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Multi-step iterative process with errors for common fixed points of a finite family of nonexpansive mappings*

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Abstract. *In this paper, we study a multi-step iterative scheme with errors involving N nonexpansive mappings in the Banach space. Some weak and strong convergence theorems for approximation of common fixed points of nonexpansive mappings are proved using this iteration scheme. The results extend and improve the corresponding results of [1].*

Key words: *nonexpansive mapping, multi-step iteration process with errors, common fixed points*

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1. Introduction and preliminaries

Let K be a nonempty convex subset of a normed linear space E, and let $\{T_i\}_{i=1}^N$ be N self-maps of K. Khan and Fukhar-ud-din [1] introduced the following iterative scheme.

The sequence$\{x_n\}$ defined by

$$\begin{cases} x_1 = x \in K, \\ x_{n+1} = a_n S y_n + b_n x_n + c_n u_n, \\ y_n = a'_n T x_n + b'_n x_n + c'_n v_n, \quad \forall n \geq 1. \end{cases} \tag{1.1}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$ and $\{c'_n\}$ are six real sequences in [0,1] with $0 < \delta \leq a_n,\ a'_n \leq 1 - \delta < 1$, $a_n + b_n + c_n = a'_n + b'_n + c'_n = 1$, and $\{u_n\}, \{v_n\}$ are bounded sequences in K.

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If $c_n = c'_n = 0$, then the above scheme means that

$$\begin{cases} x_1 = x \in K, \\ x_{n+1} = a_n S y_n + b_n x_n, \\ y_n = a'_n T x_n + b'_n x_n, \qquad \forall n \geq 1. \end{cases} \tag{1.2}$$

where $\{a_n\}$, $\{b_n\}$, $\{a'_n\}$ and $\{b'_n\}$ are four real sequences in [0,1] satisfying $a_n + b_n = a'_n + b'_n = 1$. This scheme has been studied by Das and Debata [2] and Takahashi and Tamura [3].

Now, we further generalize the scheme given in (1.1) as follows.

The sequence $\{x_n\}$ is defined by

$$\begin{cases} x_1 = x \in K, \\ x_{n+1} = a_n^{(1)} T_1 x_n^{(1)} + b_n^{(1)} x_n + c_n^{(1)} u_n^{(1)}, \\ x_n^{(1)} = a_n^{(2)} T_2 x_n^{(2)} + b_n^{(2)} x_n + c_n^{(2)} u_n^{(2)}, \\ x_n^{(2)} = a_n^{(3)} T_3 x_n^{(3)} + b_n^{(3)} x_n + c_n^{(3)} u_n^{(3)}, \\ \vdots \\ x_n^{(N-2)} = a_n^{(N-1)} T_{N-1} x_n^{(N-1)} + b_n^{(N-1)} x_n + c_n^{(N-1)} u_n^{(N-1)}, \\ x_n^{(N-1)} = a_n^{(N)} T_N x_n^{(N)} + b_n^{(N)} x_n + c_n^{(N)} u_n^{(N)}, \\ x_n^{(N)} = x_n \qquad \text{for all } n \geq 1. \end{cases} \tag{1.3}$$

The scheme is expressed in a compact form as

$$x_n^{(i-1)} = a_n^{(i)} T_i x_n^{(i)} + b_n^{(i)} x_n + c_n^{(i)} u_n^{(i)}, \quad \textit{for all } n \geq 1, i \in I, \tag{1.4}$$

where $I = \{1, 2, 3, \cdots, N\}$, $x_{n+1} = x_n^{(0)}$, $\{a_n^{(i)}\}$, $\{b_n^{(i)}\}$ and $\{c_n^{(i)}\}$ are three real sequences in [0,1] with $0 < \delta \leq a_n^{(i)} \leq 1 - \delta < 1$, $a_n^{(i)} + b_n^{(i)} + c_n^{(i)} = 1$, and $\{u_n^{(i)}\}$ and $\{v_n^{(i)}\}$ are two bounded sequences in K.

Since their introduction nonexpansive mappings have been extensively studied by many authors in different frames of work. The purpose of this paper is to study the weak and strong convergence of a multi-step iteration scheme (1.4) for N nonexpansive mappings in a uniformly convex Banach space. The results presented in this paper extend and improve the corresponding results of [1] from two nonexpansive mappings to a family of nonexpansive mappings.

To proceed in this direction, we first recall the following definitions.

