R \to R$ is denoted by m.

Take an admissible representation π of $GL(n,F)$ of finite length. Let $\alpha = (n_1, \dots, n_k)$ be an ordered partition of n. Take the standard parabolic subgroup P_α^{GL} of $GL(n,F)$ whose Levi factor M_α^{GL} is naturally isomorphic to $GL(n_1,F) \times \dots \times GL(n_k,F)$. The Jacquet module of π with respect to P_α^{GL} is denoted by $r_\alpha(\pi)$. Consider s.s. $(r_\alpha(\pi)) \in R_{n_1} \otimes \dots \otimes R_{n_k}$. Set

$$m^*(\pi) = \sum_{k=0}^{n} \text{s.s. } \left(r_{(k,n-k)}(\pi)\right) \in R \otimes R.$$

One lifts m^* $\mathbb{Z}$-linearly to all of R.

For a matrix g, denote by ${}^t g$ (resp. ${}^\tau g$) the transposed matrix of g (resp. the transposed matrix of g with respect to the second diagonal). For a representation π of $GL(n,F)$, ${}^\tau\pi^{-1}$ denotes the representation $g \mapsto \pi({}^\tau g^{-1})$. We denote by $\tilde{\pi}$ the contragredient representation of π. We have ${}^\tau\pi^{-1} \cong \tilde{\pi}$ for irreducible π.

Let π be an irreducible admissible representation of $GL(n,F)$. If π is a subquotient of $\rho_1 \times \dots \times \rho_k$ where ρ_i are irreducible cuspidal representations of $GL(n_i,F)$, then we shall call the multiset $(\rho_1, \dots, \rho_k)$ the support of π. We write $\text{supp}(\pi) = (\rho_1, \dots, \rho_k)$. If π is of finite length such that any irreducible subquotient π' of π has $\text{supp}(\pi') = (\rho_1, \dots, \rho_k)$, then we say that π has a support and we shall write $\text{supp}(\pi) = (\rho_1, \dots, \rho_k)$. We extend this definition to allow $\pi \in R_n$ with $\pi > 0$ (there is a natural order on R_n's).

We now introduce a similar notation for two series of classical groups (see [T2] and [T3] for more details). The $n \times n$ matrix having 1's on the second diagonal and all other entries 0 will be denoted by J_n. The identity matrix is denoted by I_n. For a $2n \times 2n$ matrix S, set

$${}^\times S = \begin{bmatrix} 0 & -J_n \\ J_n & 0 \end{bmatrix} {}^t S \begin{bmatrix} 0 & J_n \\ -J_n & 0 \end{bmatrix}.$$

The group $Sp(n,F)$ consists of all $2n \times 2n$ matrices over F which satisfy ${}^{\times}S\,S = I_{2n}$. We define $Sp(0,F)$ to be the trivial group. Fix the minimal parabolic subgroup $P_{\min}$ in $Sp(n,F)$ consisting of all upper triangular matrices in the group.

We denote by $SO(2n+1,F)$ the group of all $(2n+1)\times(2n+1)$ matrices X with entries in F which satisfy ${}^{\tau}X\,X = I_{2n+1}$ and $\det X{=}1$. Fix the minimal parabolic subgroup $P_{\min}$ in $SO(2n+1,F)$ consisting of all upper triangular matrices in the group.

In the sequel, we denote by S_n either the group $Sp(n,F)$ or $SO(2n+1,F)$. Parabolic subgroups which contain the minimal parabolic subgroup which we have fixed will be called standard parabolic subgroups.

For $p_i \times p_i$ matrices $X_i,\ i = 1,\dots,k$, the quasi-diagonal $(p_1+\cdots+p_k)\times(p_1+\cdots+p_k)$ matrix which has the matrices $X_1,\cdots,X_k$ on the quasi-diagonal, is denoted by q-diag $(X_1,\cdots,X_k)$.

Let $\alpha = (n_1,\dots,n_k)$ be an ordered partition of some non-negative integer $m \le n$ into positive integers. If $m = 0$, then the only partition will be denoted by (0). Set

$$M_\alpha = \left\{\text{q-diag}\,(g_1,\cdots,g_k,h,\ {}^{\tau}g_k^{-1},\cdots,\ {}^{\tau}g_1^{-1}); g_i \in GL(n_i,F), h \in S_{n-m}\right\}$$

Then, $P_\alpha = M_\alpha P_{\min}$ is a standard parabolic subgroup of S_n. The unipotent radical of P_α is denoted by N_α. Since M_α is naturally isomorphic to $GL(n_1,F)\times\ldots\times GL(n_k,F)\times S_{n-m}$, we have a natural bijection

$$\tilde{M}_\alpha \quad\leftrightarrow\quad GL(n_1,F)\tilde{}\times\cdots\times GL(n_k,F)\tilde{}\times\tilde{S}_{n-m}.$$

Let π be an admissible representation of $GL(n,F)$ and let σ be an admissible representation of S_m. We denote by $\pi \rtimes \sigma$ the representation of S_{n+m} which is parabolically induced from the representation $\pi\otimes\sigma$ of $P_{(n)}$. Here $\pi\otimes\sigma$ maps q-diag$(g,h,{}^{\tau}g^{-1}) \in M_{(n)}$ to $\pi(g)\otimes\sigma(h)$. For admissible representations $\pi,\pi_1,\,\pi_2$ of general linear groups and for a similar representation σ of S_m, the following hold:

$$\pi_1\rtimes(\pi_2\rtimes\sigma)\cong(\pi_1\times\pi_2)\rtimes\sigma, \tag{1.1}$$

$$(\pi\rtimes\sigma)^{\sim}\ \cong\ \tilde{\pi}\rtimes\tilde{\sigma}. \tag{1.2}$$

The Grothendieck group of the category of all admissible representations of S_n of finite length is denoted by $R_n(S)$. Set $R(S) = \oplus_{n\ge 0} R_n(S)$. We lift the multiplication $\rtimes$ to a multiplication $\rtimes : R\times R(S)\to R(S)$ in the usual way. In this way, $R(S)$ becomes an R-module. Denote the contragredient involution on R and $R(S)$ by $^{\sim}$. For $\pi\in R$ and $\sigma\in R(S)$, we have (in $R(S)$)

$$\pi\rtimes\sigma = \tilde{\pi}\rtimes\sigma. \tag{1.3}$$

Let $\mu : R\otimes R(S)\to R(S)$ be the $\mathbb{Z}$-bilinear mapping which satisfies $\mu(\pi\otimes\sigma) =$ s.s.$(\pi\rtimes\sigma)$, for $\pi\in R,\sigma\in R(S)$.

Since we have natural orders on Grothendieck groups, there is a natural order on R, $R(S)$ and $R \otimes R(S)$.

Let σ be a smooth representation of S_n of finite-length and let $\alpha = (n_1, \dots, n_k)$ be an ordered partition of $0 \leq m \leq n$. The Jacquet module of σ for P_α is denoted by $s_\alpha(\sigma)$. We may consider s.s. $(s_\alpha(\sigma)) \in R_{n_1} \otimes \cdots \otimes R_{n_k} \otimes R_{n-m}(S)$. Define a $\mathbb{Z}$-linear mapping $\mu^* : R(S) \to R \otimes R(S)$ on the basis of irreducible admissible representations by

$$\mu^*(\sigma) = \sum_{k=0}^{n} \text{s.s.}\left(s_{(k)}(\sigma)\right).$$

Denote by $s : R \otimes R \to R \otimes R$ the mapping $s(\sum_i x_i \otimes y_i) = \sum_i y_i \otimes x_i$. For $r_1 \otimes r_2 \in R \otimes R$ and $r \otimes t \in R \otimes R(S)$ set $(r_1 \otimes r_2) \rtimes (r \otimes t) = (r_1 \times r) \otimes (r_2 \rtimes t)$. Extend $\rtimes$ $\mathbb{Z}$-bilinearly to $\rtimes : (R \otimes R) \times (R \otimes R(S)) \to R \otimes R(S)$. Set

$$M^* = (m \otimes 1) \circ (\sim \otimes m^*) \circ s \circ m^*.$$

Then,

$$\mu^*(\pi \rtimes \sigma) = M^*(\pi) \rtimes \mu^*(\sigma) \tag{1.4}$$

for an admissible representation π of $GL(n, F)$ of finite length and a similar representation σ of S_m.

Let $\pi \otimes \sigma$ be an admissible representation of $GL(n, F) \times S_m$. We say that $\pi \otimes \sigma$ has GL-support if π has support and if σ is an irreducible cuspidal representation. Then, we write

$$\text{supp}_{GL}(\pi \otimes \sigma) = \text{supp}(\pi).$$

We extend this definition to allow $\pi \otimes \sigma \in R_n \otimes R_m(S)$ with $\pi > 0$ and σ irreducible cuspidal.

Suppose that τ is an irreducible admissible representation of S_m. Then, there exist irreducible cuspidal representations ρ_i of $GL(n_i, F)$, $i = 1, \dots, k$, and an irreducible cuspidal representation σ of $S_{m-(n_1+\cdots+n_k)}$ such that τ is a subquotient of $\rho_1 \times \cdots \times \rho_k \rtimes \sigma$. We define

$$\text{depth}_{GL}(\tau) = n_1 + \cdots + n_k.$$

If τ is a an admissible representation of S_m of finite length such that $\text{depth}_{GL}(\tau') = d$ for any irreducible subquotient τ' of τ, then we say that τ has a depth and we write $\text{depth}_{GL}(\tau) = d$. In a similar way, we define if $\tau \in R_n(S)$, $\tau > 0$, has a depth, and the depth. If an admissible representation τ of finite length has a depth, then we define

$$s_{GL}(\tau) = s_{(\text{depth}_{GL}(\tau))}(\tau).$$

In a similar way, we define $s_{GL}(\tau)$ for $\tau \in R_n(S)$, $\tau > 0$, if τ has a depth.

2. Square integrability, Langlands' classification

An irreducible representation π of a reductive p-adic group G is called essentially square integrable if there exists a continuous (not necessarily unitary character) $\chi : G \to \mathbb{C}^\times$ such that $\chi\pi$ is a square integrable representation (i.e., $\chi\pi$ has a unitary central character, and for any matrix coefficient ϕ of $\chi\pi$, $|\phi|$ is a square integrable function on G modulo center).

The set of all equivalence classes of irreducible cuspidal representations of all $GL(p, F)$, $p \geq 1$, will be denoted by $\mathcal{C}$. Let $\rho \in \mathcal{C}$ and let n be a non-negative integer. The set $[\rho, \nu^n\rho] = \{\rho, \nu\rho, \nu^2\rho, \dots, \nu^n\rho\}$ is called a segment in irreducible cuspidal representations of general linear groups, or a segment in $\mathcal{C}$. The set of all segments in $\mathcal{C}$ will be denoted by $\mathcal{S}(\mathcal{C})$. The representation $\nu^n\rho \times \nu^{n-1}\rho \times \cdots \times \nu\rho \times \rho$ has a unique irreducible subrepresentation which we denote by $\delta([\rho, \nu^n\rho])$. The representation $\delta([\rho, \nu^n\rho])$ is an essentially square integrable representation and $\Delta \mapsto \delta(\Delta)$ is a bijection of $\mathcal{S}(\mathcal{C})$ onto the set of all equivalence classes of irreducible essentially square integrable representations of all $GL(k, F)$, $k \geq 0$.

If $n < 0$, then we define $[\rho, \nu^n\rho]$ to be the empty set $\emptyset$, and we take $\delta(\emptyset)$ to be $1 \in R$. By Proposition 9.5 of [Z1] we have

$$m^*\left(\delta([\rho, \nu^n\rho])\right) = \sum_{k=-1}^{n} \delta([\nu^{k+1}\rho, \nu^n\rho]) \otimes \delta([\rho, \nu^k\rho]). \tag{2.1}$$

This formula implies that

$$s\left(m^*\left(\delta([\rho, \nu^n\rho])\right)\right) = \sum_{k=-1}^{n} \delta([\rho, \nu^k\rho]) \otimes \delta([\nu^{k+1}\rho, \nu^n\rho]).$$

Assume that $\rho \in \mathcal{C}$ is a representation of $GL(p, F)$. We have

$$r_{(p)^{n+1}}\left(\delta([\rho, \nu^n\rho])\right) = \nu^n\rho \otimes \nu^{n-1}\rho \otimes \cdots \otimes \rho,$$

where $(p)^{n+1}$ denotes $(p, p, \dots, p) \in \mathbb{Z}^{n+1}$.

Let X be a set. We shall denote by M(X) the set of all finite multisets in X (more details regarding this notation can be found on the page 169 of [Z1]; see also [Z2]). The addition among multisets is defined by $(x_1, \dots, x_k) + (x'_1, \dots, x'_{k'}) = (x_1, \dots, x_k, x'_1, \dots, x'_{k'})$. If $a, b, c \in M(X)$ and $a + b = c$, then we shall also denote a by $c - b$.

For an irreducible essentially square integrable representation δ of $GL(m, F)$, one can find a unique $e(\delta) \in \mathbb{R}$ such that $\nu^{-e(\delta)}\delta$ is unitarizable. Set $\delta^u = \nu^{-e(\delta)}\delta$. Then, $\delta = \nu^{e(\delta)}\delta^u$, where $e(\delta) \in \mathbb{R}$ and δ^u is unitarizable.

We denote by D the set of all equivalence classes of the irreducible essentially square integrable representations of all $GL(n, F)$'s with $n \geq 1$. Let $d = (\delta_1, \dots, \delta_k) \in M(D)$ where $M(D)$ denotes the set of all finite multisets in D. Take a permutation p of the set $\{1, \dots, k\}$ such that $e(\delta_{p(1)}) > e(\delta_{p(2)}) \cdots > e(\delta_{p(k)})$. The representation $\delta_{p(1)} \times \cdots \times \delta_{p(k)}$ has a unique irreducible quotient which we denote by $L(d)$. Then $d \mapsto L(d)$ is Langlands' classification for general linear groups. We shall usually write $L(d) = L((\delta_1, \ldots, \delta_k))$ simply as $L(\delta_1, \ldots, \delta_k)$.

In this paper, we shall have several occasions to use the following well-known fact proved by A.V. Zelevinsky (this fact follows from Theorem 7.1 of [Z1] and Remark 5.3 of [Ro], using either Theorem 2.3 and Corollary 3.9 of [A], or the fifth section of [ScSt]; see also [T1]). For two segments Δ', $\Delta'' \in \mathcal{S}(\mathcal{C})$, one says that they are linked if $\Delta' \cup \Delta'' \in \mathcal{S}(\mathcal{C})$ and $\Delta' \cup \Delta'' \notin \{\Delta', \Delta''\}$. Let $\Delta_1, \ldots, \Delta_k \in \mathcal{S}(\mathcal{C})$. If there exist $1 \leq i < j \leq k$ such that Δ_i and Δ_j are linked, then we shall write

$$(\Delta_1, \Delta_2, \ldots, \Delta_{i-1}, \Delta_i \cup \Delta_j, \Delta_{i+1}, \ldots, \Delta_{j-1}, \Delta_i \cap \Delta_j, \Delta_{j+1}, \ldots, \Delta_{k-1}, \Delta_k) \prec (\Delta_1, \Delta_2, \ldots, \Delta_{k-1}, \Delta_k).$$

Now $\prec$ generates a partial order on $\mathcal{S}(\mathcal{C})$. Denote the partial order thus obtained by $\leq$. Let $\Delta_1', \ldots, \Delta_{k'}' \in \mathcal{S}(\mathcal{C})$. Then $L(\delta(\Delta_1'), \ldots, \delta(\Delta_{k'}'))$ is a subquotient of $\delta(\Delta_1) \times \cdots \times \delta(\Delta_k)$ if and only if $(\Delta_1', \ldots, \Delta_{k'}') \leq (\Delta_1, \ldots, \Delta_k)$. Suppose that $(\Delta_1', \ldots, \Delta_{k'}') \leq (\Delta_1, \ldots, \Delta_k)$ and suppose that among all pairs $\Delta_i', \ \Delta_j', \quad 1 \leq i \neq j \leq k'$, there do not exist linked segments. Then $\delta(\Delta_1') \times \cdots \times \delta(\Delta_{k'}')$ is irreducible and it has multiplicity one in $\delta(\Delta_1) \times \cdots \times \delta(\Delta_k)$.

