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SPLINE COLLOCATION METHOD FOR SINGULAR PERTURBATION PROBLEM

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Abstract. We introduce piecewise interpolating polynomials as an approximation for the driving terms in the numerical solution of the singularly perturbed differential equation. In this way we obtain the difference scheme which is second order accurate in uniform norm. We verify the convergence rate of presented scheme by numerical experiments.

1. Introduction

There are a lot of difference schemes trying to solve thin-layer phenomena in 1D and 2D. The layers are caused by a dominating convection or advection term, or by singularly perturbed behaviour. The aim is to find better numerical techniques for singular perturbation problems, convection-dominated flows, reaction-diffusion problems, etc. A number of new books appear on that subject ([8, 9, 13]).

Further development of collocation techniques are towards PDEs singularly perturbed in 2D. The study of numerical behaviour of the solution of a nonlinear reaction diffusion equations with one source term (heat equation) arise in plasma physics for the computation. Sufficient conditions of blow up is obtained for the numerical solution as for the exact solution in [14]. The stability and the convergence of the scheme is proved.

ϵ-uniform numerical methods are constructed for a class of semilinear problems by classical finite difference operator on special piecewise-uniform meshes (cf. [6]).

New direction in application of splines in solving singularly perturbed problems is spline-wavelet decomposition of corresponding Sobolev spaces

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based on a special point value vanishing property of basis functions, a construction of fast discrete wavelet transform, using collocation method (cf. [3]).

The former approaches include fitted finite difference methods, finite element methods using exponential elements and method which uses a priori refined or special meshes. Development of these fields goes to different schemes for boundary layer away from boundary layer with rectangular meshes and their combinations in 2D. Also, the fitted finite difference operators on piecewise-uniform, say, Shishkin meshes are used in [7]. Then, the solutions with two sharp layers are considered: boundary layer and spike layer solution (cf. [4]).

Singularly perturbed PDEs are all pervasive in applications of mathematics to problems in the sciences and engineering, say Navier-Stokes equations of fluid flow at high Reynolds number, the drift-diffusion equations of semiconductor device physics, etc. Classical methods are inadequate for solving those problems when ϵ is very small.

The main point in the numerical computations in 2D appear to exibit blow up solutions or what is the most important global solutions of the Euler and Navier-Stokes equations, but extreme numerical instability appear near the blow up time and it is difficult to find reliable conclusions. The above results are the subject of [2]. Thus, the numerical solutions of Navier-Stokes and Euler equations are the challenges of new millennium. It is necessary to make substantial progress in that field.

Thus, the new directions in solving boundary value problems are PDEs in 2D or more in 3D and new techniques more adequate for very small ϵ.

In this paper we revise an old technique spline collocation technique for singular perturbation problem in 1D to obtain optimal difference scheme in uniform norm, with respect to small parameter ϵ. O'Riordan and Stynes in [10, 11, 12, 15] are introduced piecewise constants on each subinterval $[x_{i-1}, x_i]$ as an approximation for the functions $p(x)$ and $f(x)$ in singularly perturbed differential equation in one dimension. In this paper we use quadratic interpolating splines instead of piecewise constants as an approximation for function $f(x)$.

2. Scheme generation

Consider the following two-point boundary value problem

$$Lu \equiv -\epsilon u'' + p(x)u = f(x), \quad u(0) = \alpha_0,\ u(1) = \alpha_1 \tag{1}$$

$0 < \epsilon << 1$, α_0, α_1 are given constants, functions $p(x)$ and $f(x)$ are smooth enough and satisfy $p(x) \geq \beta > 0$ for $x \in [0, 1]$. Then, the solution $u(x)$ has a boundary layer at both end points $[0, 1]$.