A Banach space E is said to satisfy the *Opial's condition* if whenever $\{x_n\}$ is a sequence in E which converges weakly to x, then

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\| \text{ for all } y \in E, \text{with } y \neq x$$

A mapping $T : K \to E$ is called *demiclosed* with respect to $y \in E$ if for each sequence $\{x_n\}$ in K and each $x \in E$, $x_n \rightharpoonup x$, and $Tx_n \to y$ imply that $x \in K$ and $Tx = y$.

In the sequel we shall need the following lemmas.

Lemma 1.1 [Schu[4]]. *Suppose that E is a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all positive integers n. Also suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences of E such that $\limsup_{n\to\infty} \|x_n\| \leq d$, $\limsup_{n\to\infty} \|y_n\| \leq d$ and $\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = d$ hold for some $d \geq 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 1.2 [Tan and Xu[5]]. *Let $\{s_n\}$ and $\{t_n\}$ be two nonnegative sequences satisfying*

$$s_{n+1} \leq s_n + t_n \qquad \text{for all } n \geq 1.$$

If $\sum_{n=1}^{\infty} t_n < \infty$, then $\lim_{n\to\infty} s_n$ exists.

Lemma 1.3 [Browder[6]]. *Let E be a uniformly convex Banach space satisfying the Opial's condition and let K be a nonempty closed convex subset of E. Let T be a nonexpansive mapping of K into itself, then $I-T$ is demiclosed with respect to zero.*

2. Main results

In this section, let $F(T)$ denote the set of all fixed points of T.

Lemma 2.1. *Let E be a normed space and K its nonempty bounded convex subset. Let $\{T_i\}_{i=1}^N : K \to K$ be N nonexpansive mappings and let $\{x_n\}$ be the sequence as defined in (1.4) with $\sum_{n=1}^{\infty} c_n^{(i)} < \infty$, $i \in I = \{1, 2, 3, \cdots, N\}$. If $F = \cap_{i=1}^N F(T_i) \neq \emptyset$, then $\lim_{n\to\infty} \|x_n - x^*\|$ exist for all $x^* \in F = \cap_{i=1}^{\infty} F(T_i)$.*

Proof. Since K is bounded, there exists $M > 0$, such that $\|x_n - u_n^{(i)}\| \leq M$, *for all* $i \in I$. Assume that $F = \cap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Let $x^* \in F = \cap_{i=1}^{\infty} F(T_i)$. Then

$$\begin{aligned}
||x_{n+1} - x^*|| &= ||a_n^{(1)} T_1 x_n^{(1)} + b_n^{(1)} x_n + c_n^{(1)} u_n^{(1)} - x^*|| \\
&= ||a_n^{(1)} (T_1 x_n^{(1)} - x^* + c_n^{(1)} (u_n^{(1)} - x_n)) \\
&\qquad + (1 - a_n^{(1)})(x_n - x^* + c_n^{(1)} (u_n^{(1)} - x_n))|| \\
&\leq a_n^{(1)} ||T_1 x_n^{(1)} - x^*|| + (1 - a_n^{(1)})||x_n - x^*|| + c_n^{(1)} ||u_n^{(1)} - x_n|| \\
&\leq a_n^{(1)} ||x_n^{(1)} - x^*|| + (1 - a_n^{(1)})||x_n - x^*|| + c_n^{(1)} M \\
&= a_n^{(1)} ||a_n^{(2)} T_2 x_n^{(2)} + b_n^{(2)} x_n + c_n^{(2)} u_n^{(2)} - x^*|| \\
&\quad + (1 - a_n^{(1)})||x_n - x^*|| + c_n^{(1)} M \\
&= a_n^{(1)} ||a_n^{(2)} (T_2 x_n^{(2)} - x^* + c_n^{(2)} (u_n^{(2)} - x_n)) \\
&\qquad + (1 - a_n^{(2)})(x_n - x^* + c_n^{(2)} (u_n^{(2)} - x_n))|| \\
&\quad + (1 - a_n^{(1)})||x_n - x^*|| + c_n^{(1)} M \\
&\leq a_n^{(1)} (a_n^{(2)} ||x_n^{(2)} - x^*|| + (1 - a_n^{(2)})||x_n - x^*|| + c_n^{(2)} M) \\
&\quad + (1 - a_n^{(1)})||x_n - x^*|| + c_n^{(1)} M \\
&= a_n^{(1)} a_n^{(2)} ||x_n^{(2)} - x^*|| + a_n^{(1)} (1 - a_n^{(2)})||x_n - x^*|| \\
&\quad + (1 - a_n^{(1)})||x_n - x^*|| + a_n^{(1)} c_n^{(2)} M + c_n^{(1)} M
\end{aligned}$$