Suppose that $\Delta_i, \Delta_j' \in \mathcal{S}(\mathcal{C})$, $1 \leq i \leq k, 1 \leq j \leq k'$. If Δ_i is not linked to any Δ_j', for $1 \leq i \leq k, 1 \leq j \leq k'$, then $L(\delta(\Delta_1), \ldots, \delta(\Delta_k)) \times L(\delta(\Delta_1'), \ldots, \delta(\Delta_{k'}'))$ is irreducible and $L(\delta(\Delta_1), \ldots, \delta(\Delta_k)) \times L(\delta(\Delta_1'), \ldots, \delta(\Delta_{k'}')) = L(\delta(\Delta_1), \ldots, \delta(\Delta_k), \delta(\Delta_1'), \ldots, \delta(\Delta_{k'}'))$.

We recall the Casselman square integrability criterion in the case of S_n (which is a special of Theorem 4.4.6 of [C]; see also the sixth section of [T2]). Consider the standard inner product on $\mathbb{R}^n$. Set

$$\beta_i = (\underbrace{1, 1, \ldots, 1}_{i \ \text{times}}, 0, 0, \ldots, 0) \in \mathbb{R}^n, \ 1 \leq i \leq n.$$

Let π be a non-cuspidal irreducible admissible representation of S_n. Take α such that $s_\alpha(\pi)$ has a cuspidal subquotient ($s_\alpha(\pi) \neq 0$). Write $\alpha = (n_1, \ldots, n_\ell)$ and denote $n_1 + \cdots + n_\ell = m$. Take an irreducible subquotient σ of s_α and decompose $\sigma = \rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_\ell \otimes \rho$, where $\rho_i \in GL(n_i, F)\tilde{}$, $\rho \in \tilde{S}_{n-m}$. Define

$$e_*(\sigma) = (\underbrace{e(\rho_1), \ldots, e(\rho_1)}_{n_1 \ \text{times}}, \ldots, \underbrace{e(\rho_\ell), \ldots, e(\rho_\ell)}_{n_\ell \ \text{times}}, \underbrace{0, \ldots, 0}_{n-m \ \text{times}}).$$

If π is square integrable, then

$$(e_*(\sigma), \beta_{n_1}) > 0, (e_*(\sigma), \beta_{n_1+n_2}) > 0, \cdots, (e_*(\sigma), \beta_m) > 0.$$

Conversely, if all above inequalities hold for any α and σ as above, then π is square integrable. If instead of > 0, the weaker condition ≥ 0 holds in all the above relations, then π is tempered.

3. Square integrable representations corresponding to single segments

First, we shall recall the square integrable representations of the Steinberg type (Proposition 3.1 of [T4]).

Proposition 3.1. *Fix an irreducible unitarizable cuspidal representation* ρ *of* $GL(p,F)$ *and a similar representation* σ *of* S_q*. Suppose that* $\nu^\alpha\rho \rtimes \sigma$ *reduces for some* $\alpha \in (1/2)\mathbb{Z}$*,* $\alpha > 0$*. Then* $\rho \cong \tilde{\rho}$*. The representation* $\nu^{\alpha+n}\rho \times \nu^{\alpha+n-1}\rho \times \cdots \nu^{\alpha+1}\rho \times \nu^\alpha\rho \rtimes \sigma$ *has a unique irreducible subrepresentation which we denote by* $\delta([\nu^\alpha\rho, \nu^{\alpha+n}\rho], \sigma)$ $(n \geq 0)$*. This irreducible subrepresentation can be characterized as a unique irreducible subquotient* π *of* $\delta([\nu^\alpha\rho, \nu^{\alpha+n}\rho]) \rtimes \sigma$ *which satisfies* $\delta([\nu^\alpha\rho, \nu^{\alpha+n}\rho]) \otimes \sigma \leq s_{GL}(\pi)$*. We have* $s_{(p)^{n+1}}(\delta([\nu^\alpha\rho, \nu^{\alpha+n}\rho], \sigma)) = \nu^{\alpha+n}\rho \otimes \nu^{\alpha+n-1}\rho \otimes \cdots \otimes \nu^{\alpha+1}\rho \otimes \nu^\alpha\rho \otimes \sigma$ *(here* $(p)^{n+1} = (p, p, \ldots, p) \in \mathbb{Z}^{n+1}$*) and*

$$\mu^*\left(\delta([\nu^\alpha\rho, \nu^{\alpha+n}\rho], \sigma)\right) = \sum_{k=-1}^{n} \delta([\nu^{\alpha+k+1}\rho, \nu^{\alpha+n}\rho]) \otimes \delta([\nu^\alpha\rho, \nu^{\alpha+k}\rho], \sigma)$$

The representation $\delta([\nu^\alpha\rho, \nu^{\alpha+n}\rho], \sigma)$ *is square integrable and we have* $\delta([\nu^\alpha\rho, \nu^{\alpha+n}\rho], \sigma)\tilde{} \cong \delta([\nu^\alpha\rho, \nu^{\alpha+n}\rho], \tilde{\sigma})$*.*

In the above formula, we just take $\delta(\emptyset, \sigma)$ to be σ. Note that the above proposition holds without assumption $\alpha \in (1/2)\mathbb{Z}$.

We shall sometimes denote $\delta([\nu^\alpha\rho, \nu^{\alpha+n}\rho], \sigma)$ also by $\delta([\nu^\alpha\rho, \nu^{\alpha+n}\rho], \sigma)_\sigma$ (this notation is in the spirit of the notation for square integrable representations that we shall introduce in the following section).

Let ρ be an irreducible unitarizable cuspidal representation of $GL(p,F)$ and let σ be an irreducible cuspidal representation of S_q. Suppose that $\nu^\alpha\rho \rtimes \sigma$ reduces for some $\alpha \in \mathbb{R}$. If there exists $\alpha_0 \geq 0$ such that

(Cα_0) $\quad \nu^{\alpha_0}\rho \rtimes \sigma$ reduces and $\nu^\beta\rho \rtimes \sigma$ is irreducible for $\beta \in \mathbb{R}, |\beta| \neq \alpha_0$,

then we shall say that ρ and σ have reducibility at α_0, or that they satisfy (Cα_0) (we follow the notation of Jantzen's paper [Jn1]). If ρ and σ have reducibility at α_0, and σ is generic, then Theorem 8.1 of [Sd1] implies that

$$\alpha_0 \in \{0, 1/2, 1\}$$

(see also Theorem 3.3 of [Sd2]). In general, if ρ and σ satisfy (Cα_0) with $\alpha_0 \in \{0, 1/2, 1\}$, then we shall say that they have generic (or non-exceptional) reducibility (σ does not need to be generic). Otherwise, we shall say that they have exceptional (or non-generic) reducibility.

It is well-known that if $\nu^\alpha\rho \rtimes \sigma$ reduces for some $\alpha \in \mathbb{R}$, then $\rho \cong \tilde{\rho}$ (as before, ρ and σ are irreducible unitarizable cuspidal representations of $GL(p,F)$ and S_q, respectively).

An admissible representation ρ will be called selfdual if $\rho \cong \tilde{\rho}$. If an irreducible cuspidal representation is selfdual, then it is unitarizable.

Remark 3.2. We shall not use the following facts in this paper (they give some information about the role of the generic reducibilities among all reducibilities). We shall denote by ρ a selfdual irreducible cuspidal representation of $GL(p,F)$ and by σ an irreducible cuspidal representation of S_q.
(i) For a given ρ and σ, the question of determining α_0 is a hard one. The case of ρ being a character and σ being the trivial representation, has been known for a long time (this is just the question of reducibility of principal series for $SL(2,F)$ and $SO(3,F)$). The first more general case was settled by J.-L. Waldspurger (Proposition 5.1 of [W]). F. Shahidi made a big progress in this problem in [Sd2] (Theorems 1.2 and 3.3, Proposition 3.5, Proposition 3.10, ...). Further results in this direction are obtained by C. Mœglin, G. Muić, F. Murnaghan and J. Repka ([MrRp], Corollary 11.5), and M. Reeder ([Re]).
(ii) C. Mœglin has formulated a conjecture which describes α_0 in terms of the conjectural local Langlands correspondence.
(iii) C. Mœglin and M. Reeder have shown independently that exceptional reducibilities can occur ([Mg2], [Re]). The existence of exceptional reducibilities seems to be known from the summer of 1996.
(iv) The Mœglin's conjecture would imply that exceptional reducibilities are rare in the following sense. For a fixed irreducible cuspidal representation σ of S_q, the Mœglin's conjecture would imply that there exist at most finitely many selfdual $\rho \in \mathcal{C}$ such that ρ and σ have exceptional reducibility. This conjecture would also imply that there are infinitely many $\rho \in \mathcal{C}$ such that ρ and σ have generic reducibility.
(v) From the Harish-Chandra and A. Silberger's work, it follows that reducibility of $\nu^\beta \rho \rtimes \sigma$ for some $\beta \geq 0$, implies irreducibility of $\nu^\alpha \rho \rtimes \sigma$ for all $\alpha \in \mathbb{R} \backslash \{\pm\beta\}$. Therefore, (C$\alpha_0$) is just the condition that $\nu^{\alpha_0} \rho \rtimes \sigma$ reduces.
(vi) We have mentioned that if $\nu^\alpha \rho \rtimes \sigma$ reduces for same $\alpha \in \mathbb{R}$, then ρ is selfdual. The converse of this fact also holds: if ρ is selfdual, then $\nu^\alpha \rho \rtimes \sigma$ reduces for some $\alpha \in \mathbb{R}$. The argument is the following: suppose that this is not the case. The properties of the standard integral intertwining operators imply that representations $\nu^\alpha \rho \rtimes \sigma$, $\alpha \in \mathbb{R}$, are unitarizable (they form a complementary series). Recall that the matrix coefficients of a unitarizable representation are bounded. Further s.s.$(s_{(p)}(\nu^\alpha \rho \rtimes \sigma)) = \nu^\alpha \rho \otimes \sigma + \nu^{-\alpha} \rho \otimes \sigma$. Now the connection between the asymptotic of matrix coefficients of a representation and Jacquet modules (see [C], Theorem 4.3.3), leads to a contradiction. One can also get in this way an explicit upper bound for reducibility point $\alpha_0 \geq 0$ corresponding to ρ and σ.
(vii) Conjecture 9.4 of [Sd1] would imply that (Cα_0) can happen only for $\alpha_0 \in (1/2)\,\mathbb{Z}$ (this is also contained in the C. Mœglin's conjecture).

Now we shall recall of the representations defined in [T6].

Theorem 3.3. *Let ρ and σ be irreducible unitarizable cuspidal representations of $GL(p,F)$ and S_q respectively, and $\alpha \in (1/2)\,\mathbb{Z}$, $\alpha \geq 0$. Suppose that ρ and σ have reducibility at α. Fix $n, m \in (\alpha+\mathbb{Z})$ satisfying $\alpha \leq n < m$. Denote $\Delta = [\nu^{-n}\rho, \nu^m\rho]$. Then the representation $\delta(\Delta \cap \tilde{\Delta}) \rtimes \sigma$ decomposes into a direct sum of two inequivalent irreducible representations τ_1 and τ_2. Each representation $\delta(\Delta \backslash \tilde{\Delta}) \rtimes \tau_i$ contains a unique irreducible subrepresentation, which we denote by*

$$\delta(\Delta, \sigma)_{\tau_i}.$$

The representation $\delta(\Delta, \sigma)_{\tau_i}$ can be characterized as a unique irreducible subquotient π of $\delta(\Delta \backslash \tilde{\Delta}) \rtimes \tau_i$ which satisfies $\delta(\Delta) \otimes \sigma \leq s_{GL}(\pi)$. Each $\delta(\Delta, \sigma)_{\tau_i}$ is a subrepresentation of $\delta(\Delta) \rtimes \sigma$,

$$\begin{aligned} &\delta(\Delta) \otimes \sigma \leq s_{GL}(\delta(\Delta, \sigma)_{\tau_i}) \\ &\qquad \leq \sum_{i=-n}^{\alpha} \delta([\nu^i\rho, \nu^m\rho]) \times \delta([\nu^{-i+1}\rho, \nu^n\rho]) \otimes \sigma \end{aligned} \tag{3.1}$$

and $\delta(\Delta, \sigma)_{\tau_i}$ is square integrable.

Let $\delta(\Delta, \sigma)_\tau = \delta([\nu^{-n}\rho, \nu^m\rho], \sigma)_\tau$ be a representation defined either in Proposition 3.1 or Theorem 3.3. Then in both cases hold

$$\begin{aligned} s_{GL}(\delta([\nu^{-n}\rho, \nu^m\rho], \sigma)_{\tau_i}) &\leq \sum_{i=-n}^{\alpha} \delta([\nu^i\rho, \nu^m\rho]) \times \delta([\nu^{-i+1}\rho, \nu^n\rho]) \otimes \sigma \\ &\leq \sum_{i=-n}^{|n|} \delta([\nu^i\rho, \nu^m\rho]) \times \delta([\nu^{-i+1}\rho, \nu^n\rho]) \otimes \sigma \end{aligned} \tag{3.2}$$

(the above inequality is obvious in the case of $n \geq 0$, while in the case $n < 0$ one checks it directly).

4. Square integrable representations corresponding to several segments

For $\Delta \in M(\mathcal{S}(\mathcal{C}))$, set $\tilde{\Delta} = \{\tilde{\rho} \in \Delta; \rho \in \Delta\}$. We shall say that Δ is selfdual if $\Delta = \tilde{\Delta}$. We say that Δ is balanced if $e(\delta(\Delta)) = 0$. Clearly, a selfdual segment is balanced.

Let X be a set. For a finite multiset $x = (x_1, \ldots, x_k)$ in X, we shall denote by $\mathrm{Set}(x) = \{x_1, \ldots, x_k\}$ the subset of X corresponding to x (this is the set which one gets from the multiset x by forgetting the multiplicities of elements which enter x). If one considers a finite multiset x in X as a function $x : X \to \{z \in \mathbb{Z}; z \geq 0\}$ with finite support, then $\mathrm{Set}(x)$ is just the support of the function x.

In the following proposition, we collect some facts about tempered representations that we need in the construction of square integrable representations corresponding to several segments in irreducible cuspidal representations of general linear groups. In the case of char $(F)=0$, claim (i) of the following proposition follows from Theorems 4.9, 6.4, 6.5 of [Go] and Proposition 2.3 of [H]. We present a different proof of (i) in the following section, in order to have the claim (i) also proved for positive characteristic.

PROPOSITION 4.1. *Let σ be an irreducible cuspidal representation of S_q. Let $\Delta_1,\dots,\Delta_k \in \mathcal{S}(\mathcal{C})$ be a sequence of different selfdual segments. Write $\Delta_i=[\nu^{-n_i}\rho_i,\nu^{n_i}\rho_i]$, $i=1,\dots,k$, where $\rho_i\in\mathcal{C}$, $n_i\in(1/2)\mathbb{Z}$, $n_i\geq 0$. Suppose that (ρ_i,σ) has reducibility at $\alpha_i\in(1/2)\mathbb{Z}$, $\alpha_i\geq 0$ and $\delta(\Delta_i)\rtimes\sigma$ reduces for $i=1,\dots,k$. Then,*
(i) $\delta(\Delta_1)\times\dots\times\delta(\Delta_k)\rtimes\sigma$ is a multiplicity one representation of length 2^k.
(ii) The multiplicity of $\delta(\Delta_1)\times\dots\times\delta(\Delta_k)\otimes\sigma$ in $s_{GL}(\delta(\Delta_1)\times\dots\times\delta(\Delta_k)\rtimes\sigma)$ is 2^k.
(iii) Let τ be an irreducible subrepresentation of $\delta(\Delta_1)\times\dots\times\delta(\Delta_k)\rtimes\sigma$. Then, the multiplicity of $\delta(\Delta_1)\times\dots\times\delta(\Delta_k)\otimes\sigma$ in $s_{GL}(\tau)$ is one.
(iv) Let τ be as in (iii). If π is any irreducible subquotient of $s_{GL}(\tau)$ different from $\delta(\Delta_1)\times\dots\times\delta(\Delta_k)\otimes\sigma$, then

$$\text{(4.1)}\quad Set(supp_{GL}(\pi))\subseteq\Delta_1\cup\dots\cup\Delta_k \text{ and } supp_{GL}(\pi)\neq\Delta_1+\dots+\Delta_k.$$

REMARK 4.2. The condition in the above proposition that $\delta(\Delta_i)\rtimes\sigma$ reduces is equivalent to $\nu^{\alpha_i}\in\Delta_i$ (Theorem 13.2 of [T5]).