We seek solution to (1) on each subinterval (x_{i-1}, x_i) of interval $[0,1]$ as a solution of differential equation

$$(2)\qquad LS_\Delta(x) \equiv -\epsilon S''_\Delta(x) + \bar{p}_i S_\Delta(x) = \bar{f}_i,\ \ x \in I_i = (x_{i-1}, x_i),$$

$S_\Delta(x_{i-1}) = u_{i-1},\ S_\Delta(x_i) = u_i$. The value u_{i-1} and u_i will be determined from the corresponding difference scheme. The corresponding difference scheme is a consequence of the continuity condition of the first derivative of spline at observed points. This method is in principal of collocation type.

As an approximation for the function $p(x)$ we use piecewise constants $p_=(\bar{p}_{i-1}+\bar{p}_i)/2$ where $\bar{p}_i = p(x_i)$. For the function $f(x)$ we introduce quadratic interpolating spline at points x_{i-1}, $x_{i-1/2}$ and x_i ($x_{i-1/2}$ is the mid-point of subinterval $[x_{i-1}, x_i]$):

$$(3)\qquad \begin{aligned} \bar{f}_i &= f_{i-1}\Big(2/h^2(x - x_{i-1/2})(x - x_i)\Big) + f_i\Big(2/h^2(x - x_{i-1/2})(x - x_{i-1})\Big) \\ &\quad + f_{i-1/2}\Big(- 4/h^2(x - x_{i-1})(x - x_i)\Big). \end{aligned}$$

Spline

$$(4)\qquad \begin{aligned} S_\Delta(x) = 1/\sinh\rho_i\Big(&\sinh(\sqrt{p_i/\epsilon}(x - x_{i-1})) \\ &\Big(u_i - 4f_{i-1}/(p_i\rho_i^2) - f_i(1/p_i + 4/(p_i\rho_i^2)) + f_{i-1/2}/p_i(8/\rho_i^2)\Big) \\ &- \sinh(\sqrt{p_i/\epsilon}(x - x_i)) \\ &\Big(u_{i-1} - f_{i-1}(4/(p_i\rho_i^2) + 1/p_i) - f_i(4/(p_i\rho_i^2)) + f_{i-1/2}8/(p_i\rho_i^2)\Big)\Big) \\ &+ 2x^2/(p_ih^2)(f_{i-1} + f_i - 2f_{i-1/2}) \\ &+ 2x/(p_ih^2)\Big(- f_{i-1}(x_{i-1/2} + x_i) - f_i(x_{i-1/2} + x_{i-1}) \\ &+ 2f_{i-1/2}(x_{i-1} + x_i)\Big) + C \end{aligned}$$

where $\rho_i = \sqrt{p_i/\epsilon}h$, and $p_i = (\bar{p}(x_{i-1})+\bar{p}(x_i))/2$ is the solution to (2) with (3). Using continuity condition of (4) at the point x_i we obtain spline collocation method which gives the following spline difference scheme

$$R^h v_i = Q^h f_i,\ \ i = 1(1)n-1,\ \ v_0 = \alpha_0,\ v_n = \alpha_n,$$

where

$$\begin{aligned} R^h v_i &= r_i^- v_{i-1} + r_i^c v_i + r_i^+ v_{i+1}, \\ Q^h f_i &= q_i^- f_{i-1} + q_i^c f_i + q_i^+ f_{i+1} + q_{i1/2}^- f_{i-1/2} + q_{i1/2}^+ f_{i+1/2}. \end{aligned}$$

We can modify these piecewise constants in order to obtain uniform scheme in the following way:

$$
\begin{aligned}
r_i^- &= -\rho_i/\sinh\rho_i,\\
r_i^+ &= -\rho_{i+1}/\sinh\rho_{i+1},\\
r_i^c &= \rho_i\coth\rho_i+\rho_{i+1}\coth\rho_{i+1},\\
q_i^- &= 1/\bar{p}_{i-1}(4/\rho_i\tanh\rho_i/2-\rho_i/\sinh\rho_i-1),\\
q_i^+ &= 1/\bar{p}_{i+1}(4/\rho_{i+1}\tanh\rho_{i+1}/2-\rho_{i+1}/\sinh\rho_{i+1}-1),\\
q_i^c &= 1/\bar{p}_i(4/\rho_i\tanh(\rho_i/2)+\rho_i\coth\rho_i+4/\rho_{i+1}\tanh(\rho_{i+1}/2)\\
&\quad+\rho_{i+1}\coth\rho_{i+1}-6),\\
q_{i1/2}^- &= 1/\bar{p}_{i-1/2}(-8/\rho_i\tanh(\rho_{i/2})+4),\\
q_{i1/2}^+ &= 1/\bar{p}_{i+1/2}(-8/\rho_{i+1}\tanh(\rho_{i+1}/2)+4).
\end{aligned}\tag{5}
$$