$$
\begin{aligned}
&= a_n^{(1)}a_n^{(2)}||x_n^{(2)}-x^*|| + (1-a_n^{(1)}a_n^{(2)})||x_n-x^*|| + a_n^{(1)}c_n^{(2)}M + c_n^{(1)}M \\
&\le a_n^{(1)}a_n^{(2)}(a_n^{(3)}||x_n^{(3)}-x^*|| + (1-a_n^{(3)})||x_n-x^*|| + c_n^{(3)}M) \\
&\quad +(1-a_n^{(1)}a_n^{(2)})||x_n-x^*|| + a_n^{(1)}c_n^{(2)}M + c_n^{(1)}M \\
&= a_n^{(1)}a_n^{(2)}a_n^{(3)}||x_n^{(3)}-x^*|| + (1-a_n^{(1)}a_n^{(2)}a_n^{(3)})||x_n-x^*|| \\
&\quad +a_n^{(1)}a_n^{(2)}c_n^{(3)}M + a_n^{(1)}c_n^{(2)}M + c_n^{(1)}M \\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots \\
&\le a_n^{(1)}a_n^{(2)}\cdots a_n^{(N)}||x_n^{(N)}-x^*|| + (1-a_n^{(1)}a_n^{(2)}\cdots a_n^{(N)})||x_n-x^*|| \\
&\quad +a_n^{(1)}a_n^{(2)}\cdots a_n^{(N-1)}c_n^{(N)}M + a_n^{(1)}a_n^{(2)}\cdots a_n^{(N-2)}c_n^{(N-1)}M \\
&\quad +\cdots + a_n^{(1)}c_n^{(2)}M + c_n^{(1)}M \\
&\le a_n^{(1)}a_n^{(2)}\cdots a_n^{(N)}||x_n^{(N)}-x^*|| + (1-a_n^{(1)}a_n^{(2)}\cdots a_n^{(N)})||x_n-x^*|| + M\Sigma_{i=1}^N c_n^{(i)} \\
&= ||x_n-x^*|| + M\Sigma_{i=1}^N c_n^{(i)}.
\end{aligned}
$$

Since $\Sigma_{n=1}^{\infty} c_n^{(i)} < \infty$, hence, using *Lemma 1.2*, we have that

$$\lim_{n\to\infty} ||x_n - x^*|| \text{ exists for each } x^* \in F = \cap_{i=1}^N F(T_i).$$

This completes the proof of *Lemma 2.1*. □

Lemma 2.2. *Let E be a uniformly convex Banach space and K its nonempty bounded convex subset. Let $\{T_i\}_{i=1}^N : K \to K$ be nonexpansive mappings, and $\{x_n\}$ the sequence as defined in (1.4) with $\sum_{n=1}^{\infty} c_n^{(i)} < \infty$, $i = 1,2,3,\cdots,N$. If $F = \cap_{i=1}^N F(T_i) \neq \emptyset$, then $\lim_{n\to\infty} \|T_i x_n - x_n\| = 0$ for all $i \in I$.*

Proof. By *Lemma 2.1*, $\lim_{n\to\infty} ||x_n - x^*||$ exists. Suppose $\lim_{n\to\infty} ||x_n-x^*|| = d$ for some $d \ge 0$. Now,

