Common fixed point theorems of different compatible type mappings using Ciric's contraction type condition

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Abstract. *The purpose of this paper is to establish necessary and sufficient conditions for the existence of common fixed points for a compatible pair of selfmaps under Ciric's contraction type condition. These theorems improve and generalize the results of Mukherjee and Verma* [11] *and Jungck* [9] *to a pair of selfmaps. Also established the existence of common fixed points for a pair of compatible mappings of type* (B)*, and obtain a result on the existence of common fixed points for a pair of compatible mappings of type* (A) *as corollary. Greguš fixed point theorem follows as a special case to our results.*

Key words: *compatible mappings, compatible mappings of type* (A)*, compatible mappings of type* (B)*, common fixed point, linear map, affine map, Banach space, Ciric's contraction type condition, reciprocal continuity*

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1. Introduction

Finding necessary and sufficient conditions for the existence of fixed points is an interesting aspect. In 1986, Fisher and Sessa [6], established common fixed points for a pair of selfmaps in which one map is linear and nonexpansive. It was improved to affine maps by Mukherjee and Verma [11]. Further it is improved by Jungck [9] to continuous maps for a compatible pair of selfmaps. The aim of this paper is to find necessary and sufficient conditions for the existence of common fixed points for a pair of selfmaps under weak commutativity hypotheses using Ciric's contraction type condition, which improve and generalize the results of Fisher and Sessa [6], Mukherjee and Verma [11], and Jungck [9].

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Throughout this paper, X denotes a Banach space with norm $\|\cdot\|$; T and I are selfmaps of X; N is the set of all natural numbers.

Definition 1.1(Sessa [11]). *Two selfmaps T and I of X are said to be* `weakly commuting` *if* $\|TIx - ITx\| \leq \|Tx - Ix\|$ *for all* $x \in X$.

In 1986, Jungck [8] introduced the concept of compatible mappings as a generalization of weakly commuting maps.

Definition 1.2(Jungck [5]). *Two selfmaps T and I of X are said to be* `compatible` *if*

$$\lim_{n\to\infty} \|ITx_n - TIx_n\| = 0$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Ix_n = t$$

for some $t \in X$.

Clearly, every weakly commuting pair of maps is compatible, but its converse is not true [8].

Definition 1.3. *Let C be a convex subset of X. A mapping $I : C \to C$ is called* `affine` *if $I(\alpha x + \beta y) = \alpha Ix + \beta Iy$ for all $x,\ y \in C$ and $\alpha,\ \beta \geq 0$ with $\alpha + \beta = 1$.*

Pant [12] introduced the concept of reciprocal continuity for a pair of selfmaps.

Definition 1.4(Pant [12]). *Two selfmaps T and I of X are said to be* `reciprocal continuous` *if*

$$\lim_{n\to\infty} TIx_n = Tt \quad \text{and} \quad \lim_{n\to\infty} ITx_n = It$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Ix_n = t \quad \text{for some } t \in X.$$

Clearly, every continuous pair of selfmaps is reciprocal continuous, but its converse need not be true [12].

In 1986, Fisher and Sessa [6] obtained the following common fixed point theorem of Greguš type.

Theorem 1.5(Fisher and Sessa [6]). *Let T and I be weakly commuting selfmaps of a closed convex subset C of X with $T(C) \subseteq I(C)$ and satisfying the inequality*

$$\|Tx - Ty\| \leq\ a\ \|Ix - Iy\| +\ (1-a)\ \max\{\|Ix - Tx\|,\ \|Iy - Ty\|\} \qquad (1)$$

for all $x,\ y \in C$, where $0 < a < 1$. If I is linear, nonexpansive in C, then T and I have a unique common fixed point in C.

In 1988, Mukherjee and Verma [11] improved *Theorem 1.5* by using affine map in place of linear map I.

Theorem 1.6 (Mukherjee and Verma [8]). *Let T and I be weakly commuting selfmaps of a closed convex subset C of X satisfying the inequality* (1) *with $T(C) \subseteq I(C)$. If I is affine, nonexpansive in C, then T and I have a unique common fixed point in C.*

In 1990, Jungck [9] improved and generalized *Theorem 1.5*, by replacing the nonexpansive property of I by continuity and weak commutativity by compatibility in the following way.

Theorem 1.7(Jungck [9]). *Let T and I be compatible selfmaps of a closed convex subset C of X. Assume that $T(C) \subseteq I(C)$ and satisfying the inequality* (1). *If I is continuous and linear in C, then T and I have a unique common fixed point in C.*

`Ciric's contraction type condition`: there exist real numbers $a,\ b,\ c$ with $0 < a < 1,\ b \geq 0,\ a+b=1,\ 0 \leq c < \eta$ such that

$$\begin{aligned} \|Tx - Ty\| \leq\ & a\ \max\{\|Ix - Iy\|,\ c[\|Ix - Ty\| + \|Iy - Tx\|]\} \\ & + b\ \max\{\|Ix - Tx\|,\ \|Iy - Ty\|\} \end{aligned} \tag{2}$$

for all $x,\ y \in X$, where $\eta = min\{\frac{2+a}{5+a},\ \frac{2-a}{4},\ \frac{4}{9+a}\}$.

Here we observe that $\eta < \frac{1}{2}$.

By choosing I as the identity map, we obtain Ciric's contraction condition for a single selfmap T which is introduced by Ciric[2].