In the sequel we will use the notation $O(h^i)$, $i=0(1)6$, to denote a quantity bounded in absolute value by Ch^i.

3. Proof of the uniform convergence

The proof of the uniform convergence will be given for the case $p(x)=$ const, the first.

The scheme (5) has the matrix form

$$A\cdot V=F$$

where $V=[v_1,...,v_{n-1}]^T$ and F is also vector. Then,

$$|u(x_i)-v_i|\le M||A^{-1}||\max_i|\tau_i(u)|,\tag{6}$$

where $\tau_i(u)$ is the truncated error of the scheme.

Lemma 1 ([5]). *Let $u(x)\in C^4([0,1])$, and $p'(0)=p'(1)=0$. Then, the solution to (1) has the form*

$$u(x)=u_0(x)+w_0(x)+g(x)\tag{7}$$

where

$$u_0(x)=p_0\exp(-x\sqrt{p(0)/\epsilon})\tag{8}$$

$$w_0(x)=p_1\exp(-(1-x)\sqrt{p(1)/\epsilon}),\tag{9}$$

p_0, p_1 are bounded functions of ϵ independent of x and

$$|g^{(i)}(x)|\le M(1+\epsilon^{1-i/2}),\quad i=0(1)n,\tag{10}$$

M is a constant independent of ϵ.

Lemma 2. *Let $p(x)=$ const. Then, the truncated error for the boundary layer functions u_o and w_0 equals zero.*

PROOF. From (2) it is obvious that the approximation for the function $f(x)$ affects only the particular solution to (2) and the solution of the homogeneous problem is independent of $f(x)$. The truncated error for the boundary layer function (8) is $\tau_i(u) = R^h u_i - Q^h(L(u_i))$. Denote these parts of difference by τ_r and τ_q. We have $\tau_r = u_{oi}\Big(\exp(\rho_0)r_i^- + r_i^c + \exp(-\rho_0)r_i^+\Big)$ where $u_{0i} = \exp(-\sqrt{p(0)/\epsilon}x_i)$, $\tau_q = u_{0i}\big((\rho_0^2 - \rho_{i-1}^2)\rho_{i-1}^2 \exp(\rho_0)q_i^- + (\rho_0^2 - \rho_i^2)\rho_i^2 q_i^c + (\rho_0^2 - \rho_{i+1}^2)\rho_{i+1}^2 q_i^+ \exp(-\rho_0)\Big)$. When $p = \text{const}$ part $\tau_q = 0$. Using r_i^-, r_i^c, r_i^+ from (5) by simple calculation we obtain $\tau_r = 0$. It implies $\tau_i(u) = 0$. Similarly $\tau_i(w_0) = 0$. ∎

LEMMA 3. *Estimate of the matrix for the scheme (5) is*

$$||A^{-1}|| \le \begin{cases} M\epsilon/h^2 & \text{for } h^2 \le \epsilon \\ M\sqrt{\epsilon}/h & \text{for } h^2 \ge \epsilon, \end{cases} \tag{11}$$

where $||\cdot||$ *denotes the usual max norm.*

PROOF. Follows from

$$\begin{aligned} ||A^{-1}|| &= (r_i^- + r_i^c + r_i^+)^{-1} \\ &= (-\rho_i/\sinh\rho_i + \rho_i\coth\rho_i + \rho_{i+1}\coth\rho_{i+1} - \rho_{i+1}/\sinh\rho_{i+1})^{-1}. \end{aligned}$$