Recall that if Δ is balanced, but not selfdual, then $\delta(\Delta)\rtimes\sigma$ is irreducible.

PROOF. First we prove (ii). Let $\beta=\delta(\Delta_1)\times\dots\times\delta(\Delta_k)\otimes\sigma$ (clearly, β is irreducible). Write

$$\begin{aligned}(4.2)\quad & M^*_{GL}(\delta([\nu^{-n_i}\rho_i,\nu^{n_i}\rho_i]))=\\ &=\sum_{k_i=-n_i}^{n_i+1}\delta([\nu^{k_i}\rho_i,\nu^{n_i}\rho_i])\times\delta([\nu^{-k_i+1}\rho_i,\nu^{n_i}\rho_i]).\end{aligned}$$

The above sum runs over $k_i\in n_i+\mathbb{Z}$ such that $-n_i\leq k_i\leq n_i+1$ (such convention we shall also use in the sequel). Then, (1.4) implies

$$\begin{aligned}(4.3)\quad & \text{s.s.}(s_{GL}(\delta(\Delta_1)\times\dots\times\delta(\Delta_k)\rtimes\sigma))=\\ &=M^*_{GL}(\delta(\Delta_1))\times\dots\times M^*_{GL}(\delta(\Delta_k))\otimes\sigma.\end{aligned}$$

Note that for $k_i=-n_i$ or n_i+1, the term in the sum (4.2) is $\delta(\Delta_i)$. Therefore, multiplying these terms in (4.3), we get that the multiplicity of β in $s_{GL}(\delta(\Delta_1)\times\dots\times\delta(\Delta_k)\rtimes\sigma)$ is at least 2^k.

Now, we shall see that β can appear as a subquotient of $s_{GL}(\delta(\Delta_1)\times\dots\times\delta(\Delta_k)\rtimes\sigma)$ only in the above way. We shall discuss when β can be obtained as a subquotient of the product in (4.3). Choose i_1 such that $\Delta_{i_1}\not\subseteq\Delta_i$ for $i\in\{1,\dots,k\}, i\neq i_1$ (this choice is possible since Δ_i's are mutually different). If we want to get β in the product of the right hand side of (4.3), then in the i_1-place in the product, we must take a term in the sum (4.2) corresponding to $-n_{i_1}$ or $n_{i_1}+1$ (since $\nu^{-n_{i_1}}\rho_{i_1}$ is in $\text{supp}_{GL}(\beta)$, and because no other terms in the sum except these two can give $\nu^{-n_{i_1}}\rho_{i_1}$ in the GL-support, nor can other terms in the product give $\nu^{-n_{i_1}}\rho_{i_1}$ in the GL-support, thanks to the condition $\Delta_{i_1} \not\subseteq \Delta_i$ for $i \neq i_1$). This proves that in the i_1-th place β can come only from terms corresponding to $k=-n_{i_1}$ or $n_{i_1}+1$. Now, choose $i_2 \in \{1,\dots,k\}, i_2 \neq i_1$ such that $\Delta_{i_2} \not\subseteq \Delta_i$ for $i \in \{1,\dots,n\}\backslash\{i_1,i_2\}$. Then, repeating the above type of argument with the GL-support (and $\nu^{-n_2}\rho_{n_2}$), we obtain that we can get β in the product only if in the i_2-th place, we take a term corresponding to $-n_{i_2}$ or $n_{i_2}+1$ (one needs to work with $\text{supp}_{GL}(\beta)-\Delta_{i_1}$, where $-$ denotes subtraction between multisets). Choosing $i_3, i_4, \dots, i_k$ in an analogous way and continuing with the above type of argument, we obtain that β can appear only in the way that we have described. Therefore, the multiplicity of β in $s_{GL}\left(\delta(\Delta_1)\times\cdots\times\delta(\Delta_k)\rtimes\sigma\right)$ is 2^k. This proves (ii).

Theorems 4.9, 6.4, 6.5 of [Go] and Proposition 2.3 of [H] imply (i) when char$(F)=0$. In the case of positive characteristic, (i) is proved in the fifth section.

Claim (iii) follows from the fact that $\left(\prod_{i=1}^k \delta(\Delta_i)\right)\otimes\sigma$ must be a quotient of $s_{GL}(\tau)$ (which follows from Frobenius reciprocity and the unitarizability of $\delta(\Delta_1)\times\cdots\times\delta(\Delta_k)\rtimes\sigma$), using (i) and (ii).

It remains to prove (iv). The first claim in (iv) follows from (4.3) and (4.2). Now, suppose π as in (iv) comes from a term $\beta'=\tau_1'\times\cdots\times\tau_k'$ on the right-hand side of (4.3), with τ_i' an irreducible subquotient of $M^*_{GL}(\delta(\Delta_i))$. From the above considerations, this can happen only if for some i', $\tau'_{i'}$ is a subquotient of a term (in the expression (4.2) for $M^*_{GL}(\delta(\Delta_i))$) with $-n_{i'}<k_{i'}<n_{i'}+1$. Denote (for above β') the set of all such indexes i' by X (i.e., the set of all indexes i' where enters a term corresponding to $-n_{i'}<k_{i'}<n_{i'}+1$). Choose i_0 such that $\Delta_{i_0}\not\subseteq\Delta_i$ for any $i\in X\backslash\{i_0\}$. Now, it is easy to see that $\text{supp}_{GL}(\pi)\neq\Delta_1+\cdots+\Delta_k$ (consider the multiplicity of $\nu^{-n_{i_0}}\rho_{i_0}$ in $\text{supp}_{GL}(\pi)$ and in $\Delta_1+\cdots+\Delta_k$; they are different).

We do not need the following theorem in this paper (which is Theorem 13.2 of [T5]). We mention the theorem because it gives additional insight into some of the conditions in the following proposition.

Theorem 4.3. *Let $\Delta=[\nu^\alpha\rho,\nu^\beta\rho]\in\mathcal{S}(\mathcal{C})$, where $\alpha,\beta\in\mathbb{R}$, and ρ is unitarizable. Assume char$(F)=0$. Let σ be an irreducible cuspidal representation of S_q. Suppose that (ρ,σ) has reducibility in $\alpha\in(1/2)\mathbb{Z}$, $\alpha\geq 0$, if ρ is selfdual. Then, $\delta(\Delta)\rtimes\sigma$ reduces if and only if $\rho'\rtimes\sigma$ reduces for some $\rho'\in\Delta$.*

Proposition 4.4. *Let $\Delta_i=[\nu^{-n_i}\rho_i,\nu^{m_i}\rho_i]\in\mathcal{S}(\mathcal{C})$, $i=1,\dots,k$, where ρ_i are selfdual, $m_i,n_i\in(1/2)\mathbb{Z}$, and let σ be an irreducible cuspidal representation of S_q. Assume that (ρ_i,σ) has reducibility in $\alpha_i\in(1/2)\mathbb{Z}$, $\alpha_i\geq 0$, for $i=1,\dots,k$. Suppose that the following three conditions hold:*

(a) $n_i > m_i$ *for* $i = 1, \dots, k$.
(b) $\nu^{\alpha_i}\rho_i \in \Delta_i \cap \tilde{\Delta}_i$ *or* $\Delta_i \cap \tilde{\Delta}_i = \emptyset$ *and* $-n_i = \alpha_i$.
(c) If $\rho_i \cong \rho_j$ *for some* $i \neq j$, *then either* $m_i < n_j$ *or* $m_j < n_i$. *Let* $l = card(\{\Delta_i \cap \tilde{\Delta}_i; i = 1, \dots, k\} \backslash \{\emptyset\}) = card(\{i; 1 \leq i \leq k$ *and* $n_i \geq 0\})$. *Then:*
(i) The multiplicity of $\left(\prod_{i=1}^{k} \delta(\Delta_i)\right) \otimes \sigma$ *in representations* $s_{GL}\left(\left(\prod_{i=1}^{k} \delta(\Delta_i)\right) \rtimes \sigma\right)$ *and* $s_{GL}\left(\left(\prod_{i=1}^{k} (\delta(\Delta_i \backslash \tilde{\Delta}_i) \times \delta(\Delta_i \cap \tilde{\Delta}_i))\right) \rtimes \sigma\right)$ *is* 2^l.
(ii) Let τ *be an irreducible subrepresentation of* $\left(\prod_{i=1}^{k} \delta(\Delta_i \cap \tilde{\Delta}_i)\right) \rtimes \sigma$. *The multiplicity of* $\left(\prod_{i=1}^{k} \delta(\Delta_i)\right) \otimes \sigma$ *in* $s_{GL}\left(\left(\prod_{i=1}^{k} \delta(\Delta_i \backslash \tilde{\Delta}_i)\right) \rtimes \tau\right)$ *is one.*
(iii) Let τ *be as in (ii). The representation* $\left(\prod_{i=1}^{k} \delta(\Delta_i \backslash \tilde{\Delta}_i)\right) \rtimes \tau$ *has a unique irreducible subquotient* π_τ *such that* $\left(\prod_{i=1}^{k} \delta(\Delta_i)\right) \otimes \sigma$ *is a subquotient of* $s_{GL}(\pi_\tau)$. *Further, the multiplicity of* π_τ *in* $\left(\prod_{i=1}^{k} \delta(\Delta_i \backslash \tilde{\Delta}_i)\right) \rtimes \tau$ *is one. We shall denote* π_τ *by*

$$\delta(\Delta_1, \dots, \Delta_k, \sigma)_\tau.$$

The multiplicity of $\left(\prod_{i=1}^{k} \delta(\Delta_i)\right) \otimes \sigma$ *in* $s_{GL}(\delta(\Delta_1, \dots, \Delta_k, \sigma)_\tau)$ *is one.*
(iv) $\delta(\Delta_1, \dots, \Delta_k, \sigma)_\tau$ *is a subquotient of* $\delta(\Delta_1) \times \cdots \times \delta(\Delta_k) \rtimes \sigma$.
(v) If π *is a subquotient of* $\delta(\Delta_1) \times \cdots \times \delta(\Delta_k) \rtimes \sigma$ *such that* $\delta(\Delta_1) \times \cdots \times \delta(\Delta_k) \otimes \sigma$ *is a subquotient* $s_{GL}(\pi)$, *then* π *is isomorphic to some* $\delta(\Delta_1, \dots, \Delta_k, \sigma)_\tau$. *(vi) If* $\Delta'_1, \dots, \Delta'_{k'}$ *and* σ' *is some system which satisfies (a) - (c), and* τ' *is an irreducible subrepresentation of* $\left(\prod_{i=1}^{k'} \delta(\Delta'_i \cap \tilde{\Delta}'_i)\right) \rtimes \sigma'$, *then* $\delta(\Delta_1, \dots, \Delta_k, \sigma)_\tau \cong \delta(\Delta'_1, \dots, \Delta'_{k'}, \sigma')_{\tau'}$ *implies* $\{\Delta_1, \dots, \Delta_k\} = \{\Delta'_1, \dots, \Delta'_{k'}\}$ *and* $\sigma \cong \sigma'$.

Later, we shall prove that in (vi) we must also have $\tau \cong \tau'$.

Remark 4.5. (i) Let $k = 1$. If $\Delta_i \cap \tilde{\Delta}_i = \emptyset$, then Proposition 3.1 implies that $\delta(\Delta_1, \sigma)_\tau$ defined in the above proposition is just the square integrable representation $\delta(\Delta_1, \sigma) = \delta(\Delta_1, \sigma)_\sigma$ from Proposition 3.1. Suppose $\Delta_i \cap \tilde{\Delta}_i \neq \emptyset$. Now, Theorem 3.3 implies that the representation $\delta(\Delta_1, \sigma)_\tau$ defined above is the square integrable representation from Theorem 3.3.
(ii) Assume that $\mathrm{char}\,(F) = 0$. Then, the conditions (a), (b) and (c) on $\Delta_1, \dots, \Delta_k \in \mathcal{S}(\mathcal{C})$ and σ in the last proposition are equivalent to the following conditions
(α) If $1 \leq i \leq k$ and $\Delta_i \cap \tilde{\Delta}_i \neq \emptyset$, then $\delta(\Delta_i \cap \tilde{\Delta}_i) \rtimes \sigma$ reduces.
(β) If $1 \leq i \leq k$ and $\Delta_i \cap \tilde{\Delta}_i = \emptyset$, then $\nu^{-n_i}\rho_i \rtimes \sigma$ reduces.
(γ) $e\,(\delta(\Delta_i)) > 0$ for $i = 1, \dots, k$.
(δ) If $\Delta_i \cap \Delta_j \neq \emptyset$ for some $1 \leq i \neq j \leq k$, then

$$\Delta_i \cup \tilde{\Delta}_i \subsetneqq \Delta_j \cap \tilde{\Delta}_j \text{ or } \Delta_j \cup \tilde{\Delta}_j \subsetneqq \Delta_i \cap \tilde{\Delta}_i.$$

Proof. Assume that $\Delta_1, \dots, \Delta_k$ and σ satisfy conditions (a) - (c) in the proposition.

The proof of (i) is similar to the proof of (ii) and (iii) of Proposition 4.1. We shall modify that proof to the present situation. Write $\beta = \delta(\Delta_1) \times \cdots \times \delta(\Delta_k) \otimes \sigma$ (condition (c) provides that β is irreducible) and

$$M^*_{GL}(\delta([\nu^{-n_i}\rho_i,\nu^{m_i}\rho_i])) = \\ = \sum_{j=-n_i}^{m_i+1} \delta([\nu^{-j+1}\rho_i, \nu^{n_i}\rho_i]) \times \delta([\nu^{j}\rho_i, \nu^{m_i}\rho_i]) \tag{4.4}$$

(the sum is over $j \in -n_i + \mathbb{Z}$ such that $-n_i \leq j \leq m_i + 1$). Now, as before,

$$s_{GL}(\delta(\Delta_1)\times \cdots \times \delta(\Delta_k) \rtimes \sigma) = \\ = M^*_{GL}(\delta(\Delta_1)) \times \cdots \times M^*_{GL}(\delta(\Delta_k)) \otimes \sigma. \tag{4.5}$$

For $j = -n_i$, the term in the sum (4.4) is $\delta(\Delta_i)$. If $n_i < 0$, then this is the only term in the sum (4.4) where $\delta(\Delta_i)$ can appear as a subquotient (all other terms have support different from the support of $\delta(\Delta_i)$). If $n_i \geq 0$, then $-n_i < n_i + 1$, and the term for $j = n_i + 1$ in the sum is $\delta([\nu^{-n_i}\rho_i, \nu^{n_i}\rho_i]) \times \delta([\nu^{n_i+1}\rho_i, \nu^{m_i}\rho_i])$, which has $\delta(\Delta_i)$ for a subquotient (the multiplicity is one). These are the only two terms in the sum where $\delta(\Delta_i)$ can appear as a subquotient (again, all other terms have support different from the support of $\delta(\Delta_i)$). Multiplying the above $\delta(\Delta_i)$'s in (4.5), we get that multiplicity of β in $s_{GL}(\delta(\Delta_1) \times \cdots \times \delta(\Delta_k) \rtimes \sigma)$ is at least 2^l.