In *Section 2*, we prove a common fixed point theorem (*Theorem 2.2*) for a compatible pair of selfmaps, in which one map is affine and continuous satisfying the Ciric's contraction type condition (2). Also we improve *Theorem 2.2* for a pair of reciprocal continuous maps. Our theorems generalize the results of Mukherjee and Verma [11] and Jungck [9]. In *Section 3*, we prove the existence of common fixed points for a pair of compatible mappings of type (B), and obtain a result on the existence of common fixed point for a pair of compatible mappings of type (A) as corollary. Also, Greguš fixed point theorem follows as a special case to our results.

2. Main results

Proposition 2.1. *Let T and I be selfmaps of X which are compatible and satisfy the Ciric's contraction type condition* (2). *If I is continuous then $Tw = Iw$ for some $w \in X$ if and only if $A = \cap\{\overline{TK_n} : n \in N\} \neq \phi$, where $K_n = \{x \in X : \|Ix - Tx\| \leq \frac{1}{n}\}$.*

Proof. Suppose that $Tw = Iw$ for some $w \in X$. Then $w \in K_n$ for all n and thus $Tw \in TK_n \subseteq \overline{TK_n}$ for all n. Hence $Tw \in A$ so that A is nonempty.

Conversely, assume that $A \neq \phi$. If $w \in A$ then for each n, there exists $y_n \in TK_n$ such that $\|w - y_n\| < \frac{1}{n}$. Consequently, for each n, there exists $x_n \in K_n$ such that $y_n = Tx_n$ and $\|w - Tx_n\| < \frac{1}{n}$ for all n. On taking limits as $n \to \infty$, we get $Tx_n \to w$ as $n \to \infty$. Since $x_n \in K_n$, we have $\|Ix_n - Tx_n\| \leq \frac{1}{n}$. Thus

$$\lim_{n\to\infty} Ix_n\ =\ \lim_{n\to\infty} Tx_n\ =\ w. \tag{3}$$

Since T and I are compatible mappings, we have

$$\|ITx_n - TIx_n\| \to 0 \quad as \quad n \to \infty. \tag{4}$$

Since I is continuous, from (4) if follows that

$$IIx_n,\ TIx_n,\ ITx_n \to Iw \quad as \quad n \to \infty. \tag{5}$$

On taking $x = w$ and $y = Ix_n$ in (2), we get

$$\begin{aligned}\|Tw - TIx_n\| \le a\ \max\{&\|Iw - IIx_n\|,\ c[\|Iw - TIx_n\| + \|IIx_n - Tw\|]\}\\ &+ b\ \max\{\|Iw - Tw\|,\ \|IIx_n - TIx_n\|\}.\end{aligned}$$

On taking limits as $n \to \infty$ and using (4) and (5), we have

$$\begin{aligned}\|Tw - Iw\| &\le a\ \max\{\|Iw - Iw\|,\ c[\|Iw - Iw\| + \|Iw - Tw\|]\}\\ &\quad + b\ \max\{\|Iw - Tw\|,\ 0\}\\ &= (ac + b)\|Iw - Tw\|\\ &= [1 - a(1 - c)]\ \|Iw - Tw\|, \quad (\text{since } [1 - a(1 - c)] < 1)\end{aligned}$$

a contradiction. Thus $Iw = Tw$. □

Theorem 2.2. *Let T and I be compatible selfmaps of X and satisfying the condition* (2)*. If I is continuous and affine on X and $T(X) \subseteq I(X)$, then T and I have a unique common fixed point in X.*

Proof. Let x_0 in X be arbitrary. Since $T(X) \subseteq I(X)$, let $x_1,\ x_2$ and x_3 be points in X such that $Ix_1 = Tx_0,\ Ix_2 = Tx_1$ and $Ix_3 = Tx_2$ so that

$$Ix_r = Tx_{r-1} \quad \text{for} \quad r = 1,\ 2,\ 3. \tag{6}$$

On using the inequality (2), we have

$$\begin{aligned}\|Tx_r - Ix_r\| &= \|Tx_r - Tx_{r-1}\|\\ &\le a\ \max\{\|Ix_r - Ix_{r-1}\|,\ c[\ \|Ix_r - Tx_{r-1}\| + \|Ix_{r-1} - Tx_r\|\ ]\}\\ &\quad + b\ \max\{\|Ix_r - Tx_r\|,\ \|Ix_{r-1} - Tx_{r-1}\|\}\\ &\le\ a\ \max\{\|Tx_{r-1} - Ix_{r-1}\|,\ c[\ \|Ix_r - Ix_r\| + \|Ix_{r-1} - Tx_{r-1}\|\\ &\qquad + \|Tx_{r-1} - Tx_r\|\ ]\}\\ &\quad + b\ \max\{\|Ix_r - Tx_r\|,\ \|Ix_{r-1} - Tx_{r-1}\|\}.\end{aligned} \tag{7}$$

If $\|Tx_{r-1} - Ix_{r-1}\| < \|Tx_r - Ix_r\|$, then from (7), we have

$$\begin{aligned}\|Tx_r - Ix_r\| &< a\ \max\{\|Tx_r - Ix_r\|,\ 2c\ \|Tx_r - Ix_r\|\} + b\ \|Tx_r - Ix_r\|\\ &=\ (a + b)\|Tx_r - Ix_r\|,\end{aligned}$$

a contradiction. Thus from (7), we have

$$\|Tx_r - Ix_r\|\ \le\ \|Tx_{r-1} - Ix_{r-1}\| \quad \text{for} \quad r = 1, 2, 3.$$