∎

THEOREM 4. *Let* $p(x) = \text{const}$ *and* $f \in C^2([0,1])$. *Let* $\{v_i\}$, $i = 1(1)n-1$ *be the approximation of the solution to (1) obtained by (5). Then, there is a constant* C *independent of* ϵ *and* h *such that*

$$|u(x_i) - v_i| \le Ch^3 \min(h/\epsilon, 1/\sqrt{\epsilon}).$$

PROOF. From Lemma 2 we have $\tau_i(u) = \tau_i(w)$ for $p(x) = \text{const}$. Thus, we have to estimate only function $g(x)$. According to [1]

$$\tau_i(g) = \tau_i^{(0)} g_i + \tau_i^{(1)} g_i^{(1)} + \tau_i^{(2)} g_i^{(2)} + \tau_i^{(3)} g_i^{(3)} + \tau_i^{(4)} g_i^{(4)} + ... + R. \tag{12}$$

For the scheme (5) $\tau_i^{(0)} = \tau_i^{(1)} = \tau_i^{(2)} = \tau_i^{(3)} = 0$. It remains to estimate $\tau_i^{(4)}$. We have

$$\begin{aligned} \tau_i^{(4)} &= h^4/24\Big(r_i^+ + r_i^- - p(q_i^- + q_i^+) - 1/16(q_{i1/2}^- + q_{i1/2}^+)\Big) \\ &\quad + \epsilon h^2/p\Big(q_i^- + q_i^+ + 1/4(q_{i1/2}^+ + q_{i1/2}^-)\Big). \end{aligned}$$

After ordering these expressions we obtain

$$\tau_i^{(4)} = h^4/24\Big(-7/\rho\tanh(\rho/2) + 3/2\Big) + \epsilon h^2\Big(2/\rho\sinh(\rho/2) - \rho/\sinh(\rho/2)\Big),$$

where $\rho = \sqrt{p/\epsilon}h$, for $p = \text{const}$. When $h^2 \le \epsilon$ then $|\tau_i^{(4)}| \le Mh^4\rho^2$ and for $h^2 \ge \epsilon$ we obtain $|\tau_i^{(4)}| \le M(h^4 + \epsilon h^2)$, i.e. $|\tau_i^{(4)}| \le Mh^4$. Using (10), (11) and (6) we obtain

$$|u_i - v_i| \le Mh^3 \min(h/\epsilon, 1/\sqrt{\epsilon}).$$

Similarly we can estimate $\tau_i^{(4)}$ and the remainder terms. They are of the minor order. We have

$$\tau_i^{(5)} = h^5/5!\Big(r_i^+ - r_i^- - \Big(- p_{i-1}q_i^- + p_{i+1}q_i^+ + 1/32(q_{i1/2}^+ - q_{i1/2}^-)\Big)\Big) + \epsilon h^3/6\Big(- q_i^- + q_i^+ + 1/8(-q_{i1/2}^- + q_{i1/2}^+)\Big).$$

For $p(x) = \text{const}$ we have $\tau_i^{(5)} = 0$. Then,

$$\tau_i^{(6)} = h^6/6!\Big(r_i^+ + r_i^- - (p_{i-1}q_i^- + p_{i+1}q_i^+ + 1/64(p_{i-1/2}q_{i1/2}^- + p_{i+1/2}q_{i1/2}^+)\Big) + \epsilon h^4/24\Big(q_i^- + q_i^+ + 1/16(q_{i1/2}^- + q_{i1/2}^+)\Big)$$

and

$$\tau_i^{(6)} = h^6/6!(-31/(4\rho)\tanh(\rho/2) - 17/8) + \epsilon h^4/24(7/\rho\tanh(\rho/2) - 2\rho/\sinh\rho - 3/2).$$