If $n_i \geq 0$, write

$$M^*_{GL}(\delta([\nu^{n_i+1}\rho_i, \nu^{m_i}\rho_i])) = \\ = \sum_{j_i=n_i+1}^{m_i+1} \delta([\nu^{-j_i+1}\rho_i, \nu^{-n_i-1}\rho_i]) \times \delta([\nu^{j_i}\rho_i, \nu^{m_i}\rho_i]). \tag{4.6}$$

For $n_i < 0$, put

$$M^*_{GL}(\delta([\nu^{\alpha_i}\rho_i, \nu^{m_i}\rho_i])) = \\ = \sum_{j_i=\alpha_i}^{m_i+1} \delta([\nu^{-j_i+1}\rho_i, \nu^{-\alpha_i}\rho_i]) \times \delta([\nu^{j_i}\rho_i, \nu^{m_i}\rho_i]). \tag{4.7}$$

Then,

$$\text{s.s.}\left(s_{GL}\left(\left(\prod_{i=1}^{k}(\delta(\Delta_i\backslash\tilde{\Delta}_i) \times \delta(\Delta_i \cap \tilde{\Delta}_i))\right) \rtimes \sigma\right)\right) \\ = \left(\prod_{i=1}^{k} M^*_{GL}(\Delta_i\backslash\tilde{\Delta}_i)\right) \times \left(\prod_{i=1}^{k} M^*_{GL}(\Delta_i \cap \tilde{\Delta}_i)\right) \otimes \sigma \tag{4.8}$$

(take $M^*_{GL}(\Delta_i \cap \tilde{\Delta}_i)$ as it is defined in (4.2) if $n_i \geq 0$, and $M^*_{GL}(1) = 1$). In the first product on the right hand side of (4.8), one takes terms in (4.6) corresponding to $j_i = n_i + 1$ if $n_i \geq 0$ and $j_i = \alpha_i$ if $n_i < 0$, and one takes $\left(\prod_{i=1}^k \delta(\Delta_i \cap \tilde{\Delta}_i)\right) \otimes \sigma$ from $\prod_{i=1}^k M^*_{GL}(\Delta_i \cap \tilde{\Delta}_i)$ (more precisely, one takes terms in (4.2) corresponding to $k_i = -n_i$ or $n_i + 1$ if $n_i \geq 0$). In this way, one gets 2^l times $\left(\prod_{i=1}^k \left(\delta(\Delta_i \backslash \tilde{\Delta}_i) \times \delta(\Delta_i \cap \tilde{\Delta}_i)\right)\right) \otimes \sigma$ (in the Grothendieck group). The last representation contains $\beta = \left(\prod_{i=1}^k \delta(\Delta_i)\right) \otimes \sigma$ as a subquotient, and the multiplicity is one.

Now, we shall show that β can appear only in this way. First introduce

$$\xi(n_i) = \begin{cases} -n_i - 1 & \text{if } n_i \geq 0 \\ n_i = -\alpha_i & \text{if } n_i < 0 \end{cases}$$

Suppose that β is a subquotient of some

$$\gamma = \left(\prod_{i=1}^k \delta([\nu^{-j_i+1}\rho_i, \nu^{\xi(n_i)}\rho_i]) \times \delta([\nu^{j_i}\rho_i, \nu^{m_i}\rho_i])\right) \times \left(\prod_{1 \leq i \leq k, n_i \geq 0} \delta([\nu^{-k_i+1}\rho_i, \nu^{n_i}\rho_i]) \times \delta([\nu^{k_i}\rho_i, \nu^{n_i}\rho_i])\right) \otimes \sigma,$$

where $n_i + 1 \leq j_i \leq m_i + 1$ if $n_i \geq 0$ and $\alpha_i \leq j_i \leq m_i + 1$ if $n_i < 0$.

Take i_1 such that $\Delta_{i_1} \not\subseteq \Delta_i$ for $i \in \{1, \dots, k\}, i \neq i_1$ (this choice is possible because of (c) and (a)). Suppose $n_i \geq 0$. Recall that $\nu^{m_{i_1}}\rho_{i_1}$ is in $\text{supp}_{GL}(\beta)$, and $\nu^{-n_{i_1}-1}\rho_{i_1}$ is not in the GL-support (for this use (c)). This implies $j_{i_1} = n_{i_1} + 1$. Using that $\nu^{-n_{i_1}}\rho_{i_1}$ is in the GL-support of β, we get that $k_{i_1} = -n_{i_1}$ or $n_{i_1} + 1$ if $n_{i_1} \geq 0$.

Suppose $n_{i_1} < 0$. Again directly follows $j_{i_1} = \alpha_{i_1}$ for the same reasons as above.

Further, choose $i_2 \in \{1, \dots, k\}, i_2 \neq i_1$, such that $\Delta_{i_2} \not\subseteq \Delta_i$ for $i \in \{1, \dots, k\} \backslash \{i_1, i_2\}$ and repeat the above argument considering supports (more precisely, $\text{supp}\beta - \Delta_{i_1}$. One shall get $j_{i_2} = n_{i_2} + 1$ if $n_{i_2} \geq 0$ and α_{i_2} if $n_{i_2} < 0$, and further $k_{i_2} = -n_{i_2}$ or $n_{i_2} + 1$ if $n_{i_2} \geq 0$.

Continuing in this way, we get we can get β as a subquotient only in the way that we have already described above. Therefore, the multiplicity is 2^l. Using (4.9), we get a complete proof of (i).

Note that $\text{s.s.}\left(s_{GL}\left(\left(\prod_{i=1}^k \delta(\Delta_i \backslash \tilde{\Delta}_i)\right) \rtimes \tau\right)\right) = \left(\prod_{i=1}^k M^*_{GL}(\Delta_i \backslash \tilde{\Delta}_i)\right) \times s_{GL}(\tau)$ (here $\times$ in the right hand side multiplies $\prod_{i=1}^k M^*_{GL}(\Delta_i \backslash \tilde{\Delta}_i)$ with the terms on the left hand side of $\otimes$, which show up in $s_{GL}(\tau)$; more precisely, of $\text{s.s.}\,(s_{GL}(\tau))$). In the product on the right hand side of this formula, take the term in (4.6) corresponding to $j_i = n_i + 1$ if $n_i \geq 0$ and $j_i = \alpha_i$

if $n_i < 0$, and take $\left(\prod_{i=1}^{k} \delta(\Delta_i \cap \tilde{\Delta}_i)\right) \otimes \sigma$ from $s_{GL}(\tau)$ (see (iii) of Proposition 8.1). In this way, one gets $\left(\prod_{i=1}^{k} \left(\delta(\Delta_i \backslash \tilde{\Delta}_i) \times \delta(\Delta_i \cap \tilde{\Delta}_i)\right)\right) \otimes \sigma$ in s.s.$\big(s_{GL}\big(\big(\prod_{i=1}^{k} \delta(\Delta_i \backslash \tilde{\Delta}_i)\big) \rtimes \tau\big)\big)$, which contains $\beta = \left(\prod_{i=1}^{k} \delta(\Delta_i)\right) \otimes \sigma$ as a subquotient. Therefore, the multiplicity is ≥ 1. Now (i) and (i) of Proposition 3.1 (and (4.9) imply that the multiplicity is exactly one. This completes the proof of (ii).

A direct consequence of (ii) is (iii).

Write $(\prod_{i=1}^{k} \delta(\Delta_i \cap \tilde{\Delta}_i)) \rtimes \sigma = \oplus_{i=1}^{2^l} \tau_i$, where τ_i are irreducible. Then, we have the following relations in the Grothendieck group:

$$(4.9) \qquad \left(\prod_{i=1}^{k} \delta(\Delta_i \backslash \tilde{\Delta}_i)\right) \rtimes \left(\oplus_{j=1}^{2^l} \tau_j\right)$$
$$= \left(\prod_{i=1}^{k} \delta(\Delta_i \backslash \tilde{\Delta}_i)\right) \times \left(\prod_{i=1}^{k} \delta(\Delta_i \cap \tilde{\Delta}_i)\right) \rtimes \sigma$$
$$= \left(\prod_{i=1}^{k} \delta(\Delta_i \backslash \tilde{\Delta}_i) \times \delta(\Delta_i \cap \tilde{\Delta}_i)\right) \rtimes \sigma \geq \left(\prod_{i=1}^{k} \delta(\Delta_i)\right) \rtimes \sigma$$

Since the multiplicity of $\left(\prod_{i=1}^{k} \delta(\Delta_i)\right) \otimes \sigma$ in $s_{GL}\left(\left(\prod_{i=1}^{k} \delta(\Delta_i \backslash \tilde{\Delta}_i)\right) \rtimes \left(\oplus_{j=1}^{2^l} \tau_j\right)\right)$ and in $s_{GL}\left(\left(\prod_{i=1}^{k} \delta(\Delta_i)\right) \rtimes \sigma\right)$ is 2^l by (i) and (ii), and $\left(\prod_{i=1}^{k} \delta(\Delta_i)\right) \rtimes \sigma \leq \left(\prod_{i=1}^{k} \delta(\Delta_i \backslash \tilde{\Delta}_i)\right) \rtimes \left(\oplus_{j=1}^{2^l} \tau_j\right)$, (i), (ii), (iii) and Proposition 4.1 imply (iv). Similar argumentation gives (v).

We get (vi) using the fact that if two representations, parabolically induced from irreducible cuspidal representations ρ' and ρ'', have an irreducible subquotient in common, then ρ' and ρ'' must be associate (see [C], Theorem 6.3.6 and Corollary 6.3.7).

The main aim of the rest of this section is to prove that representations $\delta(\Delta_1, \ldots, \Delta_k, \sigma)_\tau$ introduced in the last proposition are square integrable. Along the way, we shall get a number of useful and interesting facts about these representations. We shall first prove three lemmas.

LEMMA 4.6. *Fix an irreducible cuspidal representation σ of S_q. Let $\rho \in \mathcal{C}$ be selfdual. Assume that (ρ, σ) has reducibility at $\alpha \in (1/2)\,\mathbb{Z}$, $\alpha \geq 0$. Let $n_i, m_i \in (1/2)\,\mathbb{Z}$, $i = 1, \ldots, k$, such that $m_i - n_j \in \mathbb{Z}$ for any $i, j \in \{1, \ldots, k\}$, and*

$$n_1 < m_1 < n_2 < m_2 < n_3 < m_3 < \cdots < m_{k-1} < n_k < m_k.$$

Let $\Delta_i = [\nu^{-n_i}\rho, \nu^{m_i}\rho]$. Suppose $\nu^\alpha \rho \in \Delta_1 \cap \tilde{\Delta}_1$ or $\Delta_1 \cap \tilde{\Delta}_1 = \emptyset$ and $-n_1 = \alpha$. Let τ be an irreducible subrepresentation of $\left(\prod_{i=1}^{k} \delta(\Delta_i \cap \tilde{\Delta}_i)\right) \rtimes \sigma$. Then:

(i) If $k \geq 2$ and $i' \in \{1,\dots,k\}$, then there exists an irreducible subrepresentation τ' of $\left(\prod_{1\leq i\leq k, i\neq i'} \delta(\Delta_i \cap \tilde{\Delta}_i)\right) \rtimes \sigma$ such that

$$\delta(\Delta_1,\dots,\Delta_k,\sigma)_\tau \leq \delta(\Delta_{i'}) \rtimes \delta(\Delta_1,\dots,\Delta_{i'-1},\Delta_{i'+1},\dots,\Delta_k,\sigma)_{\tau'}.$$

(ii) There exists a positive integer c, depending on $\Delta_1,\dots,\Delta_k$ and σ, such that

$$\begin{aligned} s_{GL}\left(\delta(\Delta_1,\dots,\Delta_k,\sigma)_\tau\right) \\ \leq c\left(\prod_{i=1}^{k}\sum_{a_i=-n_i}^{|n_i|} \delta([\nu^{-a_i+1}\rho,\nu^{n_i}\rho]) \times \delta([\nu^{a_i}\rho,\nu^{m_i}\rho])\right) \otimes \sigma. \end{aligned} \tag{4.10}$$

(iii) If π is an irreducible subquotient of $s_{GL}\left(\delta(\Delta_1,\dots,\Delta_k,\sigma)_\tau\right)$ which satisfies $\pi \not\cong \left(\prod_{i=1}^{k}\delta(\Delta_i)\right)\otimes\sigma$, then $supp_{GL}(\pi) \neq supp_{GL}\left(\left(\prod_{i=1}^{k}\delta(\Delta_i)\right)\otimes\sigma\right)$

Proof. For $k = 1$, we know from the previous section that the lemma holds. Therefore, we shall suppose that $k \geq 2$.

If $n_1 < 0$, define $\epsilon(\Delta_1) = 1$. Otherwise take $\epsilon(\Delta_1) = 0$.

First, suppose $n_1 \geq 0$ or $i' > 1$. Proposition 4.1 implies that we can write

$$\left[\prod_{1\leq i\leq k, i\neq i'} \delta(\Delta_i\cap\tilde{\Delta}_i)\right]\rtimes\sigma = \overset{2^{k-1-\epsilon(\Delta_1)}}{\underset{j=1}{\oplus}} \tau_j', \quad \left[\prod_{1\leq i\leq k} \delta(\Delta_i\cap\tilde{\Delta}_i)\right]\rtimes\sigma = \overset{2^{k-\epsilon(\Delta_1)}}{\underset{j=1}{\oplus}} \tau_j$$

$$\left[\prod_{1\leq i\leq k, i\neq i'} \delta(\Delta_i\cap\tilde{\Delta}_i)\right]\rtimes\sigma = \overset{2^{k-1-\epsilon(\Delta_1)}}{\underset{j=1}{\oplus}} \tau_j', \quad \left[\prod_{1\leq i\leq k} \delta(\Delta_i\cap\tilde{\Delta}_i)\right]\rtimes\sigma = \overset{2^{k-\epsilon(\Delta_1)}}{\underset{j=1}{\oplus}} \tau_j$$

where τ_j and τ_j' irreducible. By an argument similar to that in the proof of Proposition 4.4, in the Grothendieck group, we get

$$\left(\prod_{i=1}^{k}\delta(\Delta_i\backslash\tilde{\Delta}_i)\right) \rtimes \left(\overset{2^{k-\epsilon(\Delta_1)}}{\underset{r=1}{\oplus}}\tau_r\right) = \left(\prod_{i=1}^{k}\delta(\Delta_i\backslash\tilde{\Delta}_i)\right) \times \left(\prod_{i=1}^{k}\delta(\Delta_i\cap\tilde{\Delta}_i)\right) \rtimes \sigma$$

$$= \delta(\Delta_{i'}\backslash\tilde{\Delta}_{i'}) \times \delta(\Delta_{i'} \cap \tilde{\Delta}_{i'}) \times \left(\prod_{1\le i\le k, i\ne i'} \delta(\Delta_i\backslash\tilde{\Delta}_i)\right) \times \left(\prod_{1\le i\le k, i\ne i'} \delta(\Delta_i \cap \tilde{\Delta}_i)\right) \rtimes \sigma$$

$$\ge \delta(\Delta_{i'}) \times \left(\prod_{1\le i\le k, i\ne i'} \delta(\Delta_i\backslash\tilde{\Delta}_i)\right) \times v\left(\prod_{1\le i\le k, i\ne i'} \delta(\Delta_i \cap \tilde{\Delta}_i)\right) \rtimes \sigma$$

$$= \delta(\Delta_{i'}) \times \left(\prod_{1\le i\le k, i\ne i'} \delta(\Delta_i\backslash\tilde{\Delta}_i)\right) \rtimes \left(\bigoplus_{j=1}^{2^{k-1-\epsilon(\Delta_1)}} \tau'_j \right)$$

$$\ge \delta(\Delta_{i'}) \times \left(\sum_{j=1}^{2^{k-1-\epsilon(\Delta_1)}} \delta(\Delta_1, \ldots, \Delta_{i'-1}, \Delta_{i'+1}, \ldots, \Delta_k, \sigma)_{\tau'_j} \right)$$

$$= \sum_{j=1}^{2^{k-1-\epsilon(\Delta_1)}} \delta(\Delta_{i'}) \times \delta(\Delta_1, \ldots, \Delta_{i'-1}, \Delta_{i'+1}, \ldots, \Delta_k, \sigma)_{\tau'_j}.$$

From formula (1.4), (iii) of Proposition 4.4 and (4.4), it follows that the multiplicity of $\left(\prod_{i=1}^k \delta(\Delta_i)\right) \otimes \sigma$ in $s_{GL}\left(\delta(\Delta_{i'}) \times \delta(\Delta_1, \ldots, \Delta_{i'-1}, \Delta_{i'+1}, \Delta_k, \sigma)_{\tau'_j}\right)$ is ≥ 2. The above inequalities and (iii) of Proposition 4.4 imply that the multiplicity is 2. From the multiplicities, one concludes that each

$$\delta(\Delta_1, \ldots, \Delta_k, \sigma)_{\tau_r} \le \delta(\Delta_{i'}) \times \delta(\Delta_1, \ldots, \Delta_{i'-1}, \Delta_{i'+1}, \ldots, \Delta_k, \sigma)_{\tau'_j}$$

for some τ'_j. This proves (i) in the case that $n_1 \ge 0$ or $i' > 1$.