Therefore

$$\|Tx_r - Ix_r\| \le \|Tx_0 - Ix_0\| \text{ for } \ r = 1, 2, 3.$$

On using (2) and (8), we have

$$\begin{aligned}\|Tx_2 - Ix_1\| &= \|Tx_2 - Tx_0\|\\ &\le a\ \max\{\|Ix_2 - Ix_0\|,\ c[\ \|Ix_2 - Tx_0\| + \|Ix_0 - Tx_2\|\ ]\}\\ &\qquad + b\ \max\{\|Ix_2 - Tx_2\|,\ \|Ix_0 - Tx_0\|\}\end{aligned}$$

$$\begin{aligned}
&\leq a\ \max\{\|Ix_2 - Ix_1\| + \|Ix_1 - Ix_0\|,\\
&\qquad c[\ \|Ix_2 - Tx_0\| + \|Ix_0 - Ix_1\|\\
&\qquad\qquad +\|Ix_1 - Tx_1\| + \|Tx_1 - Tx_2\|\ ]\}\\
&\qquad\qquad +b\ \max\{\|Ix_2 - Tx_2\|,\ \|Ix_0 - Tx_0\|\}\\
&= a\ \max\{\|Tx_1 - Ix_1\| + \|Tx_0 - Ix_0\|,\\
&\qquad c[\ \|Tx_1 - Ix_1\| + \|Tx_1 - Ix_1\|\\
&\qquad\qquad +\|Ix_1 - Tx_1\| + \|Ix_2 - Tx_2\|\ ]\}\\
&\qquad\qquad +b\ \max\{\|Ix_2 - Tx_2\|,\ \|Ix_0 - Tx_0\|\}\\
&\leq a\ \max\{\|Ix_0 - Tx_0\| + \|Ix_0 - Tx_0\|,\\
&\qquad c[\ \|Ix_0 - Tx_0\| + \|Ix_0 - Tx_0\|\\
&\qquad\qquad +\|Ix_0 - Tx_0\| + \|Ix_0 - Tx_0\|\ ]\}\\
&\qquad\qquad +b\ \max\{\|Ix_0 - Tx_0\|,\ \|Ix_0 - Tx_0\|\}\\
&= a\ \max\{2\ \|Ix_0 - Tx_0\|,\ 4c\ \|Ix_0 - Tx_0\|\} + b\ \|Ix_0 - Tx_0\|\\
&= (2a + b)\ \|Tx_0 - Ix_0\|\\
&= (1 + a)\ \|Tx_0 - Ix_0\|.
\end{aligned}$$

Hence

$$\|Tx_2 - Ix_1\| = \|Tx_2 - Tx_0\| \leq (1 + a)\|Tx_0 - Ix_0\|. \tag{9}$$

Write $z = \frac{1}{2}x_2 + \frac{1}{2}x_3$.

Since I is affine and using (6), we have

$$Iz = \tfrac{1}{2}\ Ix_2 + \tfrac{1}{2}\ Ix_3 = \tfrac{1}{2}\ Tx_1 + \tfrac{1}{2}\ Tx_2. \tag{10}$$

Hence

$$\|Tz - Iz\| \leq \tfrac{1}{2}\|Tz - Tx_1\| + \tfrac{1}{2}\|Tz - Tx_2\|.$$

Write $M(x,\ y) = \max\{\|Iz - Tz\|,\ \|Tx_0 - Ix_0\|\}$, and we denote it simply by M.

On using the inequality (2), we have

$$\begin{aligned}
\|Tz - Tx_1\| \leq a\ \max\{\|Iz - Ix_1\|,\ c[\ \|Iz - Tx_1\| + \|Ix_1 - Tz\|\ ]\}&\\
+b\ \max\{\|Iz - Tz\|,\ \|Ix_1 - Tx_1\|\}.&
\end{aligned} \tag{11}$$

Thus from (8), we have

$$\begin{aligned}
\|Tz - Tx_1\| \leq a\ \max\{\|Iz - Ix_1\|,\ c[\ \|Iz - Tx_1\| + \|Ix_1 - Iz\| + \|Iz - Tz\|\ ]\}&\\
+bM.&
\end{aligned} \tag{12}$$

Now, from (8), (9) and (10), we get

$$\begin{aligned}\|Iz-Ix_1\| &\le \frac{1}{2}\|Ix_2-Ix_1\|+\frac{1}{2}\|Ix_3-Ix_1\|\\ &= \frac{1}{2}\|Tx_1-Ix_1\|+\frac{1}{2}\|Tx_2-Ix_1\|\\ &\le \frac{1}{2}\|Tx_0-Ix_0\|+\frac{1}{2}(1+a)\|Tx_0-Ix_0\|\\ &= (1+\frac{a}{2})\ \|Tx_0-Ix_0\|. \end{aligned} \tag{13}$$

Now on using (6), (8) and (10), we have

$$\|Iz-Tx_1\| = \frac{1}{2}\|Tx_2-Tx_1\| = \frac{1}{2}\|Tx_2-Ix_2\| \le \frac{1}{2}\|Tx_0-Ix_0\|. \tag{14}$$

On substituting (13) and (14) in (12), we have

$$\begin{aligned}\|Tz-Tx_1\| \le a\ \max\ &\{(1+\tfrac{a}{2})\|Tx_0-Ix_0\|,\\ &c[\ \tfrac{1}{2}\|Tx_0-Ix_0\|+(1+\tfrac{a}{2})\|Tx_0-Ix_0\|+\|Iz-Tz\|\ ]\}+bM\\ = a\ \max&\{(1+\tfrac{a}{2})\|Tx_0-Ix_0\|,\\ &c[\ (\tfrac{3+a}{2})\|Tx_0-Ix_0\|+\|Iz-Tz\|\ ]\}+bM\\ \le\ a\ \max&\{(1+\tfrac{a}{2})M,\ c\ (\tfrac{5+a}{2})M\}+bM. \end{aligned} \tag{15}$$