When $h^2 \le \epsilon$ we have $|\tau_i^{(6)}| \le Mh^6\rho^2$ and for $h^2 \ge \epsilon$ we have $|\tau_i^{(6)}| \le h^6$. Thus, $|\tau_i^{(6)} g^{(6)}| \le Mh^6/\epsilon^2\rho^2$ for $h^2 \le \epsilon$, and $|\tau_i^{(6)} g^{(6)}| \le Mh^6/\epsilon^2$ for $h^2 \ge \epsilon$. ∎

THEOREM 5. *Let $p, f \in C^2([0,1])$ and $p'(0) = p'(1) = 0$ and $\{v_i\}$, $i = 1(1)n-1$ be the approximation to (1) obtained by (5). Then, the following estimate holds:*

$$|u(x_i) - v_i| \le Mh\min(h, \sqrt{\epsilon}).$$

PROOF. **Nodal errors due to function** $g(x)$. For the function $g(x)$ we have $\tau_i^{(0)} = \tau_i^{(1)} = 0$ and $\tau_i^{(2)} = \tau_i^{(2)}(\tilde\rho_i) + (\rho_i - \tilde\rho_i + \rho_{i+1} - \tilde\rho_i)(-h^2a - b\epsilon/p_i + C)$, where $\tilde\rho = \sqrt{p/\epsilon}h$ and $\tau_i^{(2)}(\tilde\rho_i) = 0$ (cf. Theorem 4) when $p(x) = \text{const}$. We have $|\rho_i - \tilde\rho + \rho_{i+1} - \tilde\rho| \le Mh^3/\sqrt{\epsilon}$, $\rho_i/\sinh\rho_i = \tilde\rho/\sinh\tilde\rho + (\rho_i - \tilde\rho)b/\rho_i\tanh(\rho_i/2) = 1/\tilde\rho\tanh(\tilde\rho/2) + (\rho_i - \tilde\rho)a$, $\rho_i\coth\rho_i = \tilde\rho\coth\tilde\rho + (\rho_i - \tilde\rho)c$ where $b = -(\sinh\tilde\rho - \rho\cosh\tilde\rho)/\sinh^2\tilde\rho = -\tilde\rho/3 + O(\rho^2)$, $a = -1/(\tilde\rho\coth(\tilde\rho/2))^2\Big(\coth(\tilde\rho/2) - \rho/2(\coth^2(\tilde\rho/2) - 1)\Big) = \tilde\rho/12 + O(\rho^2)$, and $c = \Big(\coth\tilde\rho - \rho(\coth^2\tilde\rho - 1)\Big) = \tilde\rho/3 + O(\rho^2)$. So we obtain $|\tau_i^{(2)}| \le Mh^4/\epsilon$ for $h^2 \le \epsilon$, $|\tau_i^{(2)}| \le Mh^2$ for $h^2 \ge \epsilon$. Using matrix estimate (11) we obtain the nodal errors due to the function $g(x)$.

$$|u(x_i) - v_i| \le Mh\min(h, \sqrt{\epsilon}). \tag{13}$$

Nodal errors due to boundary layer function $u_0(x)$. According to [1]

$$\tau_r = u_{0i}\Big(r_i^- \exp(\rho_0) + r_i^c + r_i^+ \exp(-\rho_0)\Big)$$

and

$$-\tau_q = (p_0/\epsilon)\Big((p_0 - p_{i-1})q_i^- \exp(\rho_0) + (p_0 - p_i)q_i^c + (p_0 - p_{i+1})q_i^+ \exp(-\rho_0) + (p_0 - p_{i-1/2})\exp(\rho_0/2)q_{i1/2}^- + (p_0 - p_{i+1/2})\exp(-\rho_0/2)q_{i1/2}^+\Big).$$