$$
\begin{aligned}
||x_n^{(i-1)}-x^*|| &= ||a_n^{(i)}T_i x_n^{(i)} + b_n^{(i)}x_n + c_n^{(i)}u_n^{(i)} - x^*|| \\
&= ||a_n^{(i)}(T_i x_n^{(i)} - x^* + c_n^{(i)}(u_n^{(i)}-x_n)) \\
&\qquad +(1-a_n^{(i)})(x_n - x^* + c_n^{(i)}(u_n^{(i)}-x_n))|| \\
&\le a_n^{(i)}||T_i x_n^{(i)} - x^*|| + (1-a_n^{(i)})||x_n-x^*|| + c_n^{(i)}||u_n^{(i)}-x_n|| \\
&\le a_n^{(i)}||x_n^{(i)} - x^*|| + (1-a_n^{(i)})||x_n-x^*|| + c_n^{(i)}M \\
&= a_n^{(i)}||a_n^{(i+1)}T_{i+1}x_n^{(i+1)} + b_n^{(i+1)}x_n + c_n^{(i+1)}u_n^{(i+1)} - x^*|| \\
&\quad +(1-a_n^{(i)})||x_n-x^*|| + c_n^{(i)}M \\
&\le a_n^{(i)}(a_n^{(i+1)}||x_n^{(i+1)}-x^*|| + (1-a_n^{(i+1)})||x_n-x^*|| + c_n^{(i+1)}M) \\
&\quad +(1-a_n^{(i)})||x_n-x^*|| + c_n^{(i)}M \\
&= a_n^{(i)}a_n^{(i+1)}||x_n^{(i+1)}-x^*|| + a_n^{(i)}(1-a_n^{(i+1)})||x_n-x^*|| \\
&\quad +(1-a_n^{(i)})||x_n-x^*|| + a_n^{(i)}c_n^{(i+1)}M + c_n^{(i)}M \\
&= a_n^{(i)}a_n^{(i+1)}||x_n^{(i+1)}-x^*|| + (1-a_n^{(i)}a_n^{(i+1)})||x_n-x^*|| \\
&\quad +a_n^{(i)}c_n^{(i+1)}M + c_n^{(i)}M \\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots \\
&\le a_n^{(i)}a_n^{(i+1)}\cdots a_n^{(N)}||x_n^{(N)}-x^*|| + (1-a_n^{(i)}a_n^{(i+1)}\cdots a_n^{(N)})||x_n-x^*|| \\
&\quad +M\Sigma_{k=i}^N c_n^{(k)} \\
&= ||x_n-x^*|| + M\Sigma_{k=1}^N c_n^{(k)}.
\end{aligned}
$$

Taking limsup on both sides in the above inequality, we have

$$\limsup_{n\to\infty} ||x_n^{(i-1)} - x^*|| \leq d. \tag{2.1}$$

Next, consider

$$\begin{aligned} ||T_i x_n^{(i)} - x^* + c_n^{(i)}(u_n^{(i)} - x_n)|| &\leq ||T_i x_n^{(i)} - x^*|| + c_n^{(i)}||u_n^{(i)} - x_n|| \\ &\leq ||x_n^{(i)} - x^*|| + c_n^{(i)} M \end{aligned}$$

Taking limsup on both sides in the above inequality and then using (2.1), we have that

$$\limsup_{n\to\infty} ||T_i x_n^{(i)} - x^* + c_n^{(i)}(u_n^{(i)} - x_n)|| \leq d \text{ for each } i \in I. \tag{2.2}$$

Also,

$$\begin{aligned} ||x_n - x^* + c_n^{(i)}(u_n^{(i)} - x_n)|| &\leq ||x_n - x^*|| + c_n^{(i)}||u_n^{(i)} - x_n|| \\ &\leq ||x_n - x^*|| + c_n^{(i)} M \end{aligned}$$

gives that

$$\limsup_{n\to\infty} ||x_n - x^* + c_n^{(i)}(u_n^{(i)} - x_n)|| \leq d \text{ for each } i \in I. \tag{2.3}$$

Further, $\lim_{n\to\infty} ||x_{n+1} - x^*|| = d$ means that

$$\lim_{n\to\infty} ||a_n^{(1)}(T_1 x_n^{(1)} - x^* + c_n^{(1)}(u_n^{(1)} - x_n)) + (1 - a_n^{(1)})(x_n - x^* + c_n^{(1)}(u_n^{(1)} - x^*))|| = d$$

Hence, applying *Lemma 1.1*, we get that

$$\lim_{n\to\infty} ||T_1 x_n^{(1)} - x_n|| = 0$$

Next,

$$||x_n - x^*|| \leq ||x_n - T_1 x_n^{(1)}|| + ||T_1 x_n^{(1)} - x^*|| \leq ||x_n - T_1 x_n^{(1)}|| + ||x_n^{(1)} - x^*||.$$

It follows from (2.1) that

$$d \leq \liminf_{n\to\infty} ||x_n^{(1)} - x^*|| \leq \limsup_{n\to\infty} ||x_n^{(1)} - x^*|| \leq d$$

That is,

$$\lim_{n\to\infty} ||x_n^{(1)} - x^*|| = d$$

Now $\lim_{n\to\infty} ||x_n^{(1)} - x^*|| = d$ can be expressed as

$$\lim_{n\to\infty} ||a_n^{(2)}(T_2 x_n^{(2)} - x^* + c_n^{(2)}(u_n^{(2)} - x_n)) + (1 - a_n^{(2)})(x_n - x^* c_n^{(2)}(u_n^{(2)} - x^*))|| = d$$