Again, on using the inequality (2), we have

$$\begin{aligned}\|Tz-Tx_2\| \le a\ \max\{\|Iz-Ix_2\|,\ c[\ \|Iz-Tx_2\|+\|Ix_2-Tz\|\ ]\}\\ +b\ \max\{\|Iz-Tz\|,\ \|Ix_2-Tx_2\|\}.\end{aligned}$$

On using (8), we have

$$\begin{aligned}\|Tz-Tx_2\| \le a\ \max\{\|Iz-Ix_2\|,\ c[\ \|Iz-Tx_2\|+\|Ix_2-Iz\|+\|Iz-Tz\|\ ]\}\\ +bM.\end{aligned} \tag{16}$$

From (6), (8) and (10), we get the following:

$$\|Iz-Ix_2\| = \frac{1}{2}\|Ix_2-Ix_3\| = \frac{1}{2}\|Ix_2-Tx_2\| \le \frac{1}{2}\|Tx_2-Ix_0\|, \tag{17}$$

and

$$\|Iz-Tx_2\| = \frac{1}{2}\|Tx_1-Tx_2\| = \frac{1}{2}\|Ix_2-Tx_2\| \le \frac{1}{2}\|Tx_0-Ix_0\|. \tag{18}$$

On substituting (17) and (18) in (16), we get

$$\begin{aligned}\|Tz-Ix_2\| \le a\ \max\{&\frac{1}{2}\|Tx_0-Ix_0\|,\ c[\ \frac{1}{2}\|Tx_0-Ix_0\|+\frac{1}{2}\|Tx_0-Ix_0\|\\ &+\|Iz-Tz\|\ ]\}+bM\\ &\le a\ \max\{\ \frac{1}{2}M,\ 2cM\ \}+bM.\end{aligned} \tag{19}$$

On substituting (15) and (19) in (11), we have

$$\begin{aligned}\|Tz - Iz\| &\leq \frac{1}{2}[a\ \max\{\ (1+\frac{a}{2})M,\ (\frac{5+a}{2})cM\ \} + bM] \\ &\quad + \frac{1}{2}[a\ \max\{\ \frac{1}{2}M,\ 2cM\ \} + bM] \\ &= \frac{a}{2}[\ \max\{\ (1+\frac{a}{2})M,\ (\frac{5+a}{2})cM\ \}] \\ &\quad + \frac{a}{2}[\ \max\{\ \frac{1}{2}M,\ 2cM\ \}] + bM. \end{aligned} \tag{20}$$

Now the following *four* possible cases may arise in (20).

Case 1. $\max\{\ (1+\frac{a}{2})M,\ (\frac{5+a}{2})cM\ \} = (1+\frac{a}{2})M$ and $\max\{\ \frac{1}{2}M,\ 2cM\ \} = \frac{1}{2}M$. Now from (20), we have

$$\begin{aligned}\|Tz - Iz\| &\leq [\ \frac{a}{2}(1+\frac{a}{2}) + \frac{a}{2}\cdot\frac{1}{2} + b\ ]M = [\ \frac{a(2+a)}{4} + \frac{a}{4} + (1-a)\ ]M \\ &= \lambda_1 \cdot M, \end{aligned} \tag{21}$$

where $\lambda_1 = \frac{a^2-a+4}{4}$ (< 1).

Case 2. $\max\{\ (1+\frac{a}{2})M,\ (\frac{5+a}{2})cM\ \} = (1+\frac{a}{2})M$ and $\max\{\ \frac{1}{2}M,\ 2cM\ \} = 2cM$. Thus from(20), we have

$$\begin{aligned}\|Tz - Iz\| &\leq [\ \frac{a}{2}(1+\frac{a}{2}) + \frac{a}{2}\ 2c + b\ ]M = [\ \frac{a(2+a)}{4} + ac + (1-a)\ ]M \\ &= \lambda_2 \cdot M, \end{aligned} \tag{22}$$

where $\lambda_2 = \frac{a^2-2a+4+4ac}{4}$ (< 1).

Case 3. $\max\{\ (1+\frac{a}{2})M,\ (\frac{5+a}{2})cM\ \} = (\frac{5+a}{2})cM$ and $\max\{\ \frac{1}{2}M,\ 2cM\ \} = 2cM$. In this case, again from (20), then we have

$$\begin{aligned}\|Tz - Iz\| &\leq [\ \frac{a}{2}(\frac{5+a}{2})c + \frac{a}{2}2c + b\ ]M = [\ \frac{ac(5+a)}{4} + ac + 1 - a\ ]M \\ &= \lambda_3 \cdot M, \end{aligned} \tag{23}$$

where $\lambda_3 = \frac{a^2c+9ac+4-4a}{4}$ (< 1).

Case 4. $\max\{\ (1+\frac{a}{2})M,\ (\frac{5+a}{2})cM\ \} = (\frac{5+a}{2})cM$ and $\max\{\ \frac{1}{2}M,\ 2cM\ \} = \frac{1}{2}M$. It follows that

$$\frac{2+a}{5+a} \leq c \leq \frac{1}{4},$$

and since

$$c \leq \eta \leq \frac{2+a}{5+a},$$

this case doesn't arise.

