On the family of B-spline surfaces obtained by knot modification

IMRE JUHÁSZ* AND MIKLÓS HOFFMANN†

Abstract. *B-spline surfaces are piecewisely defined surfaces where the section points of the domain of definition are called knots. In [2] the authors proved some theorems in terms of knot modification of B-spline curves. Here we generalize these results for one- and two-parameter family of surfaces. An additional result concerning a higher order contact of these surfaces and an envelope is also proved.*

Key words: *B-spline surface, knots, envelope*

AMS subject classifications: 68U05

Received November 2, 2005 Accepted February 8, 2006

1. Introduction

Piecewise polynomial curves and their generalizations, the tensor-product surfaces are standard approximation methods in several applications and play a central role in computer-aided geometric design and other fields as well. The most well-known type of them is the B-spline curve and surface. The basic definitions are as follows (c.f. [10]):

Definition 1. *The recursive function* $N_j^k(u)$ *given by the equations*

$$N_j^1(u) = \begin{cases} 1 \text{ if } u \in [u_j, u_{j+1}), \\ 0 \text{ otherwise} \end{cases}$$
$$N_j^k(u) = \frac{u-u_j}{u_{j+k-1}-u_j} N_j^{k-1}(u) + \frac{u_{j+k}-u}{u_{j+k}-u_{j+1}} N_{j+1}^{k-1}(u)$$

is called a normalized B-spline basis function of order k *(degree* $k-1$*). The numbers* $u_j \leq u_{j+1} \in \mathbb{R}$ *are called knot values or simply knots, and* $0/0 \doteq 0$ *by definition.*

Definition 2. *The curve* $\mathbf{a}(u)$ *defined by*

$$\mathbf{a}(u) = \sum_{r=0}^{n} N_r^k(u)\, \mathbf{d}_r\,, u \in [u_{k-1}, u_{n+1}]$$

*Department of Descriptive Geometry, University of Miskolc, H-3515 Miskolc-Egyetemváros, Hungary, e-mail: `agtji@gold.uni-miskolc.hu`

†Institute of Mathematics and Computer Science, Károly Eszterházy College, P.O. Box 43, H-3300 Eger, Hungary, e-mail: `hofi@ektf.hu`

is called a B-spline curve of order k (degree $k-1$), where $N_r^k(u)$ is the r^{th} normalized B-spline basis function of order k, for the evaluation of which knots $u_0, u_1, \dots, u_{n+k}$ are necessary. Points $\mathbf{d}_l$ are called control points or de Boor points, while the polygon formed by these points is called a control polygon.

Definition 3. *The surface $\mathbf{s}(u,v)$ defined by*

$$\mathbf{s}(u,v) = \sum_{r=0}^{n}\sum_{s=0}^{m} N_r^k(u)\, N_s^l(v)\, \mathbf{d}_{rs}\,, \quad u \in [u_{k-1}, u_{n+1}]\,, \quad v \in [v_{l-1}, v_{m+1}] \qquad (1)$$

is called a B-spline surface of order (k,l) (degree $(k-1, l-1)$), where $N_r^k(u)$ and $N_s^l(v)$ are the r^{th} and s^{th} normalized B-spline basis functions, for the evaluation of which the knots $u_0, u_1, \dots, u_{n+k}$ and $v_0, v_1, \dots, v_{m+l}$ are necessary, respectively. Points $\mathbf{d}_{rs}$ are called control points or de Boor-points, while the mesh formed by these points is calleda control mesh.

A *patch* of the B-spline surface can be written as

$$\mathbf{s}_{i,j}(u,v) = \sum_{r=i-k+1}^{i}\ \sum_{s=j-l+1}^{j} N_r^k(u)\, N_s^l(v)\, \mathbf{d}_{rs}, \quad u \in [u_i, u_{i+1})\,, \quad v \in [v_j, v_{j+1})\,.$$

As one can observe, the surface is uniquely given by its orders (k,l), control points $\mathbf{d}_{rs}$, and knot vectors $\mathbf{u}$ and $\mathbf{v}$. It is an obvious fact, that any alteration of these data affects the shape of the surface. Modification of the orders yields discrete positions of the surface. The effect of reposition of control points for the shape of the surface has been described in several papers ([3], [4], [7], [9]). Geometric aspects of knot alteration of the surface are discussed in [1] and [8].

