A fixed point theorem for multi-maps satisfying an implicit relation on metrically convex metric spaces

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Abstract. *In this paper, we give a fixed point theorem for multi-valued mapping satisfying an implicit relation on metrically convex metric spaces. This result extends and generalizes some fixed point theorem in the literature.*

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1. Introduction

Let (X, d) be a metric space. Then X is said to be metrically convex if for every pair $x, y \in X, x \neq y$, there is a point $z \in X$ such that $d(x, y) = d(x, z) + d(z, y)$. We need the following lemma in the sequel.

Lemma 1 [[1]]. *Let K be a non-empty and closed subset of a metrically convex metric space X. Then for any $x \in K$ and $y \notin K$, there exists a point $z \in \partial K$ such that $d(x, y) = d(x, z) + d(z, y)$, where ∂K denotes the boundary of K.*

Let $CB(X)$ denote the family of all non-empty closed and bounded subsets of X. Denote for $A, B \in CB(X)$

$$D(A, B) = \inf\{d(a, b) : a \in A, b \in B\}, \tag{1}$$

$$\delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\} \tag{2}$$

and

$$H(A, B) = \max\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\}. \tag{3}$$

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Note that $D(A,B) \leq H(A,B) \leq \delta(A,B)$. Function H is a metric on $CB(X)$ and is called a Hausdorff metric. It is well known that if X is a complete metric space, then so is the metric space $(CB(X),H)$.

Itoh [4] proved a fixed point theorem for non-self maps $F : K \to CB(X)$ satisfying certain contraction condition in terms of Hausdorff metric H on $CB(X)$ under the boundary condition $F(\partial K) \subset K$. Rhoades [7] generalized this result to a wider class of non-self multi-maps on K. Recently Dhage [2] has proved a fixed point theorem for non-self multi-maps on K satisfying a slightly stronger contraction condition than that in Rhoades [7] and under a weaker boundary condition. In *Section 2* of this paper we give an implicit relation and some examples for this relation. In *Section 3*, we prove a fixed point theorem for non-self multi-maps on K satisfying an implicit relation.

2. Implicit relation

Implicit relations on metric space have been used in many articles (see [3], [5], [6], [8]).

Let R_+ be the set of all non-negative real numbers and let $\mathcal{T}$ be the set of all continuous functions $T : R_+^5 \to R$ satisfying the following conditions:

T_1 : $T(t_1,...,t_5)$ is non-decreasing in t_1 and non-increasing in $t_2,...,t_5$.

T_2 : there exist two constants $a,b \geq 0$, $2a+3b<1$ such that the inequality

$$T(u,v,v,w,v+w) \leq 0 \tag{4}$$

implies $u \leq \max\{(a+b)v+bw,(a+b)w+bv\}$.

T_3 : $T(u,0,0,u,u) > 0, T(u,0,u,0,u) > 0$ and $T(u,u,0,0,2u) > 0$, $\forall u > 0$.

Remark 1. *Note that, if $u = w$ in T_2, then the inequality $T(u,v,v,w,v+w) \leq 0$ implies $u \leq \dfrac{a+b}{1-b}v$.*

Now we give some examples.

Example 1. *Let $T(t_1,...t_5) = t_1 - \alpha \max\{t_2,t_3,t_4\} - \beta t_5$, where $\alpha,\beta \geq 0$ and $2\alpha+3\beta<1$.*

T_1 : Obvious. T_2 : Let $T(u,v,v,w,w+v) = u - \alpha\max\{w,v\} - \beta(w+v) \leq 0$. Thus $u \leq \max\{(\alpha+\beta)v+\beta w,(\alpha+\beta)w+\beta v\}$. T_3 : $T(u,0,0,u,u) = T(u,0,u,0,u) = u(1-\alpha-\beta) > 0$ and $T(u,u,0,0,2u) = u(1-\alpha-2\beta) > 0, \forall u > 0$. Therefore $T \in \mathcal{T}$.