Now, from (21), (22) and (23), we have

$$\|Tz - Iz\| \leq \lambda \cdot M,\ \text{where } \lambda = \max\{\lambda_1,\ \lambda_2,\ \lambda_3\}. \tag{24}$$

Thus it follows that

$$\|Tz - Iz\| \leq \lambda \ \max\{ \ \|Iz - Tz\|, \ Tx_0 - Ix_0\| \ \}.$$

Therefore

$$\|Tz - Iz\| \leq \lambda \cdot \|Tx_0 - Ix_0\|.$$

This implies

$$\inf \ \{ \ \|Tz - Iz\| : z = \frac{1}{2}x_2 + \frac{1}{2}x_3\} \leq \lambda \ \|Tx_0 - Ix_0\|.$$

Since $x_0 \in X$ is arbitrary, we have

$$\inf \ \{ \ \|Tz - Iz\| : z = \frac{1}{2}x_2 + \frac{1}{2}x_3\} \leq \lambda \ \inf \ \{ \ \|Tx - Ix\| : x \in X \ \}.$$

On the other hand

$$\inf\{ \ \|Tx - Ix\| : x \in X \ \} \leq \inf\{ \ \|Tz - Iz\| : z = \frac{1}{2}x_2 + \frac{1}{2}x_3\}.$$

It follows that

$$\inf\{ \ \|Tx - Ix\| : x \in X \ \} = 0. \tag{25}$$

Define $K_n = \{x \in X : \|Tx - Ix\| \leq \frac{1}{n}\}$ and

$H_n = \{x \in X : \|Tx - Ix\| \leq \frac{a+1}{(1-a)n}\}$ for $n = 1, \ 2, \ 3, \ \dots$.

Then $K_n \neq \phi$ and also that

$K_1 \supseteq K_2 \supseteq K_3 \supseteq \dots \supseteq K_n \supseteq \dots$.

Consequently, TK_n is nonempty for $n = 1, \ 2, \ 3, \dots$, and

$\overline{TK_1} \supseteq \overline{TK_2} \supseteq \overline{TK_3} \supseteq \dots \supseteq \overline{TK_n} \supseteq \dots$.

For any $x, \ y \in K_n$, by (2), we have

$$\begin{aligned}
\|Tx - Ty\| &\leq a \max\{\|Ix - Iy\|, \ c[\ \|Ix - Ty\| + \|Iy - Tx\|]\} \\
&\quad + b \ \max\{\|Ix - Tx\|, \ \|Iy - Ty\|\} \\
&\leq a \ \max\{\|Ix - Tx\| + \|Tx - Ty\| + \|Ty - Iy\|, \\
&\qquad\qquad c[\ \|Ix - Tx\| + \|Tx - Ty\| + \|Iy - Ty\| + \|Ty - Tx\| \ ]\} \\
&\quad +b \ \max\{\|Ix - Tx\|, \ \|Iy - Ty\|\} \\
&\leq a \max\{\frac{1}{n} + \|Tx - Ty\| + \frac{1}{n}\}, \ c[\ \frac{1}{n} + \|Tx - Ty\| + \frac{1}{n} + \|Tx - Ty\| \ ]\} \\
&\quad +b \ \max\{\frac{1}{n}, \ \frac{1}{n}\} \\
&\leq a \ \max\{\frac{2}{n} + \|Tx - Ty\|, \ c[\ \frac{2}{n} + 2\|Tx - Ty\| \ ]\} + \frac{b}{n}. \qquad (26)
\end{aligned}$$

Here we consider the following *two* possible cases of (26).

Case I. $\max\{\frac{2}{n} + \|Tx - Ty\|,\ c[\ \frac{2}{n} + 2\|Tx - Ty\|\ ]\} = \frac{2}{n} + \|Tx - Ty\|$. Now from in (26), we have

$$\|Tx - Ty\| \leq \frac{2a}{n} + a\|Tx - Ty\| + \ \frac{b}{n} = \frac{2a + b}{n} + a\|Tx - Ty\|.$$

Therefore

$$(1 - a)\|Tx - Ty\| \leq \frac{a + 1}{n}$$
$$\|Tx \ - \ Ty\| \leq \frac{a + 1}{(1 - a)n}. \tag{27}$$

Case II. $\max\{\frac{2}{n} + \|Tx - Ty\|,\ c[\ \frac{2}{n} + 2\|Tx - Ty\|\ ]\} =\ c[\ \frac{2}{n} + 2\|Tx - Ty\|]$. From (26), we have

$$\begin{aligned}\|Tx - Ty\| &\leq\ a\ c\ \tfrac{2}{n} + 2ac\|Tx - Ty\| +\ \tfrac{b}{n}\\ &=\ 2ac[\ \tfrac{1}{n} + \|Tx - Ty\|] +\ \tfrac{b}{n}\\ &<\ a[\tfrac{1}{n} + \|Tx - Ty\|] +\ \tfrac{b}{n}\\ &=\ \tfrac{1}{n} + a\ \|Tx - Ty\|.\end{aligned}$$

Thus

$$\|Tx - Ty\| < \frac{1}{(1 - a)n} \leq \frac{a + 1}{(1 - a)n}. \tag{28}$$

Thus in both cases we get

$$\|Tx - Ty\| \leq \frac{a + 1}{(1 - a)n},\ \text{so that } x,\ y \in H_n.$$

Hence

$$\lim_{n\to\infty} \operatorname{diam}(TK_n) = \lim_{n\to\infty} \operatorname{diam}(\overline{TK_n})\ =\ 0.$$

On using Cantor's intersection theorem, $A = \bigcap\{\overline{TK_n} : n \in N\}$ contains exactly one point w (say).