Setting the coefficients of (5) we obtain

$$\tau_r = 2u_{0i}\rho_i/\sinh\rho_i(\cosh\rho_0 - \cosh\rho_i) + O(h^4/\epsilon) = v_i h^2/\epsilon(p_0 - p_i) + O(h^4/\epsilon) \tag{14}$$

and

$$\tau_q = (p_0/\epsilon)\Big((p_0 - p_i)\Big(q_i^- \exp(\rho_0) + q_i^c + q_i^+ \exp(-\rho_0) + q_{i1/2}^- \exp(\rho_0/2) + q_{i1/2}\exp(-\rho_0/2)\Big) + (p_i - p_{i-1})q_i^- \exp(\rho_0) + (p_i - p_{i+1})q_i^+ \exp(-\rho_0) + (p_i - p_{i-1/2})q_{i1/2}^- \exp(\rho_0/2) + (p_i - p_{i+1/2})q_{i1/2}^+ \exp(-\rho_0/2)\Big).$$

Divide τ_q into two parts $\tau_q = \tau_q A + \tau_q B$ where

$$\begin{aligned}\tau_q A = (p_0/\epsilon)(p_0 - p_i)/p_i\Big(&(4/\rho_i \tanh(\rho_i/2) - \rho/\sinh\rho_i - 1)2\cosh\rho_0 \\ &+ (8/\rho_i \tanh(\rho_i/2) + 2\rho_i \coth\rho_i - 6) \\ &+ 2\cosh(\rho_0/2)(-8/\rho_i \tanh(\rho_i/2) + 4)\Big) + N \\ = (p_0/\epsilon)(p_0 - p_i)/p_i\Big(&2A\cosh\rho_0 + B + 2C\cosh(\rho_0/2)\Big).\end{aligned} \tag{15}$$

Here N denotes the part of τ_q which is negligible, i.e. it does not influence to the order of uniform convergence which is $O(h^2)$ what is our intention to prove. We denote by

$$A = 4/\rho_i \tanh(\rho_i/2) - \rho_i/\sinh\rho_i - 1,$$
$$B = 8/\rho_i \tanh(\rho_i/2) + 2\rho_i \coth\rho_i - 6,$$
$$C = -8/\rho_i \tanh(\rho_i/2) + 4.$$

In a case $h^2 \leq \epsilon$ we shall estimate these parts separately. Taylor's expansion gives $A = -1/360\rho_i^4 + O(\rho_i^5)$, $B = \rho_i^2/3 - \rho_i^4/90 + O(\rho_i^6)$, $C = \rho_i^2/3 - \rho_i^4/30 + O(\rho_i^6)$. Putting it in (15) and after ordering the terms we obtain

$$\tau_q A = (p_0 - p_i)h^2/\epsilon + O(h^4/\epsilon). \tag{16}$$

Subtracting (14) and (15) we obtain

$$|\tau_r - \tau_q A| \leq M h^4/\epsilon \ \text{ for } \ h^2 \leq \epsilon. \tag{17}$$

Further, we have

$$\tau_q = u_{0i}\Big((p_i - p_{i-1})q_i^- \exp(\rho_0) + (p_i - p_{i+1})q_i^+ \exp(-\rho_0) + (p_i - p_{i-1/2})\exp(\rho_0/2)q_{i1/2}^- + (p_i - p_{i+1/2})q_{i1/2}^+ \exp(-\rho_0/2)\Big).$$

Divide this into two parts:

$$D = u_{0i}\Big((p_i - p_{i-1})q_i^- \exp(\rho_0) + (p_i - p_{i+1})q_i^+ \exp(-\rho_0)\Big),$$
$$E = u_{0i}\Big((p_i - p_{i-1/2})q_{i1/2}^- \exp(\rho_0/2) + (p_0 - p_{i+1/2})\exp(-\rho_0/2)\Big).$$

As $|q_i^-| \le M\rho_i^4$, $|q_i^+| \le M\rho_i^4$ we obtain $|D| \le Mh^6/\epsilon^2$,

$$E = h/2p'(\xi_1)\exp(\rho_0/2)q_{i1/2}^- - h/2p'(\xi_2)q_{i1/2}^+ \exp(-\rho_0/2)$$

where $x_{i-1} \le \xi_1 \le x_i$, $x_i \le \xi_2 \le x_{i+1}$, and $|E| \le h^2/2p''(\xi)\rho_i^2/3$, where $\xi_1 \le \xi \le \xi_2$. It implies $|E| \le Mh^4/\epsilon$. The final estimate is $|\tau_q B| \le Mh^4/\epsilon$ for $h^2 \le \epsilon$. This estimate, (17) and the matrix estimate give $|u(x_i) - v_i| \le Mh^2$ for $h^2 \le \epsilon$ for boundary layer term.

