A note on medial quasigroups

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Abstract. *In this short note we prove two results about medial quasigroups. First, let φ and ψ be binary operations defined by multiplication, left and right division in a medial quasigroup. Then φ and ψ are mutually medial, i.e. $\varphi(\psi(a,b),\psi(c,d)) = \psi(\varphi(a,c),\varphi(b,d))$. Second, four points a, b, c, d in an idempotent medial quasigroup form a parallelogram if and only if $d = (a\,/\,b)(b\setminus c)$.*

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Let $(Q,\cdot)$ be a quasigroup. We will denote left and right division in Q by $\setminus$ and $/$, i.e. $a\setminus c = b \iff ab = c \iff c\,/\,b = a$. By a *formula* we mean any expression built up from a number of variables using the operations $\cdot$, $\setminus$ and $/$. More precisely:

1. the variables x, $y, \ldots$ are formulae;
2. if φ, ψ are formulae, then so are $\varphi\cdot\psi$, $\varphi\setminus\psi$ and $\varphi\,/\,\psi$.

A formula φ containing (at most) two variables gives rise to a new binary operation $Q\times Q \to Q$, which we will also denote by the letter φ.

A quasigroup is *medial* if the identity $ab\cdot cd = ac\cdot bd$ holds. It is known that $(Q,\cdot)$ is medial if and only if either (and therefore both) of the quasigroups $(Q,\setminus)$ and $(Q,/)$ are medial. Our first result states that binary operations defined by formulae in a medial quasigroup are *mutually medial*:

Theorem 1. *Let φ, ψ be binary operations defined by any two formulae in a medial quasigroup Q. Then the following identity holds:*

$$\varphi(\psi(a,b),\psi(c,d)) = \psi(\varphi(a,c),\varphi(b,d)).$$

Proof. It is easily verified that the operations $\cdot$, $\setminus$ and $/$ are mutually medial, i.e. the identities $ab\setminus cd = (a\setminus c)(b\setminus d)$, $ab\,/\,cd = (a\,/\,c)(b\,/\,d)$ and $(a\setminus b)/(c\setminus d) = (a\,/\,c)\setminus(b\,/\,d)$ hold. For example, if we denote $x = a\setminus c$ and $y = b\setminus d$, then $ax = c$

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and $by = d$. Using mediality we get $cd = ax \cdot by = ab \cdot xy \Longrightarrow ab \setminus cd = xy = (a \setminus c)(b \setminus d)$.

A binary operation defined by a formula φ is mutually medial with multiplication, left and right division:

$$\varphi(ab, cd) = \varphi(a, c) \cdot \varphi(b, d), \tag{1}$$

$$\varphi(a \setminus b, c \setminus d) = \varphi(a, c) \setminus \varphi(b, d), \tag{2}$$

$$\varphi(a \,/\, b, c \,/\, d) = \varphi(a, c) \,/\, \varphi(b, d). \tag{3}$$

This is obvious for $\varphi(a, b) = a$ and $\varphi(a, b) = b$, and follows by induction for more complicated formulae. Supposing the identities are true for φ_1 and φ_2, we see that they also hold for $\varphi = \varphi_1 \cdot \varphi_2$:

$$\begin{aligned}\varphi(ab, cd) &= \varphi_1(ab, cd) \cdot \varphi_2(ab, cd) = \varphi_1(a, c)\varphi_1(b, d) \cdot \varphi_2(a, c)\varphi_2(b, d) \\ &= \varphi_1(a, c)\varphi_2(a, c) \cdot \varphi_1(b, d)\varphi_2(b, d) = \varphi(a, c) \cdot \varphi(b, d).\end{aligned}$$

The argument is similar for identities (2), (3) and formulae $\varphi = \varphi_1 \setminus \varphi_2$, $\varphi = \varphi_1 \,/\, \varphi_2$.

Finally, mutual mediality of φ and ψ is obtained by induction on ψ:

$$\begin{aligned}\varphi(\psi(a, b), \psi(c, d)) &= \varphi(\psi_1(a, b)\psi_2(a, b), \psi_1(c, d)\psi_2(c, d)) \\ &\overset{(1)}{=} \varphi(\psi_1(a, b), \psi_1(c, d)) \cdot \varphi(\psi_2(a, b), \psi_2(c, d)) \\ &= \psi_1(\varphi(a, c), \varphi(b, d)) \cdot \psi_2(\varphi(a, c), \varphi(b, d)) \\ &= \psi(\varphi(a, c), \varphi(b, d)).\end{aligned}$$

Identity (2) is used if $\psi = \psi_1 \setminus \psi_2$, and identity (3) if $\psi = \psi_1 \,/\, \psi_2$. □

Corollary 1. *If $(Q, \cdot)$ is a medial quasigroup, then the binary operation defined by a formula φ is also medial.*

Of course, (Q, φ) need not be a quasigroup. Special cases of *Theorem 1* and *Corollary 1* have been used in [1] and [7]. Identity (1) was proved earlier by Puharev [2].

In [3], some geometric concepts have been introduced in a medial quasigruoup Q. For example, the points $a, b, c, d \in Q$ are said to form a *parallelogram*, denoted by $\mathrm{Par}(a, b, c, d)$, if there are points $p, q \in Q$ such that $pa = qb$ and $pd = qc$. This quaternary relation satisfies the axioms of *parallelogram space* (for definitions and further references see [8]). In particular, given any three points $a, b, c \in Q$ there is a unique $d \in Q$ such that $\mathrm{Par}(a, b, c, d)$. In [5], the parallelogram relation in idempotent medial quasigroups (satisfying the additional identity $aa = a$) was characterized in several more direct ways. In even more special quasigroups, explicit formulae for the fourth vertex d of a parallelogram as a function of a, b and c are known; see [1], [4], [6] and [9]. Here we give such a formula valid in a general IM-quasigroup.

Theorem 2. *Let Q be an idempotent medial quasigroup and $a, b, c, d \in Q$. Then, $\mathrm{Par}(a, b, c, d)$ holds if and only if there are $x, y \in Q$ such that $xb = a$, $by = c$ and $xy = d$.*

Proof. Let $x, y \in Q$ be elements satisfying $xb = a$, $by = c$ and $xy = d$. By taking $p = a$ and $q = x$, we see that $pa = qb$ and $pd = xb \cdot xy = x \cdot by = qc$, i.e. $\mathrm{Par}(a, b, c, d)$ holds.

Now suppose $\mathrm{Par}(a, b, c, d)$ holds and denote $x = a \,/\, b$, $y = b \setminus c$. Then, $xb = a$ and $by = c$. According to [3, Corollary 5], for any $p \in Q$ there is a unique $q \in Q$ such that $pa = qb$ and $pd = qc$. Specially, for $p = a$ we see that $a = qb \Longrightarrow q = x$ and $ad = qc = xc = x \cdot by = xb \cdot xy = a \cdot xy$. Cancelling a from the left yields $xy = d$. □

Corollary 2. *In an idempotent medial quasigroup,* $\mathrm{Par}(a, b, c, d)$ *holds if and only if* $d = (a \,/\, b)(b \setminus c)$.

Formulae for the fourth vertex of a parallelogram in hexagonal, quadratical, GS and G_2-quasigroups contain only multiplication. In fact, such formulae follow from *Corollary 2* because left and right division can be expressed by multiplication in these four classes of quasigroups. The formulae given in [1], [4] and [9] are shorter because other identities valid in these particular classes of quasigroups were taken into account.

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