Moreover, by (2.2) and (2.3) we have that

$$\limsup_{n\to\infty} ||T_2 x_n^{(2)} - x^* + c_n^{(2)}(u_n^{(2)} - x_n)|| \leq d,$$

and

$$\limsup_{n\to\infty} ||x_n - x^* + c_n^{(2)}(u_n^{(2)} - x^*)|| \le d$$

So again by *Lemma 1.1*, we obtain that

$$\lim_{n\to\infty} ||T_2 x_n^{(2)} - x_n|| = 0$$

Now,

$$||x_n - x^*|| \le ||x_n - T_2 x_n^{(2)}|| + ||T_2 x_n^{(2)} - x^*|| \le ||x_n - T_2 x_n^{(2)}|| + ||x_n^{(2)} - x^*||.$$

From (2.1) it follows that

$$d \le \liminf_{n\to\infty} ||x_n^{(2)} - x^*|| \le \limsup_{n\to\infty} ||x_n^{(2)} - x^*|| \le d$$

That is

$$\lim_{n\to\infty} ||x_n^{(2)} - x^*|| = d$$

Using the same method, we get that

$$\begin{array}{ll} \lim_{n\to\infty} ||T_3 x_n^{(3)} - x_n|| = 0, & \lim_{n\to\infty} ||x_n^{(3)} - x^*|| = d \\ \lim_{n\to\infty} ||T_4 x_n^{(4)} - x_n|| = 0, & \lim_{n\to\infty} ||x_n^{(4)} - x^*|| = d \\ \cdots\cdots & \cdots\cdots \\ \lim_{n\to\infty} ||T_N x_n^{(N)} - x_n|| = 0, & \lim_{n\to\infty} ||x_n^{(N)} - x^*|| = d \end{array}$$

i.e.

$$\lim_{n\to\infty} ||T_i x_n^{(i)} - x_n|| = 0, \quad \lim_{n\to\infty} ||x_n^{(i)} - x^*|| = d \quad \text{for all } \ i \in I$$

Now, observe that

$$\begin{aligned} ||x_n - T_i x_n|| &\le ||x_n - T_i x_n^{(i)}|| + ||T_i x_n^{(i)} - T_i x_n|| \\ &\le ||x_n - T_i x_n^{(i)}|| + ||x_n^{(i)} - x_n|| \\ &= ||x_n - T_i x_n^{(i)}|| + ||a_n^{(i+1)} T_{i+1} x_n^{(i+1)} + b_n^{(i+1)} x_n + c_n^{(i+1)} u_n^{(i+1)} - x_n|| \\ &= ||x_n - T_i x_n^{(i)}|| + ||a_n^{(i+1)} (T_{i+1} x_n^{(i+1)} - x_n) + c_n^{(i+1)} (u_n^{(i+1)} - x_n)|| \\ &\le ||x_n - T_i x_n^{(i)}|| + a_n^{(i+1)} ||T_{i+1} x_n^{(i+1)} - x_n|| + c_n^{(i+1)} M \end{aligned}$$

Hence

$$\lim_{n\to\infty} ||x_n - T_i x_n|| = 0 \quad \text{for all } \ i \in I.$$

This completes the proof of *Lemma 2.2.* □

Theorem 2.1. *Let E be a uniformly convex Banach space satisfying the Opial's condition and let K,$\{T_i\}_{i=1}^N$ and $\{x_n\}$ be taken as in* Lemma 2.1. *If $F = \cap_{i=1}^N F(T_i) \neq \varnothing$, then $\{x_n\}$ converges weakly to a common fixed point of $\{T_i\}_{i=1}^N$.*

Proof. Let $x^* \in F = \cap_{i=1}^N F(T_i)$, then as proved in *Lemma 2.1*, $\lim_{n\to\infty} ||x_n - x^*||$ exists. Now we prove that $\{x_n\}$ has a unique weak subsequential limit in $F = \cap_{i=1}^N F(T_i)$. To prove this, let z_1 and z_2 be weak limits of the subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$, respectively. By *Lemma 2.2*, $\lim_{n\to\infty} \|T_i x_n - x_n\| = 0$ and $I - T_i$ is demiclosed with respect to zero by *Lemma 1.3*, therefore we obtain $T_i z_1 = z_1$, i.e., $z_1 \in F = \cap_{i=1}^N F(T_i)$. Again in the same way, we can prove that $z_2 \in F = \cap_{i=1}^N F(T_i)$. Next, we prove the uniqueness. For this suppose that $z_1 \neq z_2$, then by the Opial's condition