Thus from *Proposition 2.1*, we have

$$Tw = Iw. \tag{29}$$

We now show that w is a common fixed point of T and I. On taking $x = w$ and $y = x_n$ in (2), we have

$$\begin{aligned}\|Tw - Tx_n\| \leq a\ \max\{&\|Iw - Ix_n\|,\ c[\ \|Iw - Tx_n\| + \|Ix_n - Tw\|]\}\\ &+ b\ \max\{\|Iw - Tw\|,\ \|Ix_n - Tx_n\|\}.\end{aligned}$$

On taking limits as $n \to \infty$ and using (4) and (29), we get

$$\begin{aligned}\|Tw - w\| &\leq a\ \max\{\|Tw - w\|,\ c[\ \|Tw - w\| + \|w - Tw\|]\}\\ &\quad + b\ \max\{\|Tw - Tw\|,\ \|w - w\|\}\\ &= a\ \max\{\|Tw - w\|,\ 2c\|Tw - w\|\}\ (\text{since } c < \tfrac{1}{2})\\ &\leq a\ \|Tw - w\| < \|Tw - w\|,\end{aligned}$$

a contradiction. Thus $Tw = w$, so that

$$Tw = Iw = w.$$

Thus w is a common fixed point of T and I. Uniqueness of the common fixed point follows from the Ciric's contraction type condition.

An alternate proof: The proof is similar upto the identity (25). Here we show that

$$\max\{\|Tx - Ty\|,\ \|Ix - Iy\|\} \leq \frac{3-a}{1-a}\ \max\{\|Ix - Tx\|,\ \|Iy - Ty\|\}. \tag{30}$$

Write $R = R(x,y) = \max\{\|Ix - Tx\|,\ \|Iy - Ty\|\}$. From the inequality (2), we have

$$\begin{aligned}
\|Tx - Ty\| &\leq a\ \max\{\|Ix - Iy\|,\ c[\ \|Ix - Ty\| + \|Iy - Tx\|]\} \\
&\quad + b\ \max\{\|Ix - Tx\|,\ \|Iy - Ty\|\} \\
&\leq a\ \max\{\|Ix - Tx\| + \|Tx - Ty\| + \|Ty - Iy\|, \\
&\qquad c[\ \|Ix - Tx\| + \|Tx - Ty\| + \|Iy - Ty\| + \|Ty - Tx\|\ ]\} \\
&\quad +b\ \max\{\|Ix - Tx\|,\ \|Iy - Ty\|\} \\
&\leq a\ \max\{R + \|Tx - Ty\| + R\},\ c[2R + 2\|Tx - Ty\|\ ]\} + bR \\
&\leq a\ \max\{2R + \|Tx - Ty\|,\ 2c[\ R + \|Tx - Ty\|\ ]\} + bR \\
&= (2a + b)R + a\|Tx - Ty\| \\
&= (1 + a)R + a\|Tx - Ty\|.
\end{aligned}$$

Hence

$$\|Tx - Ty\| \leq \frac{1+a}{1-a}\ R. \tag{31}$$

Now

$$\begin{aligned}
\|Ix - Iy\| &\leq \|Ix - Ty\| + \|Tx - Ty\| + \|Ty - Iy\| \\
&\leq R + \frac{1+a}{1-a}\ R + R \\
&= \frac{3-a}{1-a}\ R.
\end{aligned} \tag{32}$$

From (31) and (32), the inequality (30) follows.

Now, by (25), we can choose a sequence $\{x_n\} \in X$ such that

$$\|Ix_n - Tx_n\| \leq \frac{1}{n} \text{ for } n = 1, 2, 3, \ldots \tag{33}$$

From (30) and (33), we have

$$\max\{\|Ix_n - Tx_m\|,\ \|Tx_n - Tx_m\| \leq \frac{3-a}{1-a} \cdot \frac{1}{n} \text{ for } 1 \leq n \leq m.$$

Therefore, both $\{Ix_n\}$ and $\{Tx_n\}$ are Cauchy sequence in X and from (33), we have

$$\lim_{n\to\infty} Ix_n\ =\ \lim_{n\to\infty} Tx_n =\ w \text{ (say)}, w \in X. \tag{34}$$

Since T and I are compatible mappings and I is continuous, we have

$$IIx_n,\;\; TIx_n.\;\; ITx_n\;\; \to Iw \text{ as } n \to \infty. \tag{35}$$

Now we show that $Iw = w$. Suppose that $Iw \neq w$. On substituting $x = x_n$ and $y = Ix_n$ in (2), we have

$$\begin{aligned}\|Tx_n - TIx_n\| \leq\; & a\;\max\{\|Ix_n - IIx_n\|,\; c[\;\|Ix_n - TIx_n\| + \|IIx_n - Tx_n\|]\}\\ & + b\;\max\{\|Ix_n - Tx_n\|,\; \|IIx_n - TIx_n\|\}.\end{aligned}$$

On taking limits as $n \to \infty$ and using (34) and(35), we have

$$\begin{aligned}\|w - Iw\| \leq\; & a\;\max\{\|w - Iw\|,\; c[\;\|w - Iw\| + \|Iw - w\|]\}\\ & + b\;\max\{\|w - w\|,\; \|Iw - Iw\|\}\\ =\; & a\|w - Iw\| < \|w - Iw\|,\end{aligned}$$

a contradiction. Thus

$$Iw = w. \tag{36}$$

Finally, we show that $Tw = w$. Suppose that $Tw \neq w$. On taking $x = w$ and $y = x_n$ in (2), we have

$$\begin{aligned}\|Tw - Tx_n\| \leq\; & a\;]max\{\|Iw - Ix_n\|,\; c[\;\|Iw - Tx_n\| + \|Ix_n - Tw\|]\}]\\ & + b\;\max\{\|Iw - Tw\|,\; \|Ix_n - Ix_n\|\}.\end{aligned}$$

On taking limits as $n \to \infty$ and using (34) and (36), we have

$$\begin{aligned}\|Tw - w\| \leq\; & a\;\max\{\|Iw - w\|,\; c[\;\|w - w\| + \|w - Tw\|]\}\\ & + b\;\max\{\|Tw - Tw\|,\; \|w - w\|\}\\ =\; & (ac + b)\|w - Tw\|\\ =\; & [1 - a(1 - c)]\|w - Tw\|,\end{aligned}$$

a contradiction. Hence

$$Tw = w. \tag{37}$$

From (36) and (37), we have

$$Tw = Iw = w.$$

Hence w is a common fixed point of T and I. This completes the proof of *Theorem 2.2.* □

The following is an example in support of *Theorem 2.2.*

Example 2.3. *Let $X = \mathbb{R}$ with the usual metric. Define selfmaps $T,\ I$ on X by $Tx = \frac{2+x}{3}$ and $Ix = \frac{3x-1}{2},\ x \in X$.*

Clearly, I is continuous and affine, but I is not nonexpansive and linear. Observe that T and I are compatible mappings of X.