In the case $h^2 \ge \epsilon$ we have $|q_i^-| \le M$, $|q_i^+| \le M$, $|q_i^c| \le Mh/\sqrt{\epsilon}$, $|q_{i1/2}^-| \le M$, $|q_{i1/2}^+| \le M$. Thus, $|\tau_q| \le (p_0 - p_{i-1})Mu_{0i} + (p - p_i)h/\sqrt{\epsilon}Mu_{0i} + (p_0 - p_{i-1/2})Mu_{0i} + (p_0 - p_{i+1/2})Mu_{0i}$.

Using estimate $|p(0) - p(x_i)| \le Mx_i^2$ under the condition $p'(0) = 0$ for τ_q we obtain

$$|\tau_q| \le M\epsilon \ \text{ for } \ h^2 \ge \epsilon. \tag{18}$$

In the estimate of τ_r we shall find the deviation from $\tau_r(\tilde\rho)$ where $\tilde\rho = \sqrt{p/\epsilon}h$ where $p = \text{const}$. Then,

$$\tau_r = \tau_r - \tau_r(\tilde\rho) = u_{0i}\Big((r(\tilde\rho) - r_i^-)\exp(\rho_0) + (r_i^c(\tilde\rho) - r_i^c) + (r_i^+(\tilde\rho) - r_i^+)\exp(-\rho_0)\Big).$$

As $|r_i^+(\tilde\rho) - r_i^+| \le Mx_{i+1}^2$, $|r_i^-(\tilde\rho) - r_i^-| \le Mx_{i-1}^2$, $|r_i^c(\tilde\rho) - r_i^c| \le Mx_i^2$ we have

$$|\tau_r| \le M\epsilon \ \text{ for } \ h^2 \ge \epsilon. \tag{19}$$

We obtain the same estimate for

$$|\tau_i(w)| \le M\epsilon \ \text{ for } \ h^2 \ge \epsilon. \tag{20}$$

Then, (18), (19), (20) and the matrix estimate (11) give

$$|u(x_i) - v_i| \le Mh\sqrt{\epsilon} \ \text{ for } \ h^2 \ge \epsilon. \tag{21}$$

Thus, (13), (21) and the similar estimate for the function $w(x)$ show that the boundary layer term contributes $O(h\min(h, \sqrt{\epsilon}))$ to the nodal errors. □

4. Numerical experiments

To illustrate computationally the convergence of the presented scheme we consider the following singular two-point boundary value problem

$$-\epsilon u'' + u = -\cos^2 \pi x - 2\epsilon\pi^2 \cos 2\pi x, \quad u(0) = u(1) = 0, \tag{22}$$

with the exact solution

$$u(x) = (\exp(-(1-x)/\sqrt{\epsilon}) + \exp(-x/\sqrt{\epsilon}))/(1 + \exp(-1/\sqrt{\epsilon})) - \cos^2 \pi x$$

taken from [5]. Techniques which determines the rate of uniform convergence is well-known double mesh principle taken from [5]. In the notation of [5] we have the following tables.

Table 1 presents rate of uniform convergence for the scheme (5) as applied to the example (22).

Table 1.