Now, suppose $n_1 < 0$ and $i' = 1$. Write

$$\left(\prod_{2\le i\le k} \delta(\Delta_i \cap \tilde{\Delta}_i)\right) \rtimes \sigma = \left(\prod_{1\le i\le k} \delta(\Delta_i \cap \tilde{\Delta}_i)\right) \rtimes \sigma = \bigoplus_{j=1}^{2^{k-1}} \tau_j,$$

where τ_j are irreducible. Then,

$$\left(\prod_{i=1}^{k} \delta(\Delta_i\backslash\tilde{\Delta}_i)\right) \rtimes \left(\bigoplus_{i=1}^{2^{k-1}} \tau_i \right) = \delta(\Delta_1) \times \left(\prod_{i=2}^{k} \delta(\Delta_i\backslash\tilde{\Delta}_i)\right) \rtimes \left(\bigoplus_{j=1}^{2^{k-1}} \tau_j \right)$$

$$\ge \delta(\Delta_1) \times \left(\sum_{j=1}^{2^{k-1}} \delta(\Delta_2, \Delta_3, \ldots, \Delta_k, \sigma)_{\tau_j} \right) = \sum_{j=1}^{2^{k-1}} \delta(\Delta_1) \times \delta(\Delta_2, \Delta_3, \ldots, \Delta_k, \sigma)_{\tau_j}.$$

Using (1.4), (iii) of Proposition 4.4 and (4.4), we get that the multiplicity of $\left(\prod_{i=1}^k \delta(\Delta_i)\right) \otimes \sigma$ in $s_{GL}\left(\delta(\Delta_1) \times \delta(\Delta_2, \Delta_3, \ldots, \Delta_k, \sigma)_{\tau_j}\right)$ is ≥ 1. The above inequalities and (iii) of Proposition 4.4 imply that the multiplicity is 1. This implies (i) in this case ($n_1 < 0$ and $i' = 1$). Thus, the proof of (i) is complete.

Using (i), we shall prove (ii) by induction with respect to k. Let $k \geq 2$ and suppose that (ii) holds for $k' < k$. From (i), we know that

$$\delta(\Delta_1, \ldots, \Delta_k, \sigma)_\tau \leq \delta([\nu^{-n_k}\rho, \nu^{m_k}\rho]) \rtimes \delta(\Delta_1, \ldots, \Delta_{k-1}, \sigma)_{\tau'}$$

for some irreducible subquotient τ' of $\left(\prod_{i=i}^{k-1} \delta(\Delta_i \cap \tilde{\Delta}_i)\right) \rtimes \sigma$. The inductive assumption and (1.4) imply

$$\begin{aligned} s_{GL}\left(\delta(\Delta_1, \ldots, \Delta_k, \sigma)_\tau\right) \leq \; & \left(\sum_{j_k=-n_k}^{m_k+1} \delta([\nu^{-j_k+1}\rho, \nu^{n_k}\rho]) \times \delta([\nu^{j_k}\rho, \nu^{m_k}\rho]) \right) \\ & \times c_1 \left(\prod_{i=1}^{k-1} \sum_{a_i=-n_i}^{|n_i|} \delta([\nu^{-a_i+1}\rho, \nu^{n_i}\rho]) \times \delta([\nu^{a_i}\rho, \nu^{m_i}\rho]) \right) \otimes \sigma. \end{aligned} \tag{4.11}$$

Further, $\delta(\Delta_1, \ldots, \Delta_k, \sigma)_\tau \leq \delta(\Delta_1) \rtimes \delta(\Delta_2, \ldots, \Delta_k, \sigma)_{\tau''}$ for some irreducible subquotient τ'' of $\left(\prod_{i=2}^{k} \delta(\Delta_i \cap \tilde{\Delta}_i)\right) \rtimes \sigma$ implies

$$\begin{aligned} & s_{GL}(\delta(\Delta_1, \ldots, \Delta_k, \sigma)_\tau) \\ & \quad \leq \left(\sum_{j_1=-n_1}^{m_1+1} \delta([\nu^{-j_1+1}\rho, \nu^{n_1}\rho]) \times \delta([\nu^{j_1}\rho, \nu^{m_1}\rho]) \right) \\ & \qquad \times c_2 \left(\prod_{i=2}^{k} \sum_{a_i=-n_i}^{|n_i|} \delta([\nu^{-a_i+1}\rho, \nu^{n_i}\rho]) \times \delta([\nu^{a_i}\rho, \nu^{m_i}\rho]) \right) \otimes \sigma. \end{aligned} \tag{4.12}$$

The above formula shows that $\nu^{-m_k}\rho, \nu^{-m_k+1}\rho, \ldots, \nu^{-n_k-1}\rho$ are not in $\mathrm{supp}_{GL}(\pi)$ for any irreducible subquotient π of $s_{GL}(\delta(\Delta_1, \ldots, \Delta_k, \sigma)_\tau)$. Therefore, we can sharpen the estimate (4.11) to the following estimate:

$$\begin{aligned} s_{GL}\left(\delta(\Delta_1, \ldots, \Delta_k, \sigma)_\tau\right) \leq & \left(\sum_{j_k=-n_k}^{n_k+1} \delta([\nu^{-j_k+1}\rho, \nu^{n_k}\rho]) \times \delta([\nu^{j_k}\rho, \nu^{m_k}\rho]) \right) \\ & \times c_1 \left(\prod_{i=1}^{k-1} \sum_{a_i=-n_i}^{|n_i|} \delta([\nu^{-a_i+1}\rho, \nu^{n_i}\rho]) \times \delta([\nu^{a_i}\rho, \nu^{m_i}\rho]) \right) \otimes \sigma. \end{aligned}$$

Note $n_k \geq 0$, since $k \geq 2$. The above formula implies that to prove (ii), it is enough to prove that if $\pi \otimes \sigma$ is a common irreducible subquotient of

$$\begin{aligned} & \delta([\nu^{-n_k}\rho, \nu^{n_k}\rho]) \times \delta([\nu^{n_k+1}\rho, \nu^{m_k}\rho]) \\ & \qquad \times \left(\prod_{i=1}^{k-1} \sum_{a_i=-n_i}^{|n_i|} \delta([\nu^{-a_i+1}\rho, \nu^{n_i}\rho]) \times \delta([\nu^{a_i}\rho, \nu^{m_i}\rho]) \right) \otimes \sigma \end{aligned} \tag{4.13}$$

and the right hand side of (4.12), then $\pi \otimes \sigma$ is an irreducible subquotient of the right hand side of (4.10).

First write (4.13) in a slightly different way:

$$\delta([\nu^{-n_k}\rho, \nu^{n_k}\rho]) \times \delta([\nu^{n_k+1}\rho, \nu^{m_k}\rho]) \times \Bigg(\sum_{a_1=-n_1}^{|n_1|} \sum_{a_2=-n_2}^{n_2} \dots \tag{4.14}$$

$$\sum_{a_{k-1}=-n_{k-1}}^{n_{k-1}} \prod_{i=1}^{k-1} \Bigg(\delta([\nu^{-a_i+1}\rho, \nu^{n_i}\rho]) \times \delta([\nu^{a_i}\rho, \nu^{m_i}\rho]) \Bigg) \Bigg) \otimes \sigma. \tag{4.15}$$

Now, we shall point out some properties of the factors in the line (4.15). Consider segments $\Delta_i' = [\nu^{-a_i+1}\rho, \nu^{n_i}\rho]$, $\Delta_i'' = [\nu^{a_i}\rho, \nu^{m_i}\rho]$ for $i = 1, \dots, k-1$, where $-n_i \leq a_i \leq |n_i|$, and consider all multisets $a = (\Delta_1', \Delta_1'', \Delta_2', \Delta_2'', \dots, \Delta_{k-1}', \Delta_{k-1}'')$ that we get in this way (if some $\Delta_i' = \emptyset$, then we omit $\emptyset$ from the above definition of a). Set $\Delta_k^\dagger = [\nu^{-n_k}\rho, \nu^{n_k}]$, $\Delta_k^{\dagger\dagger} = [\nu^{n_k+1}\rho, \nu^{m_k}\rho]$. Using the conditions on n_i and m_i in the lemma, one checks directly that the following properties hold:

1. $\Delta_i', \Delta_i'' \subseteq \Delta_k^\dagger \subseteq \Delta_k$ for any $1 \leq i \leq k-1$.
2. For any $1 \leq i \leq k-1$, neither Δ_i' nor Δ_i'' is linked with $\Delta_k^\dagger$, $\Delta_k^{\dagger\dagger}$ or Δ_k.
3. Linking $\Delta_k^\dagger$ and $\Delta_k^{\dagger\dagger}$ one gets Δ_k.

Let $\pi \otimes \sigma$ be a common irreducible subquotient of the right hand side of (4.12) and of (4.13) (let π be a subquotient of $\delta(\Delta_k^\dagger) \times \delta(\Delta_k^{\dagger\dagger}) \times \prod_{i=1}^{k-1}(\delta(\Delta_i') \times \delta(\Delta_i''))$ coming from (4.13)). Write $\pi = L(\delta(\Gamma_1), \dots, \delta(\Gamma_t))$ with $\Gamma_i \in \mathcal{S}(\mathcal{C})$. Because $\pi \otimes \sigma$ is a subquotient of (4.13), (1) - (3) directly imply that $(\Gamma_1, \dots, \Gamma_t) = a + (\Delta_k^\dagger, \Delta_k^{\dagger\dagger})$ or $a + (\Delta_k)$ for some multiset a in $\mathcal{S}(\mathcal{C})$. Note that if $L(\delta(\Gamma_1'), \dots, \delta(\Gamma_{t'}')) \otimes \sigma$ is a subquotient of the right hand side of (4.12), and some Δ_i' ends with $\nu^{m_k}\rho$, then Δ_i' contains also $\nu^{n_k}\rho$. Since $\pi \otimes \sigma$ is a subquotient of the right hand side of (4.12), $(\Gamma_1, \dots, \Gamma_t) = a + (\Delta_k)$. Now, (1) implies $\pi = \delta(\Delta_k) \times \pi'$. Since $\delta(\Delta_k) \times \pi'$ is a subquotient of $\delta(\Delta_k^\dagger) \times \delta(\Delta_k^{\dagger\dagger}) \times \prod_{i=1}^{k-1}(\delta(\Delta_i') \times \delta(\Delta_i''))$, (1) - (3) imply that it is a subquotient of $\delta(\Delta_k) \times \prod_{i=1}^{k-1}(\delta(\Delta_i') \times \delta(\Delta_i''))$. All irreducible subquotients of the last representations are $\delta(\Delta_k) \times \pi''$, where π'' runs over all irreducible subquotients of $\prod_{i=1}^{k-1}(\delta(\Delta_i') \times \delta(\Delta_i''))$. This implies that $\pi' = \pi''$ for some π'' as above. Thus, $\pi \otimes \sigma$ is an irreducible subquotient of the right hand side of (4.10). This completes the proof of (ii).

We shall prove (iii) by induction. Suppose that (iii) holds for $k-1$. Let π be an irreducible subquotient of $s_{GL}(\delta(\Delta_1, \dots, \Delta_k, \sigma)_\tau)$ such that $\mathrm{supp}_{GL}(\pi) = \sum_{i=1}^k \Delta_i$ $(= \mathrm{supp}_{GL}\left(\left(\prod_{i=1}^k \delta(\Delta_i)\right) \otimes \sigma\right))$. Since $\nu^{-n_k}\rho$ is in $\mathrm{supp}_{GL}(\pi)$, π can appear as a subquotient of a product γ on the right hand side of (4.10) only if $a_k = -n_k$. Since Δ_k contains each segment which shows up in the right hand side of (4.10), $\pi = \delta(\Delta_k) \times \pi'$, where π' is an irreducible representation of some $GL(p', F) \times S_q$. Since

$$\begin{aligned} &\text{s.s.}(s_{GL}(\delta(\Delta_1, \ldots, \Delta_k, \sigma)_\tau)) \\ &\qquad \leq \left(\sum_{j_k=-n_k}^{n_k+1} \delta([\nu^{-j_k+1}\rho, \nu^{n_k}\rho]) \times \delta([\nu^{j_k}\rho, \nu^{m_k}\rho]) \right) \\ &\qquad\quad \times s_{GL}\left(\delta(\Delta_1, \ldots, \Delta_{k-1}, \sigma)_{\tau'}\right) \end{aligned} \tag{4.16}$$

for some τ',

$$\begin{aligned} \delta(\Delta_k) \times \pi' \leq \delta([\nu^{-j_k+1}\rho, \nu^{n_k}\rho]) \\ \times \delta([\nu^{j_k}\rho, \nu^{m_k}\rho]) \times s_{GL}\left(\delta(\Delta_1, \ldots, \Delta_{k-1}, \sigma)_{\tau'}\right). \end{aligned} \tag{4.17}$$

Since $\nu^{-n_k}\rho$ is in the support of the left hand side of (4.17), $j_k = -n_k$ or $j_k = n_k + 1$. Since neither $\Delta_k \cap \tilde{\Delta}_k$ nor $\Delta_k \backslash \tilde{\Delta}_k$ is linked with any other segment which shows up in the upper bound for $s_{GL}\left(\delta(\Delta_1, \ldots, \Delta_{k-1}, \sigma)_{\tau'}\right)$, one directly gets $\delta(\Delta_k) \times \pi' \leq \delta(\Delta_k) \times s_{GL}\left(\delta(\Delta_1, \ldots, \Delta_{k-1}, \sigma)_{\tau'}\right)$. This implies $\pi' \leq s_{GL}\left(\delta(\Delta_1, \ldots, \Delta_{k-1}, \sigma)_{\tau'}\right)$. Since $\text{supp}_{GL}(\pi) = \Delta_k + \text{supp}_{GL}(\pi') = \sum_{i=1}^{k} \Delta_i$, then $\text{supp}_{GL}(\pi') = \sum_{i=1}^{k-1} \Delta_i$. Now, the inductive assumption implies $\pi' = \left(\prod_{i=1}^{k-1} \delta(\Delta_i)\right) \otimes \sigma$. This finishes the proof of (iii). Therefore, the proof of lemma is complete.

Let $\rho', \rho'' \in \mathcal{C}$. We shall say that they are are strongly $\mathbb{Z}$-disconnected if there does not exist $\Delta \in \mathcal{S}(\mathcal{C})$ such that $\rho', \rho'' \in \Delta$ or $\rho', (\rho'')\tilde{} \in \Delta$. For $\Gamma_1, \Gamma_2 \in \mathcal{S}(\mathcal{C})$, we say that they are strongly $\mathbb{Z}$-disconnected if any $\rho_1 \in \Gamma_1$ is strongly $\mathbb{Z}$-disconnected with any $\rho_2 \in \Gamma_2$. The following lemma is related to [Jn2] (it is a very special case of a general ideas studied there).

Lemma 4.7. *Let $\rho'_1, \ldots, \rho'_{k'}$, $\rho''_1, \ldots, \rho''_{k''} \in \mathcal{C}$ and let σ be an irreducible cuspidal representation of S_q. Suppose:*
(a) Any ρ'_i is strongly $\mathbb{Z}$-disconnected with any ρ''_j.
(b) π' is an irreducible subquotient of $\rho'_1 \times \cdots \times \rho'_{k'} \rtimes \sigma$ and π'' is an irreducible subquotient of $\rho''_1 \times \cdots \times \rho''_{k''} \rtimes \sigma$. Write s.s.$(s_{GL}(\pi')) = \gamma' \otimes \sigma$ and s.s.$(s_{GL}(\pi'')) = \gamma'' \otimes \sigma$.
(c) π is a representation which satisfies $\pi \leq \rho'_1 \times \cdots \times \rho'_{k'} \rtimes \pi''$ and $\pi \leq \rho''_1 \times \cdots \times \rho''_{k''} \rtimes \pi'$. Then, there exists a positive integer d such that $s_{GL}(\pi) \leq d(\gamma' \times \gamma'' \otimes \sigma)$.