If a control point or a knot value is modified, points of the curve or surface move along special curves called paths. If two knot values are simultaneously altered, surface points move along path-surfaces. In terms of B-spline curves, these paths are examined for control point reposition in [10] and for knot alteration in [6]. In terms of surfaces, paths and path-surfaces are discussed in [1]. Extending the domain of definition of paths to $[-\infty, \infty]$ we proved some results for curves in [2]. In the next section these results are generalized for surfaces.

2. Extended paths and path-surfaces

The u and v isoparametric curves of surface (1) are B-spline curves of order k and l, respectively. Fixing the parameter value $\tilde{u} \in [u_{k-1}, u_{n+1}]$ we obtain the isoparametric curve

$$\mathbf{b}(v) = \sum_{s=0}^{m} N_s^l(v)\, \mathbf{b}_s(\tilde{u})\,, v \in [v_{l-1}, v_{m+1}] \qquad (2)$$

control points of which are

$$\mathbf{b}_s(\tilde{u}) = \sum_{r=0}^{n} N_r^k(\tilde{u})\, \mathbf{d}_{rs} \qquad (3)$$

(The u isoparametric curves of the surface can be described in an analogous way.)

When a knot value u_p is modified, the shape of each u isoparametric curve of the surface changes on the range $[u_{p-k+1}, u_{p+k-1}] \cap [u_{k-1}, u_{n+1}]$, i.e. a strip of the surface is modified. Thus the alteration of the knot u_p effects at most $2(k-1)(m-l+2)$ patches on the rectangular domain

$$D_1 = ([u_{p-k+1}, u_{p+k-1}] \cap [u_{k-1}, u_{n+1}]) \times [v_{l-1}, v_{m+1}] \, .$$

The effect of the alteration of a knot v_q on the shape of the surface can be described analogously. If knots u_p and v_q are simultaneously altered, then u_p effects the surface on the domain D_1, v_q effects on

$$D_2 = ([v_{q-l+1}, v_{q+l-1}] \cap [v_{l-1}, v_{m+1}]) \times [u_{k-1}, u_{n+1}]$$

therefore surface patches on the domain $D_1 \cup D_2$ are modified but both knots influence only patches on the domain $D_1 \cap D_2$ which results in the modification of at most $4(k-1)(l-1)$ patches.

Modifying the knot u_p individual points of the surface move along curves called *paths* and are denoted by $\mathbf{s}(u, v, u_p)$. For these paths the following holds (c.f. [1]):

Theorem 1. *Modifying the knot u_p paths of points of patches*

$$\begin{array}{l} \mathbf{s}_{p-g-1,h}(u,v) \;\; \textit{and} \;\; \mathbf{s}_{p+g,h}(u,v) \\ g = 0, 1, \ldots, k-2; \;\; h = l-1, l, \ldots, m; (u,v) \in D_1 \end{array}$$

are rational polynomial curves of order $k-g$.

Analogous results can be obtained when modifying a knot v_q.

If knots u_p and v_q are simultaneously altered, their joint effect modifies the patches on the domain $D_1 \cap D_2$. Points of these patches move on surfaces depending on the parameters u_p and v_q, which will be refered to as *path–surfaces* and denoted by $\mathbf{s}(u, v, u_p, v_q)$. For these path-surfaces the following results have been proved in [1]:

Theorem 2. *If knots u_p and v_q are simultaneously modified, path-surfaces of points of patches*

$$\begin{array}{cc} \mathbf{s}_{p-g-1,q-h-1}(u,v) & \mathbf{s}_{p-g-1,q+h}(u,v) \\ \mathbf{s}_{p+g,q-h-1}(u,v) & \mathbf{s}_{p+g,q+h}(u,v) \\ \multicolumn{2}{c}{g = 0, 1, \ldots, k-2; \;\; h = 0, 1, \ldots, l-2} \end{array}$$

are rational polynomial surfaces in u_p and v_q of order $(k-g, l-h)$.

Corollary 1. *Path-surfaces of points of patches*

$$\begin{array}{ll} \mathbf{s}_{p-k+1,q-h-1}(u,v) & \mathbf{s}_{p-k+1,q+h}(u,v) \\ \mathbf{s}_{p+k-2,q-h-1}(u,v) & \mathbf{s}_{p+k-2,q+h}(u,v) \\ h = 0, 1, \ldots, l-2 & \end{array}$$

and

$$\begin{array}{ll} \mathbf{s}_{p-g-1,q-l+1}(u,v) & \mathbf{s}_{p+g,q-l+1}(u,v) \\ \mathbf{s}_{p-g-1,q+l-2}(u,v) & \mathbf{s}_{p+g,q+l-2}(u,v) \\ g = 0, 1, \ldots, k-2 & \end{array}$$

are ruled surfaces, but these are not cylinders in general, since the direction of their generators varies point by point.