Example 2. *Let $T(t_1,...,t_5) = t_1 - m\max\{t_2,t_3,t_4,\frac{1}{2}t_5\}$, where $0 \leq m < \frac{1}{2}$.*

T_1 : Obvious. T_2 : Let $T(u,v,v,w,w+v) = u - m\max\{w,v\} \leq 0$. Thus $u \leq \max\{mw,mv\}$ and so T_2 is satisfying with $a = m$, $b = 0$. T_3 : $T(u,0,0,u,u) = T(u,0,u,0,u) = T(u,u,0,0,2u) = u(1-m) > 0, \forall u > 0$. Therefore $T \in \mathcal{T}$.

Example 3. *Let $T(t_1,...,t_5) = t_1 - (\alpha t_2 + \beta t_3 + \gamma t_4)$, where $\alpha,\beta,\gamma \geq 0, 2\alpha+2\beta+\gamma<1$ and $\alpha+\beta-\gamma \geq 0$.*

T_1 : Obvious. T_2 : Let $T(u,v,v,w,w+v) = u - (\alpha v+\beta v+\gamma w) \leq 0$. Thus $u \leq (\alpha+\beta)v+\gamma w \leq \max\{(\alpha+\beta)v+\gamma w,(\alpha+\beta)w+\gamma v\}$ and so T_2 is satisfying with

$a = \alpha+\beta-\gamma, b = \gamma$. $T_3 : T(u,0,0,u,u) = u(1-\gamma) > 0$, $T(u,0,u,0,u) = u(1-\beta) > 0$ *and* $T(u,u,0,0,2u) = u(1-\alpha) > 0, \forall u > 0$. *Therefore* $T \in \mathcal{T}$.

Example 4. *Let* $T(t_1, ...t_5) = t_1 - \alpha t_2 - \beta \max\{t_3, t_4\} - \gamma t_5$, *where* $\alpha, \beta, \gamma \geq 0$ *and* $2\alpha + 2\beta + 3\gamma < 1$.

T_1 : *Obvious.* T_2 : *Let* $T(u,v,v,w,w+v) = u-\alpha v-\beta\max\{w,v\}-\gamma(w+v) \leq 0$. *Thus* $u \leq \max\{(\alpha+\beta+\gamma)v+\beta w, (\alpha+\beta+\gamma)w+\beta v\}$ *and so* T_2 *is satisfying with* $a = \alpha+\beta+\gamma, b = \gamma$. $T_3 : T(u,0,0,u,u) = T(u,0,u,0,u) = u(1-\beta-\gamma) > 0$ *and* $T(u,u,0,0,2u) = u(1-\alpha-2\gamma) > 0, \forall u > 0$. *Therefore* $T \in \mathcal{T}$.

3. Main result

Now we give our main theorem.

Theorem 1. *Let* (X,d) *be a metrically convex complete metric space and* K *a non-empty closed subset of* X. *Let* $F : K \to CB(X)$ *be a multi-map satisfying*

$$T(\delta(Fx,Fy), d(x,y), D(x,Fx), D(y,Fy), D(x,Fy)+D(y,Fx)) \leq 0, \tag{5}$$

for all $x,y \in K$, *where* $T \in \mathcal{T}$. *Further, if* $Fx \cap K \neq \phi$ *for each* $x \in \partial K$, *then* F *has a fixed point* $p \in K$ *such that* $Fp = \{p\}$ *and* F *is continuous at* p *in the Hausdorff metric on* X.