Proof. From (1.4) and (b), it follows that

$$
\begin{aligned}
s_{GL}(\pi) &\leq \left(\prod_{i=1}^{k'} (\rho_i' + (\rho_i')\tilde{}) \right) \times \gamma'' \otimes \sigma, \\
s_{GL}(\pi) &\leq \left(\prod_{j=1}^{k''} (\rho_j'' + (\rho_j'')\tilde{}) \right) \times \gamma' \otimes \sigma.
\end{aligned} \tag{4.18}
$$

Let β be an irreducible subquotient of $s_{GL}(\pi)$. Then, (4.18) and (a) imply $\beta = \alpha' \times \phi'' \otimes \sigma = \alpha'' \times \phi' \otimes \sigma$, where α' is an irreducible subquotient of γ', α'' an irreducible subquotient of γ'', ϕ' an irreducible subquotient of $\prod_{i=1}^{k'} (\rho_i' + (\rho_i')\tilde{})$ and ϕ'' an irreducible subquotient of $\prod_{j=1}^{k''} (\rho_j'' + (\rho_j'')\tilde{})$. Obviously, $\mathrm{supp}(\phi')$ consists only of elements from $\{\rho_i', \tilde{\rho}_i'; 1 \leq i \leq k'\}$, while $\mathrm{supp}(\phi'')$ consists only of elements from $\{\rho_j'', \tilde{\rho}_j''; 1 \leq j \leq k''\}$. Also, $\mathrm{supp}(\alpha')$ consists only of elements from $\{\rho_i', \tilde{\rho}_i'; 1 \leq i \leq k'\}$ and $\mathrm{supp}(\alpha'')$ consists only of elements from $\{\rho_j'', \tilde{\rho}_j''; 1 \leq j \leq k''\}$. Further, $\mathrm{supp}_{GL}(\beta) = \mathrm{supp}(\alpha') + \mathrm{supp}(\phi'') = \mathrm{supp}(\alpha'') + \mathrm{supp}(\phi')$. Now, (a) implies $\mathrm{supp}(\alpha') = \mathrm{supp}(\phi')$ and $\mathrm{supp}(\phi'') = \mathrm{supp}(\alpha'')$.

Now, we use the following fact from the representation theory of general linear groups. Let $X_1, X_2 \subseteq \mathcal{C}$. Suppose that any element of X_1 is strongly $\mathbb{Z}$-disconnected with any element of X_2 (a weaker condition would be enough for what follows). Let $\lambda_1, \lambda_1', \lambda_2, \lambda_2'$ be irreducible representations of general linear groups such that $\mathrm{supp}(\lambda_1)$, $\mathrm{supp}(\lambda_1')$ consist only of elements from X_1 and $\mathrm{supp}(\lambda_2)$, $\mathrm{supp}(\lambda_2')$ consist only of elements from X_2. Then, $\lambda_1 \times \lambda_2 \cong \lambda_1' \times \lambda_2'$ implies $\lambda_1 \cong \lambda_1'$ and $\lambda_2 \cong \lambda_2'$ (this follows easily from Proposition 14 and Remark 5.3 of [Ro], applying Theorem 2.3 and Corollary 3.9 of [A], or the fifth section of [ScSt]; see also [Z2]). The above fact implies $\alpha' \cong \phi'$ and $\phi'' \cong \alpha''$. Therefore, $\beta \cong \alpha' \times \alpha'' \otimes \sigma$. This implies $\beta \leq (\gamma_1 \times \gamma_2) \otimes \sigma$. From this, the claim of the lemma follows.

Lemma 4.8. *Suppose that $\Delta_1, \ldots, \Delta_k, \sigma$ and τ satisfy the assumptions of Proposition 4.4. Then*
(i) Let $1 \leq i' \leq k$. There exists an irreducible subrepresentation τ' of $\left(\prod_{i=1}^{i'} \delta(\Delta_i \cap \tilde{\Delta}_i)\right) \rtimes \sigma$ such that

$$
\delta(\Delta_1, \Delta_2, \ldots, \Delta_k, \sigma)_\tau \leq \left(\prod_{i=1}^{i'} \delta(\Delta_i) \right) \rtimes \delta(\Delta_{i'+1}, \Delta_{i'+2}, \ldots, \Delta_k, \sigma)_{\tau'}
$$

(note that now the order of Δ_i's is again arbitrary).

(ii) For some positive integer c, the following holds:

$$s_{GL}\big(\ \delta(\Delta_1,\dots,\Delta_k,\sigma)_\tau\big) \leq c\left(\prod_{i=1}^{k}\left(\sum_{a_i=-n_i}^{|n_i|}\delta([\nu^{-a_i+1}\rho_i,\nu^{n_i}\rho_i])\times\delta([\nu^{a_i}\rho_i,\nu^{m_i}\rho_i])\right)\right)\otimes\sigma.$$

(iii) If π is an irreducible subquotient of $s_{GL}\left(\delta(\Delta_1,\dots,\Delta_k,\sigma)_\tau\right)$ which satisfies $\pi\not\cong\left(\prod_{i=1}^{k}\delta(\Delta_i)\right)\otimes\sigma$, then $supp_{GL}(\pi)\neq\sum_{i=1}^{k}\Delta_i$.

(iv) $\left(\prod_{i=1}^{k}\delta(\Delta_i)\right)\otimes\sigma$ is a direct summand in $s_{GL}\left(\delta(\Delta_1,\dots,\Delta_k,\sigma)_\tau\right)$.

Proof. Let $l_1=\text{card}(\{i;i'+1\leq i\leq n \text{ and } \Delta_i\cap\tilde{\Delta}_i\neq\emptyset\}$, $l_2=\text{card}(\{i;1\leq i\leq i' \text{ and } \Delta_i\cap\tilde{\Delta}_i\neq\emptyset\}$ and $l=\text{card}(\{i;1\leq i\leq n \text{ and } \Delta_i\cap\tilde{\Delta}_i\neq\emptyset\}$. Then, $l_1+l_2=l$. By Proposition 4.1, we can write

$$\left(\prod_{i=i'+1}^{k}\delta(\Delta_i\cap\tilde{\Delta}_i)\right)\rtimes\sigma=\oplus_{i=1}^{2^{l_1}}\tau'_i,\quad \left(\prod_{i=1}^{k}\delta(\Delta_i\cap\tilde{\Delta}_i)\right)\rtimes\sigma=\oplus_{i=1}^{2^{l}}\tau_i.$$

Now, in the Grothendieck group, we have

$$\left(\prod_{i=1}^{k}\delta(\Delta_i\backslash\tilde{\Delta}_i)\right)\times\left(\oplus_{j=1}^{2^l}\tau_j\right)\rtimes\sigma=\left(\prod_{i=1}^{k}\delta(\Delta_i\backslash\tilde{\Delta}_i)\right)\times\left(\prod_{i=1}^{k}\delta(\Delta_i\cap\tilde{\Delta}_i)\right)\rtimes\sigma$$

$$=\left(\prod_{i=1}^{k}\delta(\Delta_i\backslash\tilde{\Delta}_i)\right)\times\left(\prod_{i=1}^{i'}\delta(\Delta_i\cap\tilde{\Delta}_i)\right)\times\left(\prod_{i=i'+1}^{k}\delta(\Delta_i\cap\tilde{\Delta}_i)\right)\rtimes\sigma$$

$$=\left(\prod_{i=1}^{k}\delta(\Delta_i\backslash\tilde{\Delta}_i)\right)\times\left(\prod_{i=1}^{i'}\delta(\Delta_i\cap\tilde{\Delta}_i)\right)\rtimes\left(\oplus_{j=1}^{2^{l_1}}\tau'_j\right)$$

$$\geq\left(\prod_{i=1}^{i'}\delta(\Delta_i)\right)\times\left(\prod_{i=i'+1}^{k}\delta(\Delta_i\backslash\tilde{\Delta}_i)\right)\rtimes\left(\oplus_{j=1}^{2^{l_1}}\tau'_j\right)$$

$$=\left(\prod_{i=1}^{i'}\delta(\Delta_i)\right)\rtimes\left(\oplus_{j=1}^{2^{l_1}}\left(\prod_{i=i'+1}^{k}\delta(\Delta_i\backslash\tilde{\Delta}_i)\right)\rtimes\tau'_j\right)$$

$$\geq\left(\prod_{i=1}^{i'}\delta(\Delta_i)\right)\rtimes\left(\sum_{j=1}^{2^{l_1}}\delta(\Delta_{i'+1},\dots,\Delta_k,\sigma)_{\tau'_j}\right)$$

$$\geq\sum_{j=1}^{2^{l_1}}\left(\prod_{i=1}^{i'}\delta(\Delta_i)\right)\rtimes\delta(\Delta_{i'+1},\dots,\Delta_k,\sigma)_{\tau'_j}.$$

The multiplicity of $\left(\prod_{j=1}^{k} \delta(\Delta_i)\right) \otimes \sigma$ in $s_{GL}\bigg(\left(\prod_{i=1}^{k} \delta(\Delta_i \backslash \tilde{\Delta}_i)\right) \times \left(\oplus_{i=1}^{2^l} \tau_i\right) \rtimes \sigma\bigg)$ is 2^l by (ii) of Proposition 4.4. From (ii) of Proposition 4.4 and (1.4), one easily gets that the multiplicity of $\left(\prod_{j=1}^{k} \delta(\Delta_i)\right) \otimes \sigma$ in $s_{GL}\left(\sum_{j=1}^{2^{l_1}} \left(\prod_{i=1}^{i'} \delta(\Delta_i)\right) \rtimes \delta(\Delta_{i'+1}, \dots, \Delta_k, \sigma)_{\tau'_j}\right)$ is at least $2^{l_1} 2^{l_2} = 2^l$. The above inequalities imply that the multiplicity is exactly 2^l. Now, we can conclude that (i) holds.

We prove (ii) by induction. For $k = 1$, (ii) holds. Let $k > 1$. If $\Delta_i \cap \Delta_j \neq \emptyset$ for all $1 \leq i < j \leq k$, then Lemma 4.6 implies (ii). Therefore, we can suppose that $\Delta_i \cap \Delta_j = \emptyset$ for some $1 \leq i < j \leq n$. This implies that we can make a partition $\{\Delta_1, \dots, \Delta_k\}$ into a union $X \cup Y$ of two non-empty sets of segments in a such a way that any segment in X is strongly $\mathbb{Z}$-disconnected with any segment in Y. Now, using (i) and applying Lemma 4.7, the inductive assumption implies (ii).

From (iii) of Lemma 4.6, using Lemma 4.7, one easily obtains (iii). We can also prove (iii) directly in a similar way as we proved (iii) in Lemma 4.6 (after renumeration, one can assume that $\Delta_k \not\subseteq \Delta_i$ for $i = 1, \dots, k-1$; after this one proceeds in the same way as in Lemma 4.6).

Finally, (ii) of Lemma 4.6 and (iii) imply (iv) (use Theorem 7.3.2 of [C]).

Theorem 4.9. *Let $\Delta_1, \dots, \Delta_k, \sigma$ and τ be as in Proposition 4.4. Then,*
(i) $\delta(\Delta_1, \dots, \Delta_k, \sigma)_\tau$ are square integrable representations.
(ii) If π is a subrepresentation of $\left(\prod_{i=1}^{k} \delta(\Delta_i)\right) \rtimes \sigma$, then $\pi \cong \delta(\Delta_1, \dots, \Delta_k, \sigma)_\tau$ for some τ. Also, each $\delta(\Delta_1, \dots, \Delta_k, \sigma)_\tau$ is isomorphic to a subrepresentation of $\left(\prod_{i=1}^{k} \delta(\Delta_i)\right) \rtimes \sigma$.
(iii) $(\delta(\Delta_1, \dots, \Delta_k, \sigma)_\tau)\tilde{} \cong \delta(\Delta_1, \dots, \Delta_k, \tilde{\sigma})_{\tilde{\tau}}$.

Proof. One gets (i) from (i) of the last lemma using the square integrability criterion (one needs the description of Jacquet modules of the right hand side of the inequality in (i) of the last lemma; to get these Jacquet modules, apply Proposition 9.5 of [Z1] and the Hopf algebra structure on R).

If π is an irreducible subrepresentation of $\left(\prod_{i=1}^{k} \delta(\Delta_i)\right) \rtimes \sigma$, then Frobenius reciprocity implies $\left(\prod_{i=1}^{k} \delta(\Delta_i)\right) \otimes \sigma \leq s_{GL}(\pi)$. Now, (v) of Proposition 4.4 implies that π is isomorphic to some $\delta(\Delta_1, \dots, \Delta_k, \sigma)_\tau$. Further, (iv) of Lemma 4.8 and Frobenius reciprocity imply that each representation $\delta(\Delta_1, \dots, \Delta_k, \sigma)_\tau$ is a subrepresentation of $\left(\prod_{i=1}^{k} \delta(\Delta_i)\right) \rtimes \sigma$. This proves (ii).

We shall use now notation analogous to the notation which we have introduced for general linear groups and groups S_q, with the difference that the lower triangular matrices are fixed to play the role of the standard minimal parabolic subgroup. Then, this new notation will be the same as our standard notation, except that we shall underline this new notation. So, we are going to work with $\underline{\times}, \underline{\rtimes}, \underline{s}_{GL}, \dots$. More details regarding this notation can be found in section 4 of [T2] and section 6 of [T3]. From $\delta(\Delta_1,\dots,\Delta_k,\sigma)_\tau \hookrightarrow \delta(\Delta_1)\times\cdots\times\delta(\Delta_k)\rtimes\sigma$, Propositions 4.1 of [T2] and 6.1 of [T3], we get $\delta(\Delta_1,\dots,\Delta_k,\sigma)_\tau \hookrightarrow \delta(\Delta_1)\tilde{}\,\underline{\times}\cdots\underline{\times}\,\delta(\Delta_k)\tilde{}\,\underline{\rtimes}\sigma$. Therefore, there exists an epimorphism $\underline{s}_{GL}\left(\delta(\Delta_1,\dots,\Delta_k,\sigma)_\tau\right) \twoheadrightarrow \delta(\Delta_1)\tilde{}\,\underline{\times}\cdots\underline{\times}\,\delta(\Delta_k)\tilde{}\otimes\sigma$. Thus, $\delta(\Delta_1)\underline{\times}\dots,\underline{\times}\delta(\Delta_k)\otimes\tilde{\sigma}\hookrightarrow\left(\underline{s}_{GL}\left(\delta(\Delta_1,\dots,\Delta_k,\sigma)_\tau\right)\right)\tilde{}$. Since

$$\left(\underline{s}_{GL}\left(\delta(\Delta_1,\dots,\Delta_k,\sigma)_\tau\right)\right)\tilde{} \cong s_{GL}\left(\left(\delta(\Delta_1,\dots,\Delta_k,\sigma)_\tau\right)\tilde{}\right)$$

by Corollary 4.2.5 of [C], and $\delta(\Delta_1)\underline{\times}\dots,\underline{\times}\delta(\Delta_k) = \delta(\Delta_1)\times\cdots\times\delta(\Delta_k)$, we get that $\delta(\Delta_1)\times\cdots\times\delta(\Delta_k)\otimes\tilde{\sigma}$ is a subrepresentation of $s_{GL}\left(\left(\delta(\Delta_1,\dots,\Delta_k,\sigma)_\tau\right)\tilde{}\right)$.