Corollary 2. *Path-surfaces of patches*

$$\begin{matrix} \mathbf{s}_{p-k+1,q-l+1}(u,v) & \mathbf{s}_{p-k+1,q+l-2}(u,v) \\ \mathbf{s}_{p+k-2,q-l+1}(u,v) & \mathbf{s}_{p+k-2,q+l-2}(u,v) \end{matrix}$$

are double ruled surfaces, i.e. both isoparametric curves of them are straight lines. Moreover, these double ruled surfaces are affine transforms of the surface obtained by the bilinear combination of the corresponding control point quadruples.

Summarizing the preceding results one can observe that simultaneously modifying knot values u_p and v_q a topologically quadrilateral part of the surface is affected containing at most $4(k-1)(l-1)$ number of patches around the patch $\mathbf{s}_{p,q}(u,v)$. Points of these patches move on rational polynomial surfaces the degree of which decreases in a central symmetrical way as we consider further patches in both parameter directions. Along the sides of this array of patches path-surfaces are ruled surfaces, while at the four corners one can find bilinear path-surfaces.

We can extend the domain of paths and path-surfaces, i.e. we can let $u_p \notin [u_{p-1}, u_{p+1}]$ and $v_q \notin [v_{q-1}, v_{q+1}]$. This leads to the extension of paths and path-surfaces for which the following two limit theorems hold.

Theorem 3. *Extended paths obtained by the modification of the knot v_q have the following properties:*

$$\lim_{v_q \to -\infty} \mathbf{s}(u,v,v_q) = \sum_{r=0}^{n} N_r^k(u)\,\mathbf{d}_{rq}, \lim_{v_q \to \infty} \mathbf{s}(u,v,v_q) = \sum_{r=0}^{n} N_r^k(u)\,\mathbf{d}_{r,q-l}$$
$$u \in [u_{k-1}, u_{n+1}], v \in [v_{q-1}, v_{q+1})$$

Proof. Fixing the parameter value $\tilde{u} \in [u_{k-1}, u_{n+1}]$ we obtain the isoparametric curve (2) the control points of which are of the form (3). For the extended paths of points of the isoparametric curve $\mathbf{b}(v)$ which are obtained by modifying the knot v_q, theorem 5 of [2] is valid, therefore

$$\lim_{v_q \to -\infty} \mathbf{b}(v,v_q) = \mathbf{b}_q(\tilde{u}) \text{ and } \lim_{v_q \to \infty} \mathbf{b}(v,v_q) = \mathbf{b}_{q-l}(\tilde{u})$$
$$v \in [v_{q-1}, v_{q+1})$$

where $\mathbf{b}_q(\tilde{u})$ and $\mathbf{b}_{q-l}(\tilde{u})$ can be written in the form (3). When modifying the value $\tilde{u}$ these control points describe the B-spline curves

$$\mathbf{b}_q(u) = \sum_{r=0}^{n} N_r^k(u)\,\mathbf{d}_{rq}, u \in [u_{k-1}, u_{n+1}] \tag{4}$$

and

$$\mathbf{b}_{q-l}(u) = \sum_{r=0}^{n} N_r^k(u)\,\mathbf{d}_{r,q-l}, u \in [u_{k-1}, u_{n+1}]$$

respectively, which completes the proof. □

For the extension of paths obtained by alterating a knot u_p, we can show a similar property, i.e.

$$\lim_{u_p\to-\infty}\mathbf{s}\left(u,v,u_p\right)=\sum_{s=0}^{m}N_s^l\left(v\right)\mathbf{d}_{ps},\ \lim_{u_p\to\infty}\mathbf{s}\left(u,v,u_p\right)=\sum_{s=0}^{m}N_s^l\left(v\right)\mathbf{d}_{p-k,s}$$
$$u\in\left[u_{p-1},u_{p+1}\right),v\in\left[v_{l-1},v_{m+1}\right].$$