Proof. Let be arbitrary and consider a sequence $\{x_n\}$ in K as follows: Let $x_0 = x$ and take a point $x_1 \in Fx_0 \cap K$ if $Fx_0 \cap K \neq \phi$. Otherwise choose a point $x_1 \in \partial K$ such that

$$d(x_0, x_1') = d(x_0, x_1) + d(x_1, x_1') \tag{6}$$

for some $x_1' \in Fx_0 \subset X \backslash K$. Similarly, pick $x_2 \in Fx_1 \cap K$ if $Fx_1 \cap K \neq \phi$, otherwise choose a point $x_2 \in \partial K$ such that

$$d(x_1, x_2') = d(x_1, x_2) + d(x_2, x_2') \tag{7}$$

for some $x_2' \in Fx_1 \subset X \backslash K$. Continuing this way we have

$$x_{n+1} \in Fx_1 \cap K \text{ if } Fx_1 \cap K \neq \phi, \tag{8}$$

or $x_{n+1} \in \partial K$ satisfying

$$d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+1}') = d(x_n, x_{n+1}') \tag{9}$$

for some $x_{n+1}' \in Fx_n \subset X \backslash K$.

By the construction of $\{x_n\}$ we can write

$$\{x_n\} = P \cup Q \subset K, \tag{10}$$

where

$$P = \{x_n \in \{x_n\} : x_n \in Fx_{n-1}\} \tag{11}$$

and

$$Q = \{x_n \in \{x_n\} : x_n \in \partial K, x_n \notin Fx_{n-1}\}. \tag{12}$$

Then for any two consecutive terms x_n, x_{n+1} of the sequence $\{x_n\}$, we observe that there are only the following three possibilities:

(i) $x_n, x_{n+1} \in P$,

(ii) $x_n \in P, x_{n+1} \in Q$, and

(iii) $x_n \in Q$, $x_{n+1} \in P$.

First we show that $\{x_n\}$ is a Cauchy sequence in K. Now for any $x_n, x_{n+1} \in \{x_n\}$, we have the following estimates:

Case 1: Suppose that $x_n, x_{n+1} \in P$. Now since $x_{n-1}, x_n \in K$, we can use the inequality (5), then we have

$$\begin{aligned} T\,(\delta(Fx_{n-1}, Fx_n), d(x_{n-1}, x_n), D(x_{n-1}, Fx_{n-1}), \\ D(x_n, Fx_n), D(x_{n-1}, Fx_n) + D(x_n, Fx_{n-1})) \leq 0 \end{aligned} \tag{13}$$

and so

$$T(d(x_n, x_{n+1}), d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_n)+d(x_n, x_{n+1})) \leq 0. \tag{14}$$

From *Remark 1* there exist two constants $a, b \geq 0$, $2a+3b < 1$ such that $d(x_n, x_{n+1}) \leq \dfrac{a+b}{1-b} d(x_{n-1}, x_n)$, where $\dfrac{a+b}{1-b} < \frac{1}{2}$ since $2a + 3b < 1$.

Case 2: Let $x_n \in P$ and $x_{n+1} \in Q$. Then $d(x_n, x_{n+1}) + d(x_{n+1}, x'_{n+1}) = d(x_n, x'_{n+1})$ for some $x'_{n+1} \in Fx_n$. Clearly,

$$\begin{cases} d(x_n, x_{n+1}) \leq d(x_n, x'_{n+1}) \\ d(x_n, x'_{n+1}) \leq \delta(Fx_{n-1}, Fx_n). \end{cases} \tag{15}$$

Now following arguments similar to those in Case 1, we obtain

$$d(x_n, x'_{n+1}) \leq \frac{a+b}{1-b} d(x_{n-1}, x_n). \tag{16}$$

From (15) and (16) it follows that

$$d(x_n, x_{n+1}) \leq \frac{a+b}{1-b} d(x_{n-1}, x_n). \tag{17}$$

Case 3: Suppose that $x_n \in Q$ and $x_{n+1} \in P$. Note that then $x_{n-1} \in P$ and there is a point $x'_n \in Fx_{n-1}$ such that

$$d(x_{n-1}, x_n) + d(x_n, x'_n) = d(x_{n-1}, x'_n). \tag{18}$$

Now,

$$\begin{aligned} d(x_n, x_{n+1}) &\leq d(x_n, x'_n) + d(x'_n, x_{n+1}) \\ &\leq d(x_{n-1}, x'_n) + \delta(Fx_{n-1}, Fx_n). \end{aligned} \tag{19}$$