Recall that $\delta(\Delta_1,\dots,\Delta_k,\sigma)_\tau$ is a subquotient of $\left(\prod_{i=1}^k \delta(\Delta_i\backslash\tilde{\Delta}_i)\right)\rtimes\tau$. Therefore, $\left(\delta(\Delta_1,\dots,\Delta_k,\sigma)_\tau\right)\tilde{}$ is a subquotient of $\left(\prod_{i=1}^k \delta(\Delta_i\backslash\tilde{\Delta}_i)\tilde{}\right)\rtimes\tilde{\tau}$. This last representation has the same Jordan-Hölder factors as $\left(\prod_{i=1}^k \delta(\Delta_i\backslash\tilde{\Delta}_i)\right)\rtimes\tilde{\tau}$ (use (1.3)). From the definition of $\delta(\Delta_1,\dots,\Delta_k,\tilde{\sigma})_{\tilde{\tau}}$ in Proposition 4.4, we get $\delta(\Delta_1,\dots,\Delta_k,\tilde{\sigma})_{\tilde{\tau}} \cong \left(\delta(\Delta_1,\dots,\Delta_k,\sigma)_\tau\right)\tilde{}$.

Now, we shall get some interesting additional information about the representations $\delta(\Delta_1,\dots,\Delta_k,\sigma)_\tau$

LEMMA 4.10. *Suppose that $\Delta_1,\dots,\Delta_k,\sigma$ and τ satisfy the assumptions of Proposition 4.4. Then,*
(i) The multiplicity of $\left(\prod_{i=1}^k \delta(\Delta_i\backslash\tilde{\Delta}_i)\right)\otimes\tau$ in $\mu^*\left(\left(\prod_{i=1}^k \delta(\Delta_i)\right)\rtimes\sigma\right)$ is one.
(ii) The representation $\delta(\Delta_1,\dots,\Delta_k,\sigma)_\tau$ is a subrepresentation of $\left(\prod_{i=1}^k \delta(\Delta_i\backslash\tilde{\Delta}_i)\right)\rtimes\tau'$, for some irreducible subrepresentation τ' of $\left(\prod_{i=1}^k \delta(\Delta_i\cap\tilde{\Delta}_i)\right)\rtimes\sigma$.

PROOF. First, we compute:

$$\begin{aligned} &M^*(\delta([\nu^{-n_i}\rho_i,\nu^{m_i}\rho_i]) = \\ &\quad= (m\otimes 1)\circ(\sim\otimes m^*)\circ s\circ m^*(\delta([\nu^{-n_i}\rho_i,\nu^{m_i}\rho_i])) \\ &\quad= (m\otimes 1)\circ(\sim\otimes m^*) \\ &\qquad\circ\, s\left(\sum_{a_i=-n_i-1}^{m_i}\delta([\nu^{a_i+1}\rho_i,\nu^{m_i}\rho_i])\otimes\delta([\nu^{-n_i}\rho_i,\nu^{a_i}\rho_i])\right) \end{aligned} \tag{4.19}$$

$$= (m \otimes 1) \circ (\sim \otimes m^*) \Big(\sum_{a_i=-n_i-1}^{m_i} \delta([\nu^{-n_i}\rho_i, \nu^{a_i}\rho_i]) \otimes \delta([\nu^{a_i+1}\rho_i, \nu^{m_i}\rho_i]) \Big)$$

$$= \sum_{a_i=-n_i-1}^{m_i} \sum_{b_i=a_i}^{m_i} \delta([\nu^{-a_i}\rho_i, \nu^{n_i}\rho_i]) \times \delta([\nu^{b_i+1}\rho_i, \nu^{m_i}\rho_i]) \otimes \delta([\nu^{a_i+1}\rho_i, \nu^{b_i}\rho_i]).$$

By (1.4), we have

$$\mu^*\Big(\Big(\prod_{i=1}^{k} \delta(\Delta_i)\Big) \rtimes \sigma\Big) = \tag{4.20}$$
$$= \Big(\prod_{i=1}^{k} \Big(\sum_{a_i=-n_i-1}^{m_i} \sum_{b_i=a_i}^{m_i} \delta([\nu^{-a_i}\rho_i, \nu^{n_i}\rho_i]) \times$$
$$\times \delta([\nu^{b_i+1}\rho_i, \nu^{m_i}\rho_i]) \otimes \delta([\nu^{a_i+1}\rho_i, \nu^{b_i}\rho_i])\Big)\Big)$$
$$\rtimes (1 \otimes \sigma).$$

Conditions (a) - (c) in Proposition 4.4 imply that $\beta = \Big(\prod_{i=1}^{k} \delta(\Delta_i \backslash \tilde{\Delta}_i)\Big) \otimes \tau$ is irreducible. Suppose that β is a subquotient of the right hand side of (4.20) Then, β is a subquotient of some

(4.21)

$$\Big(\prod_{i=1}^{k} \delta([\nu^{-a_i}\rho_i, \nu^{n_i}\rho_i]) \times \delta([\nu^{b_i+1}\rho_i, \nu^{m_i}\rho_i]) \otimes \delta([\nu^{a_i+1}\rho_i, \nu^{b_i}\rho_i])\Big) \rtimes (1 \otimes \sigma),$$

where

$$-n_i - 1 \le a_i \le m_i \text{ and } a_i \le b_i \le m_i. \tag{4.22}$$

Write (4.21) as $\gamma \otimes \gamma'$. Since $\beta = \Big(\prod_{i=1}^{k} \delta(\Delta_i \backslash \tilde{\Delta}_i)\Big) \otimes \tau$ is irreducible, and it is a subquotient of $\gamma \otimes \gamma'$, $\prod_{i=1}^{k} \delta(\Delta_i \backslash \tilde{\Delta}_i)$ is a subquotient of γ. In particular,

$$\operatorname{supp}(\gamma) = \operatorname{supp}\Big(\prod_{i=1}^{k} \delta(\Delta_i \backslash \tilde{\Delta}_i)\Big), \tag{4.23}$$

i.e.,

$$\sum_{i=1}^{k} \big([\nu^{-a_i}\rho_i, \nu^{n_i}\rho_i] + [\nu^{b_i+1}\rho_i, \nu^{m_i}\rho_i]\big) = \sum_{i=1}^{k} (\Delta_i \backslash \tilde{\Delta}_i)$$
$$\Big(= \sum_{i=1}^{k} [\nu^{n_i+1}\rho_i, \nu^{m_i}\rho_i]\Big).$$

Choose i_1 such that $\Delta_{i_1} \not\subseteq \Delta_i$ for any $1 \leq i \leq$, $i_1 \neq i$. Suppose $n_{i_1} \geq 0$. Since we have $\nu^{n_{i_1}+1}\rho_i, \nu^{n_{i_1}+2}\rho_i, \ldots, \nu^{m_{i_1}}\rho_i$ in the support of γ, $b_{i_1}+1 \leq n_{i_1}+1$ (i.e., $b_{i_1} \leq n_{i_1}$). Since $\nu^{-n_{i_1}-1}\rho_i$ is not in the support of γ, $b_{i_1}+1 \geq n_{i_1}+1$ (i.e., $b_{i_1} \geq n_{i_1}$) and $-a_{i_1} > n_{i_1}$ (i.e., $-n_{i_1} > a_{i_1}$) . Thus, $a_{i_1} = -n_{i_1}-1$ (which now follows from (4.22)), and $b_{i_1} = n_{i_1}$. For $n_{i_1} < 0$ we must have $b_{i_1}+1 = -n_{i_1}$ and $a_{i_1}+1 = -n_{i_1}-1$.

From (4.23), it follows that

$$\sum_{1\leq i\leq k, i\neq i_1} \left([\nu^{-a_i}\rho_i, \nu^{n_i}\rho_i] + [\nu^{b_i+1}\rho_i, \nu^{m_i}\rho_i]\right) = \sum_{1\leq i\leq k, i\neq i_1} (\Delta_i \backslash \tilde{\Delta}_i).$$

Choose $i_2 \in \{1,\ldots,k\}\backslash\{i_1\}$ such that $\Delta_{i_2} \not\subseteq \Delta_i$ for any $i \in \{1,\ldots,k\}\backslash\{i_1,i_2\}$. Now, in the same way as above, one gets that we must have $a_{i_2} = -n_{i_2}-1$ and $b_{i_2} = n_{i_2}$ in (4.21) if $n_{i_2} \geq 0$, and $b_{i_2}+1 = -n_{i_2}$, $a_{i_2}+1 = -n_{i_2}-1$ if $n_{i_2} < 0$. Continuing this process, we get that $\left(\prod_{i=1}^k \delta(\Delta_i\backslash\tilde{\Delta}_i)\right) \otimes \tau$ must be a subquotient of

$$\left(\prod_{i=1}^k \left(\delta([\nu^{-\xi(n_i)}\rho, \nu^{m_i}\rho]) \otimes \delta([\nu^{-n_i}\rho, \nu^{n_i}\rho])\right)\right) \rtimes (1 \otimes \sigma),$$

where $\xi(n_i)$ are defined in the proof of Proposition 4.4 (we have shown that this is the only term on the right hand side of (4.20) which can have β as a subquotient). From this and Proposition 4.1, we get (i).

Now, we shall list some obvious properties of the segments that we have considered.

1. Among the segments $\Delta_i \cap \tilde{\Delta}_i, \Delta_i\backslash\tilde{\Delta}_i$, $1 \leq i \leq k$, the only pairs of linked segments are $\Delta_i \cap \tilde{\Delta}_i, \Delta_i\backslash\tilde{\Delta}_i$ when $\Delta_i \cap \tilde{\Delta}_i \neq \emptyset$ (this follows easily from the conditions on the segments Δ_i).
2. $\delta(\Delta_i) \hookrightarrow \delta(\Delta_i\backslash\tilde{\Delta}_i) \times \delta(\Delta_i \cap \tilde{\Delta}_i)$ (this follows from Proposition 9.5 of [Z1] and Frobenius reciprocity).

From (1) and (2), we obtain

$$\begin{aligned}\left[\prod_{i=1}^k \delta(\Delta_i)\right] \rtimes \sigma &\hookrightarrow \left[\prod_{i=1}^k \delta(\Delta_i\backslash\tilde{\Delta}_i)\right] \times \left[\prod_{i=1}^k \delta(\Delta_i \cap \tilde{\Delta}_i)\right] \rtimes \sigma \\ &\cong \left[\prod_{i=1}^k \delta(\Delta_i\backslash\tilde{\Delta}_i)\right] \rtimes \left(\oplus_{j=1}^{2^l} \tau_j\right),\end{aligned}$$

where $\left(\prod_{i=1}^k \delta(\Delta_i \cap \tilde{\Delta}_i)\right) \rtimes \sigma = \oplus_{j=1}^{2^l} \tau_j$ is the decomposition into a sum of irreducible representations. Therefore, using (ii) of Theorem 4.9, we get that each $\delta(\Delta_1,\ldots,\Delta_k,\sigma)_\tau$ is isomorphic to a subrepresentation of some $\left(\prod_{i=1}^k \delta(\Delta_i\backslash\tilde{\Delta}_i)\right) \rtimes \tau_j$. This completes the proof of the lemma.

Proposition 4.11. *Suppose that $\Delta_1, \ldots, \Delta_k, \sigma$ and τ are as in Proposition 4.4. Then,*
(i) The multiplicity of $\left(\prod_{i=1}^k \delta(\Delta_i \backslash \tilde{\Delta}_i)\right) \otimes \tau$ *in* $\mu^*\left(\left(\prod_{i=1}^k \delta(\Delta_i \backslash \tilde{\Delta}_i)\right) \rtimes \tau\right)$ *and*

$$\mu^*\left(\left(\prod_{i=1}^k \delta(\Delta_i \backslash \tilde{\Delta}_i)\right) \times \left(\prod_{i=1}^k \delta(\Delta_i \cap \tilde{\Delta}_i)\right) \rtimes \sigma\right)$$

is one.
(ii) $\delta(\Delta_1, \ldots, \Delta_k, \sigma)_\tau$ *is the unique irreducible subrepresentation of* $\left(\prod_{i=1}^k \delta(\Delta_i \backslash \tilde{\Delta}_i)\right) \rtimes \tau$.
(iii) If $\tau' \not\cong \tau''$, *then* $\delta(\Delta_1, \ldots, \Delta_k, \sigma)_{\tau'} \not\cong \delta(\Delta_1, \ldots, \Delta_k, \sigma)_{\tau''}$.

Proof. Let $\beta = \prod_{i=1}^k \delta(\Delta_i \backslash \tilde{\Delta}_i)$ and

$$\gamma = \left(\prod_{i=1}^k \delta(\Delta_i \backslash \tilde{\Delta}_i)\right) \times \left(\prod_{i=1}^k \delta(\Delta_i \cap \tilde{\Delta}_i)\right) \rtimes \sigma$$

(note that β is irreducible). To prove (i), it is enough to prove that the multiplicity of $\beta \otimes \tau$ in $\mu^*(\gamma)$ is one (use Frobenius reciprocity). For $n_j \geq 0$ compute:

$$M^*\left(\delta([\nu^{n_j+1}\rho_j, \nu^{m_j}\rho_j])\right) = (m \otimes 1) \circ (\sim \otimes m^*) \circ s \circ m^*\left(\delta([\nu^{n_j+1}\rho_j, \nu^{m_j}\rho_j])\right)$$

$$= (m \otimes 1) \circ (\sim \otimes m^*) \circ s\left(\sum_{a_j=n_j}^{m_j} \delta([\nu^{a_j+1}\rho_j, \nu^{m_j}\rho_j]) \otimes \delta([\nu^{n_j+1}\rho_j, \nu^{a_j}\rho_j])\right)$$

$$= (m \otimes 1) \circ (\sim \otimes m^*)\left(\sum_{a_j=n_j}^{m_j} \delta([\nu^{n_j+1}\rho_j, \nu^{a_j}\rho_j]) \otimes \delta([\nu^{a_j+1}\rho_j, \nu^{m_j}\rho_j])\right)$$

$$= \sum_{a_j=n_j}^{m_j} \sum_{b_j=a_j}^{m_j} \delta([\nu^{-a_j}\rho_j, \nu^{-n_j-1}\rho_j]) \times \delta([\nu^{b_j+1}\rho_j, \nu^{m_j}\rho_j]) \otimes \delta([\nu^{a_j+1}\rho_j, \nu^{b_j}\rho_j]).$$

For $n_j < 0$ we have

$$(4.24) \quad M^*\left(\delta([\nu^{\alpha_j}\rho_j, \nu^{m_j}\rho_j])\right) = M^*(\delta(\Delta_j \backslash \tilde{\Delta}_j)) = M^*(\delta(\Delta_j))$$

$$= \sum_{a_j=-n_i-1}^{m_j} \sum_{b_j=a_j}^{m_j} \delta([\nu^{-a_j}\rho_j, \nu^{n_j}\rho_j]) \times \delta([\nu^{b_j+1}\rho_j, \nu^{m_j}\rho_j]) \otimes \delta([\nu^{a_j+1}\rho_j, \nu^{b_j}\rho_j]).$$

If $n_i \geq 0$, then

(4.25)

$$M^*\left(\delta([\nu^{-n_i}\rho_i, \nu^{n_i}\rho_i])\right) = (m \otimes 1) \circ (\sim \otimes m^*) \circ s \circ m^*\left(\delta([\nu^{-n_i}\rho_i, \nu^{n_i}\rho_i])\right)$$

$$= (m \otimes 1) \circ (\sim \otimes m^*) \circ s\left(\sum_{a_i'=-n_i-1}^{n_i} \delta([\nu^{a_i'+1}\rho_i, \nu^{n_i}\rho_i]) \otimes \delta([\nu^{-n_i}\rho_i, \nu^{a_i'}\rho_i])\right)$$

$$= (m \otimes 1) \circ (\sim \otimes m^*)\Bigg(\sum_{a_i'=-n_i-1}^{n_i} \delta([\nu^{-n_i}\rho_i, \nu^{a_i'}\rho_i]) \otimes \delta([\nu^{a_i'+1}\rho_i, \nu^{n_i}\rho_i])\Bigg)$$

$$= \sum_{a_i'=-n_i-1}^{n_i} \sum_{b_i'=a_i'}^{n_i} \delta([\nu^{-a_i'}\rho_i, \nu^{n_i}\rho_i]) \times \delta([\nu^{b_i'+1}\rho_i, \nu^{n_i}\rho_i]) \otimes \delta([\nu^{a_i'+1}\rho, \nu^{b_i'}\rho]).$$

For $n_i < 0$, $M^*(\delta(\emptyset)) = 1 \otimes 1$.