Theorem 4. *If u_p and v_q simultaneously tend to infinity, then points of paths converge to a control point of the surface, more precisely*

$$\lim_{\substack{u_p\to-\infty\\ v_q\to-\infty}}\mathbf{s}\left(u,v,u_p,v_q\right)=\mathbf{d}_{pq},\ \lim_{\substack{u_p\to-\infty\\ v_q\to\infty}}\mathbf{s}\left(u,v,u_p,v_q\right)=\mathbf{d}_{p,q-l}$$
$$\lim_{\substack{u_p\to\infty\\ v_q\to-\infty}}\mathbf{s}\left(u,v,u_p,v_q\right)=\mathbf{d}_{p-k,q},\ \lim_{\substack{u_p\to\infty\\ v_q\to\infty}}\mathbf{s}\left(u,v,u_p,v_q\right)=\mathbf{d}_{p-k,q-l}$$
$$u\in\left[u_{p-1},u_{p+1}\right),v\in\left[v_{q-1},v_{q+1}\right)$$

Proof. Applying Theorem 5 of [2] to curve (4) the equality

$$\lim_{u_p\to-\infty}\mathbf{b}_q\left(u\right)=\mathbf{d}_{pq},u\in\left[u_{p-1},u_{p+1}\right)$$

is obtained. Thus considering the equality

$$\lim_{\substack{u_p\to-\infty\\ v_q\to-\infty}}\mathbf{s}\left(u,v,u_p,v_q\right)=\lim_{u_p\to-\infty}\left(\lim_{v_q\to-\infty}\mathbf{s}\left(u,v,u_p,v_q\right)\right)$$

the proof is completed.

The other three cases can be shown analogously. □

3. Envelope of paths and path-surfaces

In [1] we proved that by modifying a knot u_p of a surface the one-parameter family of surfaces

$$\mathbf{s}\left(u,v,u_p\right)=\sum_{r=0}^{n}\sum_{s=0}^{m}N_r^k\left(u,u_p\right)N_s^l\left(v\right)\mathbf{d}_{rs}$$
$$u\in\left[u_{k-1},u_{n+1}\right],v\in\left[v_{l-1},v_{m+1}\right],u_p\in\left[u_{p-1},u_{p+1}\right] \qquad (5)$$

with family parameter u_p possessing an envelope which is a B-spline surface itself and which is of the following form

$$\mathbf{e}\left(w,v\right)=\sum_{r=p-k+1}^{p-1}\sum_{s=0}^{m}N_r^{k-1}\left(w\right)N_s^l\left(v\right)\mathbf{d}_{rs},\ w\in\left[w_{p-1},w_p\right],v\in\left[v_{l-1},v_{m+1}\right]. \qquad (6)$$

If two knots, u_p and v_q, are simultaneously modified, then the two-parameter family of B-spline surfaces

$$\mathbf{s}\left(u,v,u_p,v_q\right)=\sum_{r=0}^{n}\sum_{s=0}^{m}N_r^k\left(u,u_p\right)N_s^l\left(v,v_q\right)\mathbf{d}_{rs}$$
$$u\in\left[u_{k-1},u_{n+1}\right],v\in\left[v_{l-1},v_{m+1}\right],u_p\in\left[u_{p-1},u_{p+1}\right],v_q\in\left[v_{q-1},v_{q+1}\right] \qquad (7)$$

with family parameters u_p and v_q also possessing an envelope:

$$\mathbf{e}(w,z)=\sum_{r=p-k+1}^{p-1}\sum_{s=q-l+1}^{q-1}N_r^{k-1}(w)\,N_s^{l-1}(z)\,\mathbf{d}_{rs},\; w\in[w_{p-1},w_p]\,,z\in[z_{q-1},z_q]\,. \quad (8)$$

First we will prove that the contact of the members of families and envelopes is of higher order. Then we verify that these surfaces are also envelopes of the corresponding paths and path-surfaces.

Theorem 5. *For the elements of the one-parameter family of surfaces (5) with family parameter u_p, and for surface (6) the equalities*

$$\frac{\partial^i}{\partial w^i}\mathbf{e}(w,v)\Big|_{w=u_p}=\frac{k-1-i}{k-1}\frac{\partial^i}{\partial u^i}\mathbf{s}(u,v,u_p)\Big|_{u=u_p},i\geq 0 \quad (9)$$

and

$$\frac{\partial^i}{\partial v^i}\mathbf{e}(u_p,v)=\frac{\partial^i}{\partial v^i}\mathbf{s}(u_p,v,u_p)\,,i\geq 0 \quad (10)$$

are fulfilled.