ϵ/k	1	2	3	4	5	p_ϵ
1	4.01	4.00	4.00	3.96	3.54	3.90
2^{-1}	4.01	4.00	4.00	3.98	5.15	4.23
2^{-2}	4.01	4.00	4.00	4.00	3.51	3.91
2^{-3}	4.01	4.00	4.00	3.99	3.80	3.96
2^{-4}	4.01	4.00	4.00	4.00	4.01	4.00
2^{-5}	4.02	4.01	4.00	4.00	3.96	4.00
2^{-6}	4.03	4.01	4.00	4.00	3.99	4.01
2^{-7}	4.05	4.01	4.00	4.00	4.00	4.01
2^{-8}	4.09	4.02	4.01	4.00	4.00	4.02
2^{-9}	4.15	4.02	4.01	4.00	4.00	4.04
2^{-10}	4.24	4.08	4.02	4.01	4.00	4.07
10^{-5}	3.40	3.93	4.38	4.30	4.11	4.02
10^{-6}	3.01	3.23	3.56	4.13	4.40	3.67

Table 2 shows the difference between the exact and the approximate solution to (22) obtained by (5).

Table 2.

ϵ/N	32	64	128	256	512	1024
2^{-6}	0.801E-06	0.502E-07	0.313E-08	0.196E-09	0.122E-10	0.695E-12
2^{-10}	0.831E-06	0.493E-07	0.303E-08	0.189E-09	0.115E-10	0.737E-12
10^{-5}	0.131E-05	0.830E-07	0.401E-08	0.205E-09	0.119E-10	0.729E-12
10^{-6}	0.630E-06	0.646E-07	0.534E-08	0.301E-09	0.144E-10	0.774E-12

In Table 3 is given the rate of the uniform convergence convergence and in Table 4 are given the errors in $E_\infty = \max_i |u(x_i) - v_i|$ norm for the example with non-constant function $p(x)$ for modified scheme (5)

$$-\epsilon u'' + (1+x)^2 u = 4(3x^2 - 3x + 2)(1+x)^2, \;\; u(0) = -1, \; u(1) = 0.$$

TABLE 3.

ϵ/k	1	2	3	4	5	p_ϵ
2^0	1.99	2.00	2.00	2.00	2.00	2.00
2^{-1}	1.99	2.00	2.00	2.00	2.00	2.00
2^{-2}	1.99	2.00	2.00	2.00	2.00	2.00
2^{-3}	1.99	2.00	2.00	2.00	2.00	2.00
2^{-4}	1.95	2.00	2.00	2.00	2.00	1.99
2^{-5}	1.98	1.99	2.00	2.00	2.00	1.99
2^{-6}	1.90	1.99	2.00	2.00	2.60	2.10
2^{-7}	1.94	1.97	1.99	2.00	2.00	1.98
2^{-8}	1.89	1.97	1.98	2.00	2.00	1.97
2^{-9}	1.68	1.92	1.97	2.00	2.02	1.92
2^{-10}	1.16	1.90	1.97	1.99	2.00	1.80

TABLE 4.

ϵ/N	32	64	128	256	512
1	0.400E-04	0.999E-05	0.250E-05	0.624E-06	0.156E-06
2^{-1}	0.570E-04	0.143E-04	0.357E-05	0.893E-06	0.223E-06
2^{-2}	0.718E-04	0.180E-04	0.451E-05	0.113E-05	0.281E-06
2^{-3}	0.869E-04	0.218E-04	0.545E-05	0.136E-05	0.341E-06
2^{-4}	0.108E-03	0.270E-04	0.675E-05	0.169E-05	0.422E-06
2^{-5}	0.132E-03	0.332E-04	0.831E-05	0.208E-05	0.519E-06
2^{-6}	0.155E-03	0.390E-04	0.976E-05	0.244E-05	0.611E-06
2^{-7}	0.170E-03	0.435E-04	0.109E-04	0.273E-05	0.684E-06
2^{-8}	0.182E-03	0.465E-04	0.118E-04	0.252E-05	0.738E-06
2^{-9}	0.183E-03	0.485E-04	0.124E-04	0.311E-05	0.778E-06
2^{-10}	0.187E-03	0.502E-04	0.122E-04	0.322E-05	0.807E-06

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