On the other hand, since $x_{n-1}, x_n \in K$, we can use inequality (5), then we have

$$\begin{aligned} T\,(\delta(Fx_{n-1}, Fx_n), d(x_{n-1}, x_n), D(x_{n-1}, Fx_{n-1}), D(x_n, Fx_n), \\ D(x_{n-1}, Fx_n) + D(x_n, Fx_{n-1})) \leq 0. \end{aligned} \tag{20}$$

Thus we have

$$T\,(d(x'_n,x_{n+1}),d(x_{n-1},x_n),d(x_{n-1},x'_n),d(x_n,x_{n+1}),\\ d(x_{n-1},x_{n+1})+d(x_n,x'_n))\le 0. \tag{21}$$

Using (18) we have

$$T\,(d(x'_n,x_{n+1}),d(x_{n-1},x'_n),d(x_{n-1},x'_n),d(x_n,x_{n+1}),\\ d(x_{n-1},x_n)+d(x_n,x_{n+1})+d(x_n,x'_n))\le 0 \tag{22}$$

and so

$$T\,(d(x'_n,x_{n+1}),d(x_{n-1},x'_n),d(x_{n-1},x'_n),d(x_n,x_{n+1}),\\ d(x_{n-1},x'_n)+d(x_n,x_{n+1}))\le 0. \tag{23}$$

From T_2 there exist two constants $a,b\ge 0$, $2a+3b<1$ such that

$$d(x'_n,x_{n+1})\le\max\left\{\begin{array}{c}(a+b)d(x_{n-1},x'_n)+bd(x_n,x_{n+1}),\\ (a+b)d(x_n,x_{n+1})+bd(x_{n-1},x'_n)\end{array}\right\}. \tag{24}$$

Therefore using (19) we have

$$d(x_n,x_{n+1})\le d(x_{n-1},x'_n)+\max\left\{\begin{array}{c}(a+b)d(x_{n-1},x'_n)+bd(x_n,x_{n+1}),\\ (a+b)d(x_n,x_{n+1})+bd(x_{n-1},x'_n)\end{array}\right\} \tag{25}$$

Now from (16) in Case 2 applied to $n-1$, we have

$$d(x_{n-1},x'_n)\le\frac{a+b}{1-b}d(x_{n-2},x_{n-1}) \tag{26}$$

and hence from (25)

$$\begin{aligned} d(x_n,x_{n+1}) &\le \frac{a+b}{1-b}d(x_{n-2},x_{n-1})\\ &\quad+\max\left\{\begin{array}{c}\dfrac{(a+b)^2}{1-b}d(x_{n-2},x_{n-1})+bd(x_n,x_{n+1}),\\ (a+b)d(x_n,x_{n+1})+\dfrac{b(a+b)}{1-b}d(x_{n-2},x_{n-1})\end{array}\right\}\\ &=\max\left\{\begin{array}{c}\dfrac{(a+b)(1+a+b)}{1-b}d(x_{n-2},x_{n-1})+bd(x_n,x_{n+1}),\\ (a+b)d(x_n,x_{n+1})+\dfrac{(1+b)(a+b)}{1-b}d(x_{n-2},x_{n-1})\end{array}\right\}. \end{aligned}$$

This implies

$$d(x_n,x_{n+1})\le\max\{\frac{(a+b)(1+a+b)}{(1-b)^2},\frac{(1+b)(a+b)}{(1-b)(1-a-b)}\}d(x_{n-2},x_{n-1}). \tag{27}$$