Proof. Each u isoparametric curve of the surface $\mathbf{s}(u,v)$ generates a one-parameter family of curves of order k when a knot u_p is altered, for which equality (9) is obviously fulfilled due to the results of [1].

On the basis of equations (2) and (3), the v isoparametric curve on the surface $\mathbf{s}(u,v,\tilde{u}_p)$, $\tilde{u}_p\in[u_{p-1},u_{p+1}]$, that belongs to the fixed u parameter value $u=\tilde{u}_p$, is determined by control points

$$\mathbf{b}_s(\tilde{u}_p)=\sum_{r=0}^{n}N_r^k(\tilde{u}_p)\,\mathbf{d}_{rs}.$$

Fixing the parameter value $w=\tilde{u}_p$ of the surface $\mathbf{e}(w,v)$, we obtain a v isoparametric curve on the surface which curve is determined by control points

$$\mathbf{c}_s(\tilde{u}_p)=\sum_{r=p-k+1}^{p-1}N_r^{k-1}(\tilde{u}_p)\,\mathbf{d}_{rs}.$$

According to Theorem 3 of [6], $\mathbf{b}_s(\tilde{u}_p)=\mathbf{c}_s(\tilde{u}_p)$, therefore the isoparametric curves $\mathbf{e}(\tilde{u}_p,v)$ and $\mathbf{s}(\tilde{u}_p,v,\tilde{u}_p)$ are identical for all permissible $\tilde{u}_p$, i.e. equality (10) holds. □

Corollary 3. *Surface $\mathbf{e}(w,v)$ touches the elements of one-parameter family of surfaces $\mathbf{s}(u,v,u_p)$ along their isoparametric curves $\mathbf{s}(\tilde{u}_p,v,\tilde{u}_p)$, therefore $\mathbf{e}(w,v)$ is an envelope of the family.*

The effect of the alteration of a knot v_q on the shape of the surface, analogously can be described. A similar statement holds for simultaneous modification of two knots.

Theorem 6. *For the elements of two-parameter family of surfaces (7) with family parameters u_p and v_q, and for surface (8) equalities*

$$\frac{\partial^i}{\partial w^i}\mathbf{e}(w,z)\Big|_{w=u_p}=\frac{k-1-i}{k-1}\frac{\partial^i}{\partial u^i}\mathbf{s}(u,v,u_p,v_q)\Big|_{u=u_p},i\geq 0$$

and

$$\frac{\partial^i}{\partial z^i}\mathbf{e}\left(w,z\right)\Big|_{z=v_q} = \frac{l-1-i}{l-1}\frac{\partial^i}{\partial v^i}\mathbf{s}\left(u,v,u_p,v_q\right)\Big|_{v=v_q}, i \geq 0$$

are fulfilled.

Proof. Consecutive application of *Theorem 5* verifies the statement. □

Now we prove that surfaces (6) and (8) are also envelopes of the paths and path-surfaces obtained by altering one or two knots, respectively.

Theorem 7. *Modifying the knot u_p of the surface $\mathbf{s}\left(u,v\right)$, members of two-parameter family of paths $\mathbf{s}\left(u,v,u_p\right)$ (with family parameters u and v) and surface (6) have G^1-contact, i.e. $\forall(\widetilde{u},\widetilde{v}),(\widetilde{u}\in[u_{p-1},u_{p+1}],\widetilde{v}\in[v_{l-1},v_{m+1}])$ they have a point in common at $u_p=\widetilde{u}$ and the tangent line of the path $\mathbf{s}\left(\tilde{u},\tilde{v},u_p\right)$ at this point lies in the tangent plane of surface (6).*