Now, for any $x,\ y\ \in X$,

$$\|Tx - Ty\| = |\frac{x-y}{3}| = \frac{2}{9}\|Ix - Iy\|,$$

so that the mappings T and I satisfy the inequality (2) with $a = \frac{2}{9}$, $b = \frac{7}{9}$ and $c \leq \frac{20}{47}$.

On using *Proposition 2.1* and *Theorem 2.2*, we formulate the following theorem.

Theorem 2.4. *Let T and I be compatible selfmaps of X and satisfying the condition (2). If I is continuous and affine in X and $T(X) \subseteq I(X)$, then T and I have a unique common fixed point in X if and only if*

$$A = \cap\{\overline{TK_n} : n \in N\} \neq \phi,$$

where $K_n = \|x \in X : \|Ix - Tx\| \leq \frac{1}{n}\}$.

Corollary 2.5. *Let T and I be compatible selfmaps of X and satisfying the inequality*

$$\begin{aligned} \|Tx - Ty\| \leq a\ \|Ix - Iy\| + b\ \max\{\|Ix - Ix\|,\ \|Iy - Ty\|\} \\ +c\ [\|Ix - Ty\| + \|Iy - Tx\|] \end{aligned} \tag{38}$$

for all $x,\ y \in C$, where $0 < a < 1,\ b \geq 0,\ c \geq 0,\ a + c > 0$ and $a + b + 4c = 1$. If I is continuous and affine on X and $T(X) \subseteq I(X)$, then T and I have a unique common fixed point in X.

Proof. Set $a + 4c = a_1$. Then $a_1 + b = 1$ and we have

$$\begin{aligned} \|Tx - Ty\| &\leq a\ \|Ix - Iy\| + b\ \max\{\|Ix - Tx\|,\ \|Iy - Ty\|] \\ &\quad +c \cdot \frac{4}{1} \cdot \frac{1}{4}[\|Ix - Ty\| + \|Iy - Tx\|] \\ &\leq (a + 4c) \max\{\|Ix - Iy\|,\ \frac{1}{4}[\|Ix - Iy\| + \|Iy - Tx\|]\} \\ &\quad +b\ \max\{\|Ix - Tx\},\ \|Iy - Ty\|\}. \end{aligned}$$

Since $\frac{1}{4} \leq \ \min\{\frac{2+a}{5+a},\ \frac{2-a}{4},\ \frac{4}{9+a}\}$ and $a_1 + b = 1$, where $a_1 = a + 4c$, the conclusion of this corollary follows from *Theorem 2.2*.

On choosing $c = 0$ in (2), we have the following corollary.

Corollary 2.6. *Let T and I be compatible selfmaps of X and satisfying the condition (1). Suppose that I is continuous, affine and $T(X) \subseteq I(X)$. Then T and I have a unique common fixed point in X.*

Corollary 2.7(Fisher [5]). *Let T be a selfmap of a closed convex subset C of X and satisfying the condition*

$$\|Tx - Ty\| \leq a\ \|x - y\| + b\ max\{\|Tx - x\|,\ \|Ty - y\|\} \tag{39}$$

for all $x,\ y \in C$, where $0 < a < 1$ with $a + b = 1$. Then T has a unique fixed point in C.

Proof. Follows by choosing I as the identity map of C in *Corollary 3.3*. □

In the following, we prove a common fixed point theorem for a compatible pair of selfmaps T and I, which are reciprocal continuous on X.

Theorem 2.8. *Let T and I be compatible selfmaps of X, which are reciprocal continuous on X, satisfying the Ciric's contraction type condition* (2)*. If I is affine on X and $T(X) \subseteq I(X)$, then T and I have a unique common fixed point in X if and only if $A = \cap\{\overline{TK_n} : n \in N\} \neq \phi$, where $K_n = \|x \in X : \|Ix - Tx\| \leq \frac{1}{n}\}$.*

Proof. If w is a common fixed point of T and I, then $A \neq \phi$ follows trivially by *Proposition 2.1*.

Conversely, assume that $A \neq \phi$. If $w \in A$ then for each n, there exists $y_n \in TK_n$ such that $\|w - y_n\| < \frac{1}{n}$. Consequently, for each n, there exists $x_n \in K_n$ such that $y_n = Tx_n$ and $\|w - Tx_n\| < \frac{1}{n}$ for all n. On taking limits as $n \to \infty$, we get $Tx_n \to w$ as $n \to \infty$.