Note that $q=\max\{\frac{(a+b)(1+a+b)}{(1-b)^2},\frac{(1+b)(a+b)}{(1-b)(1-a-b)}\}<1$. To see this, $2a+3b<1$ yields

$$\begin{aligned} &a+b<1-2b-a\\ &\Rightarrow a+b+ab+b^2<1-2b-a+ab+b^2\\ &\Rightarrow\frac{(a+b+ab+b^2)}{1-2b-a+ab+b^2}<1\\ &\Rightarrow\frac{(a+b)(1+b)}{(1-b)(1-a-b)}<1. \end{aligned} \tag{28}$$

Similarly, again from $2a + 3b < 1$ we have

$$\begin{aligned} & 1 > 3b \\ & \Rightarrow \frac{3}{2} > \frac{1}{1-b} \\ & \Rightarrow 1 > \frac{1}{2(1-b)} + \frac{1}{4} \\ & \Rightarrow 1 > (\frac{1}{1-b} + \frac{1}{2})\frac{1}{2} \\ & \Rightarrow 1 > (\frac{1}{1-b} + \frac{a+b}{1-b})\frac{a+b}{1-b} \\ & \Rightarrow 1 > \frac{(1+a+b)(a+b)}{(1-b)^2}. \end{aligned} \tag{29}$$

Now for any $n \in N$, we have

$$d(x_{2n}, x_{2n+1}) \leq q d(x_{2n-2}, x_{2n}) \leq q^n d(x_0, x_1). \tag{30}$$

Since n is arbitrary, one has

$$d(x_n, x_{n+1}) \leq q^n d(x_0, x_1). \tag{31}$$

Then from Cases 1-3, it easily follows that $\{x_n\}$ is a Cauchy sequence in K. As K is closed, it is complete and hence $\lim_n x_n = p$ exists. We show that p is a fixed point of F. Without loss of generality, we may assume that $x_{n+1} \in Fx_n$ for some $n \in N$. Then using (5) we have

$$T(\delta(Fx_n, Fp), d(x_n, p), D(x_n, Fx_n), D(p, Fp), D(x_n, Fp) + D(p, Fx_n)) \leq 0, \tag{32}$$

and letting $n \to \infty$ we have

$$T(D(p, Fp), 0, 0, D(p, Fp), D(p, Fp)) \leq 0. \tag{33}$$

From T_3 we have $D(p, Fp) = 0$ and so $p \in Fp$.

Further, we have

$$T(\delta(Fp, Fp), d(p, p), D(p, Fp), D(p, Fp), D(p, Fp) + D(p, Fp)) \leq 0, \tag{34}$$

and so

$$T(\delta(Fp, Fp), 0, 0, 0, 0) \leq 0. \tag{35}$$

Again from T_1 and T_3 we have $\delta(Fp, Fp) = 0$ and so $Fp = \{p\}$.

To show the uniqueness of p, let $q (\neq p)$ be another fixed point of F. Then

$$T(\delta(Fp, Fq), d(p, q), D(p, Fp), D(q, Fq), D(p, Fq) + D(q, Fp)) \leq 0, \tag{36}$$

and so

$$T(d(p, q), d(p, q), 0, 0, 2d(p, q)) \leq 0. \tag{37}$$

Again from T_3 we have $p = q$.

Finally, we prove the continuity of F at p. Let $\{z_n\} \subset X$ be any sequence such that $z_n \to p$ as $n \to \infty$. Now

$$T(\delta(Fz_n, Fp), d(z_n, p), D(z_n, Fz_n), D(p, Fp), D(z_n, Fp) + D(p, Fz_n)) \leq 0 \quad (38)$$

and letting $n \to \infty$ we have

$$T(\lim_n H(Fz_n, Fp), 0, \lim_n H(Fp, Fz_n), 0, \lim_n H(Fp, Fz_n)) \leq 0. \quad (39)$$

From T_3 we have $\lim_n H(Fz_n, Fp) = 0$, showing that F is continuous at p. This completes the proof. □

Remark 2. *Theorem 1 of [2] follows from* Example 1 *and* Theorem 1.

Remark 3. *We can have some new results from other examples and* Theorem 1.

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