First consider the case $n_i \geq 0$ for all $i = 1, \dots, k$:

(4.26)

$$\mu^*(\gamma) = \mu^*\Bigg(\Bigg(\prod_{j=1}^{k}(\delta([\nu^{n_j+1}\rho_j, \nu^{m_j}\rho_j])\Bigg) \times \Bigg(\prod_{i=1}^{k}\delta([\nu^{-n_i}\rho_i, \nu^{n_i}\rho_i])\Bigg) \rtimes \sigma\Bigg)$$

$$= \prod_{j=1}^{k}\prod_{i=1}^{k}\Bigg(\sum_{a_j=n_j}^{m_j}\sum_{b_j=a_j}^{m_j} \delta([\nu^{-a_j}\rho_j, \nu^{-n_j-1}\rho_j]) \times \delta([\nu^{b_j+1}\rho_j, \nu^{m_j}\rho_j])$$

$$\times \sum_{a_i'=-n_i-1}^{n_i}\sum_{b_i'=a_i'}^{n_i} \delta([\nu^{-a_i'}\rho_i, \nu^{n_i}\rho_i]) \times \delta([\nu^{b_i'+1}\rho_i, \nu^{n_i}\rho_i])$$

$$\otimes\ \delta([\nu^{a_j+1}\rho_j, \nu^{b_j}\rho_j]) \times \delta([\nu^{a_i'+1}\rho_i, \nu^{b_i'}\rho_i])\Bigg) \rtimes \sigma$$

(see (5.2) for the computation of $M^*(\Delta_i \cap \tilde{\Delta}_i)$). Suppose that $\beta \otimes \tau$ is a subquotient of some

$$\gamma' \otimes \gamma'' = \prod_{j=1}^{k}\prod_{i=1}^{k}\Bigg(\delta([\nu^{-a_j}\rho_j, \nu^{-n_j-1}\rho_j]) \times \delta([\nu^{b_j+1}\rho_j, \nu^{m_j}\rho_j]) \times$$

$$\times\ \delta([\nu^{-a_i'}\rho_i, \nu^{n_i}\rho_i]) \times \delta([\nu^{b_i'+1}\rho_i, \nu^{n_i}\rho_i]) \otimes$$

$$\otimes\ \delta([\nu^{a_j+1}\rho_j, \nu^{b_j}\rho_j]) \times \delta([\nu^{a_i'+1}\rho_i, \nu^{b_i'}\rho_i])\Bigg) \rtimes \sigma,$$

where a_j, a_j, a_i' and b_i' are as in (4.26). Since $\beta \otimes \tau$ is a subquotient of $\gamma' \otimes \gamma''$, β is a subquotient of γ'. Then, $\mathrm{supp}(\gamma') = \mathrm{supp}(\beta)$.

Choose $i_1 \in \{1, \dots, k\}$ such that $\Delta_{i_1} \not\subseteq \Delta_i$ for $i \in \{1, \dots, k\} \backslash \{i_1\}$. Then, $\mathrm{supp}(\gamma') = \mathrm{supp}(\beta)$ implies $b_{i_1} + 1 \leq n_{i_1} + 1$, since $\nu^{n_{i_1}+1}\rho_{i_1}$ is in $\mathrm{supp}(\beta)$. Since $n_{i_1} \leq a_{i_1} \leq b_{i_1} \leq n_{i_1}$, we get $a_{i_1} = b_{i_1} = n_{i_1}$. Since $\nu^{n_{i_1}}\rho_{i_1}$ is not in $\mathrm{supp}(\beta)$, we have $n_{i_1} \leq -a_{i_1}'$ and $n_{i_1} \leq b_{i_1}' + 1$ (i.e. $a_{i_1}' \leq -n_{i_1} - 1$ and $n_{i_1} \leq b_{i_1}'$). This implies $a_{i_1}' = -n_{i_1} - 1$ and $n_{i_1} = b_{i_1}'$. One continues arguing as in the proof of Proposition 4.1 (consider supports), and gets that for all i, $a_i = b_i = n_i$, $a_i' = -n_i - i$, $b_i' = n_i$. Then, $\gamma' \otimes \gamma'' = \Big(\prod_{i=1}^{k}\delta(\Delta_i \backslash \tilde{\Delta}_i)\Big) \otimes \Big(\prod_{i=1}^{k}\delta(\Delta_i \cap \tilde{\Delta}_i)\Big) \rtimes \sigma$. Proposition 4.1 implies that the multiplicity of $\beta \otimes \tau$ in $\gamma' \otimes \gamma''$ is one. This finishes the proof of (i) in this case. The other case follows in the similar way. Suppose for example $\Delta_{i_1} \not\subseteq \Delta_i$ for $i \in \{1, \dots, k\} \backslash \{i_1\}$ and

$n_{i_1} < 0$. Then we must have $b_{i_1} + 1 \leq -n_{i_1}$ (since $\nu^{-n_{i_1}}\rho_{i_1}$ is in $\mathrm{supp}(\beta)$). Now $-n_{i_1} - 1 \leq a_{i_1} \leq b_{i_1} \leq -n_{i_1} - 1$, which implies $a_{i_1} = b_{i_1} = -n_{i_1} - 1$. There is no need to consider a'_{i_1} and b'_{i_1} since $\Delta_{i_1} \cap \tilde{\Delta}_{i_1} = \emptyset$. We can deal with the case of $n_i < 0$ at any step in the same way, since the supports determine the indexes (and we are subtracting supports). Therefore, we shall get again the multiplicity one.

Lemma 4.10 implies that the representation $\delta(\Delta_1, \ldots, \Delta_k, \sigma)_\tau$ is a subrepresentation of $\left(\prod_{i=1}^k \delta(\Delta_i \backslash \tilde{\Delta}_i)\right) \rtimes \tau_j$, for some irreducible subrepresentation τ_j of $\left(\prod_{i=1}^k \delta(\Delta_i \cap \tilde{\Delta}_i)\right) \rtimes \sigma$. Frobenius reciprocity implies $\left(\prod_{i=1}^k \delta(\Delta_i \backslash \tilde{\Delta}_i)\right) \otimes \tau_j \leq \mu^*\left(\delta(\Delta_1, \ldots, \Delta_k, \sigma)_\tau\right)$. Suppose $\tau \neq \tau_j$. By (iii) of Proposition 4.4, $\delta(\Delta_1, \ldots, \Delta_k, \sigma)_\tau$ is a subquotient of $\left(\prod_{i=1}^k \delta(\Delta_i \backslash \tilde{\Delta}_i)\right) \rtimes \tau$. This implies that the multiplicity of $\left(\prod_{i=1}^k \delta(\Delta_i \backslash \tilde{\Delta}_i)\right) \otimes \tau_j$ in

$$\mu^*\left(\left(\prod_{i=1}^k \delta(\Delta_i \backslash \tilde{\Delta}_i)\right) \times \left(\prod_{i=1}^k \delta(\Delta_i \cap \tilde{\Delta}_i)\right) \rtimes \sigma\right)$$

would be at least two, which contradicts (i)). Therefore, $\delta(\Delta_1, \ldots, \Delta_k, \sigma)_\tau$ is (an irreducible) subrepresentation of $\left(\prod_{i=1}^k \delta(\Delta_i \backslash \tilde{\Delta}_i)\right) \rtimes \tau$. Frobenius reciprocity and (i) imply that this is the only irreducible representation, which is the claim of (ii).

Finally, one gets (iii) from $\left(\prod_{i=1}^k \delta(\Delta_i \backslash \tilde{\Delta}_i)\right) \otimes \tau \leq \mu^*\left(\delta(\Delta_1, \ldots, \Delta_k, \sigma)_\tau\right)$ (which follows from (ii)), and (i) of Lemma 4.10.

Note that we can also get (i) of Lemma 4.10 easily from (i) of the above proposition.

Information about the Langlands parameters of the square integrable representations $\delta(\Delta_1, \ldots, \Delta_k, \sigma)_\tau$ (defined in Proposition 4.4) in the conjectural local Langlands correspondence for the groups S_q can be found in the sixth section of [Mi1] (at least for generic σ). A number of information can be found in other papers, for example [GrP]. One should look at these papers for more details. One can find in [GrP] a discussion of a precise form of the Langlands parameterization conjectured by D. Vogan ([Vo1]), in the case of $SO(2n+1, F)$. There seems to exist naturally defined characters of the corresponding group of components attached to the representations $\delta(\Delta_1, \ldots, \Delta_k, \sigma)_\tau$ (which should exist by the Vogan description).

Remark 4.12. (i) Suppose $\mathrm{char}\,(F) = 0$. G. Muić has proved that each generic irreducible square integrable representation of the groups S_q is equivalent to some of the representations $\delta(\Delta_1, \ldots, \Delta_k, \sigma)_\tau$ introduced in Proposition 4.4 ([Mi2], Proposition 2.1; one can find there precise description).

(ii) We constructed in [T4] square integrable representations which are not equivalent to the representations constructed in Proposition 4.4.

(iii) C. Jantzen has pointed out to us an example of an irreducible square integrable representations which does not belong to the square integrable representations $\delta(\Delta_1, \ldots, \Delta_k, \sigma)$ of Proposition 4.4 (the first of them shows up for $SO(13, F)$). These square integrable representations are members of a wider family of square integrable representations, which can be introduced in a similar way as the representations $\delta(\Delta_1, \ldots, \Delta_k, \sigma)_\tau$ (and whose square integrability can be proved similarly).

5. Appendix

In this appendix we shall present a proof of (i) in Proposition 4.1 which works also if char $(F) > 0$.

Proof (i) of in Proposition 4.1. We shall prove (i) by induction. Suppose (i) holds for k (note that (ii) and Frobenius reciprocity imply that (i) hold for $k = 1$). After renumeration, we can assume that $\Delta_{k+1} \not\subseteq \Delta_i$ for $1 \leq i \leq k$. Now, (ii) implies that for the intertwining algebra,

$$\dim_{\mathbb{C}}\left(\operatorname{End}\left(\left(\prod_{i=1}^{k+1} \delta(\Delta_i)\right) \rtimes \sigma\right)\right) \leq 2^{k+1}. \tag{5.1}$$

Let τ be any irreducible subrepresentation of $\left(\prod_{i=1}^{k} \delta(\Delta_i)\right) \rtimes \sigma$. Using (1.4) and (2.1), we compute:

(5.2)

$$\begin{aligned}
\mu^*(\delta(\Delta_{k+1}) \rtimes \tau) &= M^*\left(\delta(\Delta_{k+1})\right) \rtimes \mu^*(\tau) \\
&= M^*\left(\delta([\nu^{-n_{k+1}}\rho_{k+1}, \nu^{n_{k+1}}\rho_{k+1}])\right) \rtimes \mu^*(\tau) \\
&= (m \otimes 1) \circ (\sim \otimes m^*) \circ s \circ m^*\left(\delta([\nu^{-n_{k+1}}\rho_{k+1}, \nu^{n_{k+1}}\rho_{k+1}])\right) \rtimes \mu^*(\tau) \\
&= (m \otimes 1) \circ (\sim \otimes m^*) \circ s\Bigg(\sum_{a_{k+1}=-n_{k+1}-1}^{n_{k+1}} \delta([\nu^{a_{k+1}+1}\rho_{k+1}, \nu^{n_{k+1}}\rho_{k+1}]) \\
&\qquad\qquad \otimes \delta([\nu^{-n_{k+1}}\rho_{k+1}, \nu^{a_{k+1}}\rho_{k+1}])\Bigg) \rtimes \mu^*(\tau) \\
&= (m \otimes 1) \circ (\sim \otimes m^*)\Bigg(\sum_{a_{k+1}=-n_{k+1}-1}^{n_{k+1}} \delta([\nu^{-n_{k+1}}\rho_{k+1}, \nu^{a_{k+1}}\rho_{k+1}]) \\
&\qquad\qquad \otimes \delta([\nu^{a_{k+1}+1}\rho_{k+1}, \nu^{n_{k+1}}\rho_{k+1}])\Bigg) \rtimes \mu^*(\tau)
\end{aligned}$$

$$= \Big(\sum_{a_{k+1}=-n_{k+1}-1}^{n_{k+1}} \sum_{b_{k+1}=a_{k+1}}^{n_{k+1}} \delta([\nu^{-a_{k+1}}\rho_{k+1}, \nu^{n_{k+1}}\rho_{k+1}])$$

$$\times \delta([\nu^{b_{k+1}+1}\rho_{k+1}, \nu^{n_{k+1}}\rho_{k+1}]) \otimes \delta([\nu^{a_{k+1}+1}\rho_{k+1}, \nu^{b_{k+1}}\rho_{k+1}])\Big) \rtimes \mu^*(\tau)$$

From the above formula, we see that the multiplicity of $\delta(\Delta_{k+1}) \otimes \tau$ in $\mu^*(\delta(\Delta_{k+1}) \rtimes \tau)$ is 2, since $\delta(\Delta_{k+1}) \otimes \tau$ can come only from terms corresponding to indexes $a_{k+1} = -n_{k+1} - 1$, $b_{k+1} = a_{k+1} = -n_{k+1} - 1$, and $a_{k+1} = n_{k+1}$, $b_{k+1} = a_{k+1} = n_{k+1}$ (consider the term $\nu^{-n_{k+1}}\rho_{k+1}$, which cannot come from $\mu^*(\tau)$). Using the same type of analysis as above and the inductive assumption, we see that the multiplicity of $\delta(\Delta_{k+1}) \otimes \tau$ in $\mu^*\left(\left(\prod_{i=1}^{k+1}\delta(\Delta_i\right) \rtimes \sigma\right) = M^*(\delta(\Delta_{k+1})) \rtimes \mu^*\left(\left(\prod_{i=1}^{k}\delta(\Delta_i\right) \rtimes \sigma\right)$ is two.

Note that the inductive assumption and (5.1) imply that for the proof of (i), it is enough to prove that $\delta(\Delta_{k+1}) \rtimes \tau$ reduces. Suppose that it does not reduce. We know $\delta(\Delta_{k+1}) \rtimes \sigma = \Psi_1 \oplus \Psi_2$, for irreducible Ψ_1 and Ψ_2. Therefore, $\delta(\Delta_{k+1}) \rtimes \tau \leq \left(\prod_{i=1}^{k}(\delta(\Delta_i)\right) \rtimes \Psi$ for some $\Psi \in \{\Psi_1, \Psi_2\}$. This implies that the multiplicity of $\delta(\Delta_{k+1}) \otimes \tau$ in $\mu^*\left(\left(\prod_{i=1}^{k}(\delta(\Delta_i)\right) \rtimes \Psi\right)$ is two.

Now, in the same way as (5.2), we get

$$\mu^*\Big(\Big[\prod_{i=1}^{k}\delta(\Delta_i)\Big] \rtimes \Psi_i\Big) = \Big[\prod_{i=1}^{k} M^*(\delta(\Delta_i))\Big] \rtimes \mu^*(\Psi_i) =$$

$$= \Big[\prod_{i=1}^{k}\Big[\sum_{a_i=-n_i-1}^{n_i}\sum_{b_i=a_i}^{n_i}\delta([\nu^{-a_i}\rho_i, \nu^{n_i}\rho_i]) \times \tag{5.3}$$

$$\times \delta([\nu^{b_i+1}\rho_i, \nu^{n_i}\rho_i]) \otimes \delta([\nu^{a_i+1}\rho_i, \nu^{b_i}\rho_i])\Big]\Big] \rtimes \mu^*(\Psi_i).$$

Frobenius reciprocity implies that the multiplicity of $\delta(\Delta_{k+1}) \otimes \sigma$ in $s_{GL}(\Psi_i)$ is one. This and (5.3) imply that the multiplicity of $\delta(\Delta_{k+1}) \otimes \tau$ in (5.3) is ≥ 1, for $i = 1, 2$. Using the first part of the proof, we get that the multiplicity must be one. This implies that the multiplicity of $\delta(\Delta_{k+1}) \otimes \tau$ in $\mu^*(\delta(\Delta_{k+1}) \rtimes \tau)$ is ≤ 1. This contradicts the multiplicity of $\delta(\Delta_{k+1}) \otimes \tau$ in (5.2). Thus, (i) holds. ∎

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