$$\begin{aligned}\lim_{n\to\infty} ||x_n - z_1|| &= \lim_{n_i\to\infty} ||x_{n_i} - z_1|| < \lim_{n_i\to\infty} ||x_{n_i} - z_2|| \\ &= \lim_{n\to\infty} ||x_n - z_2|| = \lim_{n_j\to\infty} ||x_{n_j} - z_2|| \\ &< \lim_{n_j\to\infty} ||x_{n_j} - z_1|| = \lim_{n\to\infty} ||x_n - z_1||\end{aligned}$$

This is a contradition. Hence $\{x_n\}$ converges weakly to a point in $F = \cap_{i=1}^N F(T_i)$. This completes the proof of *Theorem 2.1*. □

Now we will prove a strong convergence theorem.

Two mappings $S, T{:}K \to K$, where K is a subset of E, are said to satisfy condition (A) if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0,\ f(r) > 0$ for all $r \in (0,\infty)$ such that $\frac{1}{2}(||x - Tx|| + ||x - Sx||) \geq f(d(x,F))$ for all$x \in K$ where $d(x,F) = \inf\{||x - x^*|| : x^* \in F = F(S) \cap F(T)\}$.

Khan and Fukhar-ud-din [1] approximated a common fixed point of two nonexpansive mappings S and T by iterating scheme (1.1). We modify this condition for mappings $\{T_i\}_{i=1}^N : K \to K$ as follow:

N mappings $\{T_i\}_{i=1}^N : K \to K$ where K is a subset of E, are said to satisfy condition (A') if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0,\ f(r) > 0$ for all $r \in (0,\infty)$ such that $\frac{1}{N}\Sigma_{i=1}^N ||x - T_i x|| \geq f(d(x,F))$ for all $x \in K$ where $d(x,F) = \inf\{||x - x^*|| : x^* \in \cap_{i=1}^N F(T_i)\}$.

Note that condition (A') reduces to condition (A) when $N = 2$. We shall use condition (A') to study the strong convergence of $\{x_n\}$ defined in (1.4).

Theorem 2.2. *Let E be a uniformly convex Banach space and let K, $\{x_n\}$ be taken as in* Lemma 2.1. *Let $\{T_i\}_{i=1}^N : K \to K$ be nonexpansive mappings satisfying condition (A'). If $F = \cap_{i=1}^N F(T_i) \neq \emptyset$, then $\{x_n\}$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^N$.*

Proof. By *Lemma 2.1*, suppose that $\lim_{n\to\infty} ||x_n - x^*|| = d$ for all $x^* \in F = \cap_{i=1}^N F(T_i)$. If $d = 0$, there is nothing to prove. Assume $d > 0$, by *Lemma 2.2* $\lim_{n\to\infty} ||T_i x_n - x_n|| = 0,\ i \in I$. Moreover,

$$||x_{n+1} - x^*|| \leq ||x_n - x^*|| + M\Sigma_{i=1}^N c_n^{(i)},$$

gives that

$$\inf_{x^*\in F} ||x_{n+1} - x^*|| \leq \inf_{x^*\in F} ||x_n - x^*|| + M\Sigma_{i=1}^N c_n^{(i)}.$$

That is,

$$d(x_{n+1}, F) \leq d(x_n, F) + M\Sigma_{i=1}^N c_n^{(i)}$$

gives that $\lim_{n\to\infty} d(x_n, F)$ exists by virtue of *Lemma 1.2*. Now by condition (A'), $\lim_{n\to\infty} f(d(x_n, F)) = 0$. Since f is a nondecreasing function and $f(0) = 0$, therefore $\lim_{n\to\infty} d(x_n, F) = 0$. Now we can take a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and a sequence $\{p_j\} \subset F$ such that $||x_{n_j} - p_j|| < 2^{-j}$. Then following the method of proof of Tan and Xu [5], we get that $\{p_j\}$ is a Cauchy sequence in F and so it converges. Let $p_j \to p$. Since F is closed, therefore $p \in F$ and then $x_{n_j} \to p$. As $\lim_{n\to\infty} ||x_n - x^*||$ exists, $x_n \to p \in F = \cap_{i=1}^{\infty} F(T_i)$. This completes the proof of *Theorem 2.2.* □

Remark 2.1. Theorems 2.1-2.2 *extend and improve the corresponding results of [1].*

References

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