Proof. Fixing the parameter value $\widetilde{v}$ of $\mathbf{s}\left(u,v\right)$ the isoparametric curve $\mathbf{s}\left(u,\widetilde{v}\right)$ is obtained. Altering the knot u_p of this curve between u_{p-1} and u_{p+1} the paths $\mathbf{s}\left(u,\widetilde{v},u_p\right)$ fulfill the equation

$$\mathbf{e}\left(\widetilde{u},\widetilde{v}\right)=\mathbf{s}\left(\widetilde{u},\widetilde{v},\widetilde{u}\right), \quad \forall\widetilde{u}\in[u_{p-1},u_{p+1}]$$

due to *Theorem 5.* Applying Theorem 4 of [2] one can easily show that for every fixed parameters $(\widetilde{u},\widetilde{v})$

$$\frac{\partial}{\partial w}\mathbf{e}\left(w,v\right)\Big|_{w=\widetilde{u},v=\widetilde{v}} = (2-k)\,\frac{\partial}{\partial u_p}\mathbf{s}\left(\widetilde{u},\widetilde{v},u_p\right)\Big|_{u_p=\widetilde{u}}$$

holds which completes the proof. □

Theorem 8. *Modifying simultaneously knot values u_p and v_q of the surface $\mathbf{s}\left(u,v\right)$, members of two-parameter family of path-surfaces $\mathbf{s}\left(u,v,u_p,v_q\right)$ (with family parameters u and v) and surface (8) have G^1-contact, i.e. $\forall(\widetilde{u},\widetilde{v})$, $(\widetilde{u}\in[u_{p-1},u_{p+1}],\widetilde{v}\in[v_{q-1},v_{q+1}])$ they have a point in common at $u_p=\widetilde{u},v_q=\widetilde{v}$ and their tangent planes coincide at this point.*

Proof. Consecutive application of *Theorem 7* yields the equations

$$\begin{aligned}
\mathbf{e}\left(\widetilde{u},\widetilde{v}\right)&=\mathbf{s}\left(\widetilde{u},\widetilde{v},\widetilde{u},\widetilde{v}\right)\\
\frac{\partial}{\partial w}\mathbf{e}\left(w,z\right)\Big|_{w=\widetilde{u},z=\widetilde{v}} &= (2-k)\,\frac{\partial}{\partial u_p}\mathbf{s}\left(\widetilde{u},\widetilde{v},u_p,v_q\right)\Big|_{u_p=\widetilde{u},v_q=\widetilde{v}}\\
\frac{\partial}{\partial z}\mathbf{e}\left(w,z\right)\Big|_{w=\widetilde{u},z=\widetilde{v}} &= (2-l)\,\frac{\partial}{\partial v_q}\mathbf{s}\left(\widetilde{u},\widetilde{v},u_p,v_q\right)\Big|_{u_p=\widetilde{u},v_q=\widetilde{v}} \qquad \forall(\widetilde{u},\widetilde{v}),
\end{aligned}$$

which completes the proof. □

Corollary 4. *Surface (8) is an envelope of two-parameter family of path-surfaces $\mathbf{s}\left(u,v,u_p,v_q\right)$.*

Acknowledgements

The authors wish to thank the Hungarian National Research Fund (OTKA) grant No. T 048523 for the financial support of this research.

References

[1] M. Hoffmann, I. Juhász, *Geometric Aspects of Knot Modification of B-spline Surfaces*, Journal for Geometry and Graphics **6**(2002), 141-149.

[2] M. Hoffmann, I. Juhász, *On the paths of B-spline curves obtained by the modification of a knot*, Publicationes Mathematicae **65**(2004), 193-203.

[3] S.-M. Hu, Y.-F. Li, T. Ju, X. Zhu, *Modifying the shape of NURBS surfaces with geometric constraints*, Computer-Aided Design **33**(2001), 903-912.

[4] J. Ishida, *The general B-spline interpolation method and its application to the modification of curves and surfaces*, Computer-Aided Design **29**(1997), 779-790.

[5] I. Juhász, M. Hoffmann, *Modifying a knot of B-spline curves*, Computer Aided Geometric Design **20**(2003), 243-245.

[6] I. Juhász, M. Hoffmann, *The effect of knot modifications on the shape of B-spline curves*, Journal for Geometry and Graphics **5**(2001), 111-119.

[7] M. Kimura, T. Saito, M. Shinya, *Surface deformation with differential geometric structures*, Computer Aided Geometric Design **13**(1996), 243-256.

[8] Y. Li, G. Wang, *On Knot Modification of B-Spline and NURBS Surface*, Journal of Computer-Aided Design & Computer Graphics **17**(2005), 986-989.

[9] L. Piegl, *Modifying the shape of rational B-spline. Part 2: Surfaces*, Computer Aided Design, **21**(1989), 538-546.

[10] L. Piegl, W. Tiller, *The NURBS Book*, Springer-Verlag, 1995.