Since $x_n \in K_n$, we have $\|Ix_n - Tx_n\| \leq \frac{1}{n}$. Thus

$$\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Tx_n = w. \tag{40}$$

Since T and I are reciprocally continuous mappings, we have

$$\lim_{n\to\infty} TIx_n = Tw \text{ and } \lim_{n\to\infty} ITx_n = Tw.$$

Now since T and I are compatible mappings

$$Tw = \lim_{n\to\infty} TIx_n = \lim_{n\to\infty} ITx_n = Iw. \tag{41}$$

Now on substituting $x = w$ and for each n, substituting $y = Ix_n$ in (2) and using (40) and (41), as in the alternate proof of *Theorem 2.2*, it is easy to see that $Tw = w$. Thus from (41), w is a common fixed point of T and I. □

Example 2.9. *Let $X = \mathbb{R}$ with the usual metric. Define selfmaps T and I on X by*

$$Tx = \begin{cases} \frac{1}{2}, & if \;\; x \leq 0 \;\; and \;\; x = \frac{5}{2} \\ \frac{1+x}{2}, & if \;\; x > 0 \;\; and \;\; x \neq \frac{5}{2} \end{cases} \qquad and \quad Ix = \frac{3x-1}{2}, \quad x \in X.$$

Clearly, I is affine, but I is not nonexpansive and linear. The mappings T and I are reciprocal continuous and compatible on X.

Observe that the inequality (2) holds with $a = \frac{1}{3}$, $b = \frac{2}{3}$ and for any $c \geq 0$ with $c \leq \frac{7}{16}$. Thus, all the hypotheses of Theorem 2.4 *is satisfied and has a unique fixed point 1.*

Now, for $x = 2$,

$$\|TI(2) - IT(2)\| = \frac{5}{4} \nleq 1 = \|T(2) - I(2)\|.$$

Thus T and I are not weakly commuting, so that Theorem 1.6 *is not applicable. Since I is not linear,* Theorem 1.7 *is also not applicable.*

Hence, from this example, we conclude that *Theorem 2.4* is a generalization of *Theorem 1.6* and *Theorem 1.7*.

3. Compatible mappings of type (A), compatible mappings of type (B) and common fixed point theorems

Definition 3.1(Lal et al. [10]). T*wo selfmaps T and I of X are said to be* `compatible mappings of type (A)`, *if*

$$\lim_{n\to\infty} \|TIx_n - IIx_n\| = 0 \text{ and } \lim_{n\to\infty} \|ITx_n - TTx_n\| = 0,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Tx_n = t, \text{ for some } t \in X.$$

Here we note that compatible mappings and compatible mappings of type (A) are independent (Lal et al. [10]).

Pathak et al. [13] introduced the concept of compatible mappings of type (B) as a generalization of compatible mappings of type (A).

Definition 3.2(Pathak et al.[13]). *Two selfmaps T and I of X are said to* `compatible mappings of type (B)`, *if*

$$\lim_{n\to\infty} \|ITx_n - TTx_n\| \leq \tfrac{1}{2} [\lim_{n\to\infty} \|ITx_n - It\| + \lim_{n\to\infty} \|It - IIx_n\|]$$

and

$$\lim_{n\to\infty} \|TIx_n - IIx_n\| \leq \tfrac{1}{2} [\lim_{n\to\infty} \|TIx_n - Tt\| + \lim_{n\to\infty} \|Tt - TTx_n\|],$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Tx_n = t, \text{ for some } t \in X.$$

Clearly, every compatible mappings of type (A) are compatible mappings of type (B), but its converse need not be true (Pathak et al. [13]).

Proposition 3.3(Pathak et al. [13]). *Two selfmaps T and I of X are compatible mappings of type (B). Suppose that $\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Tx_n = t$, for some $t \in X$. Then $\lim_{n\to\infty} TTx_n = It$, if I is continuous at t.*

Proposition 2.1 remains true, if we replace compatible mappings by compatible mappings of type (B).

Proposition 3.4. *Let T and I be selfmaps of X which are compatible mappings of type (B) and satisfy the Ciric's contraction type condition* (2). *If I is continuous then $Tw = Iw$ for some $w \in X$ if and only if $A = \cap\{\overline{TK_n} : n \in N\} \neq \phi$, where $K_n = \{x \in X : \|Ix - Tx\| \leq \frac{1}{n}\}$.*

Proof. Follows as on the lines of *Proposition 2.1* and using *Proposition 3.4.* □

Theorem 3.5. *Let T and I be selfmaps of X, which are compatible mappings of type (B) and satisfying the condition* (2). *If I is continuous and affine on X and $T(X) \subseteq I(X)$, then T and I have a unique common fixed point in X.*

Proof. Follows as on the lines of proof of *Theorem 2.2* and *Proposition 3.4.* □

Theorem 3.6. *Let T and I be selfmaps of X, which are compatible mappings of type (B) and satisfying the condition (2). If I is continuous and affine in X and $T(X) \subset I(X)$, then T and I have a unique common fixed point in X if and only if $A = \cap\{\overline{TK_n} : n \in N\} \neq \phi$, where $K_n = \{x \in X : \|Ix - Tx\| \leq \frac{1}{n}\}$.*

Corollary 3.7. *Let T and I be selfmaps of X, which are compatible mappings of type (A) and satisfying the condition (2). If I is continuous and affine in X and $T(X) \subset I(X)$, then T and I have a unique common fixed point in X if and only if $A = \cap\{\overline{TK_n} : n \in N\} \neq \phi$, where $K_n = \{x \in X : \|Ix - Tx\| \leq \frac{1}{n}\}$.*

Proof. Since compatible mappings of type (A) implies compatible mappings of type (B), proof follows from *Theorem 3.6.* □

Corollary 3.8(Greguš [7]). *Let T be a selfmap of a closed convex subset C of X and satisfying the inequality*

$$\|Tx - Ty\| \leq p\ \|x - y\| + q\ \|Tx - x\| + r\ \|Ty - y\|$$

for all $x,\ y \in C$, where $0 < p < 1,\ q \geq 0,\ r \geq 0$ with $p + q + r = 1$. Then T has a unique fixed point in C.

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