

Vektorsko rješenje jednog zadatka

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Sažetak. U članku se analizira jedan planimetrijski zadatak objavljen u MFL-u, br. 174 (1993-94). Zadatak se rješava elementarno primjenom vektorskog računa.

Ključne riječi: planimetrija, vektori

Vector solution of one task

Abstract. The paper analyzes one planimetry task published in MFL, No. 174 (1993-94). The task is solved by using vector calculus.

Key words: planimetry, vectors

Vjerni čitatelji naših matematičkih časopisa pronaći će u MFL-u br. 174 (1993-94) zgodan planimetrijski zadatak i dva njegova rješenja. Naime, radi se o zadatku br. 2265 i ovdje bih želio izložiti još jedno, vektorsko, rješenje toga zadatka. Rješenje je također elementarno i lako razumljivo svim srednjoškolicima.

Zadatak 1. Iz vrha A kvadrata $ABCD$ unutar njega povučene su dvije zrake na koje su spuštene okomice BK , BL , DM i DN iz vrhova B i D . Dokažite da su dužine \overline{KL} i \overline{MN} okomite i da imaju jednake duljine.

Rješenje. Da bi se uvjerali u okomitost vektora \overline{MN} i \overline{LK} dovoljno je pokazati da je

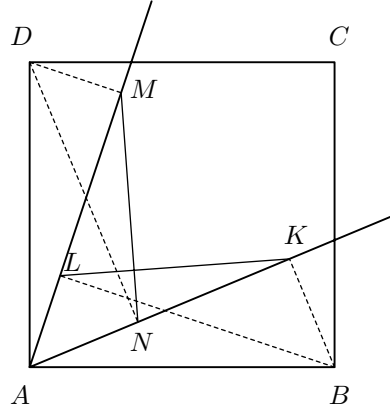
$$\overline{MN} \cdot \overline{LK} = 0.$$

Sa slike je očito sljedeće: $\overline{AK} = \lambda \cdot \overline{AN}$, $\overline{AN} = \lambda_1 \cdot \overline{AB} + \lambda_2 \cdot \overline{AD}$, $\overline{AM} = \mu \cdot \overline{AL}$ te $\overline{AL} = \mu_1 \cdot \overline{AB} + \mu_2 \cdot \overline{AD}$.

Sada računamo:

$$\begin{aligned} \overline{MN} \cdot \overline{LK} &= \\ &= (-\overline{AM} + \overline{AN})(-\overline{AL} + \overline{AK}) \\ &= (\lambda_1 \overline{AB} + \lambda_2 \overline{AD} - \mu(\mu_1 \overline{AB} + \mu_2 \overline{AD})) \\ &\quad \cdot (\lambda(\lambda_1 \overline{AB} + \lambda_2 \overline{AD}) - \mu_1 \overline{AB} - \mu_2 \overline{AD}) \\ &= ((\lambda_1 - \mu\mu_1)\overline{AB} + (\lambda_2 - \mu\mu_2)\overline{AD}) \\ &\quad \cdot ((\lambda\lambda_1 - \mu_1)\overline{AB} + (\lambda\lambda_2 - \mu_2)\overline{AD}) \\ &= ((\lambda_1 - \mu\mu_1)(\lambda\lambda_1 - \mu_1) + (\lambda_2 - \mu\mu_2)(\lambda\lambda_2 - \mu_2)) |\overline{AB}|^2 \end{aligned} \quad (1)$$

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Slika 1.

Budući da (1) za sada više ne možemo pojednostavljivati, iskoristimo neke uvjete iz zadatka pa ćemo joj se kasnije vratiti.

$$\text{i) } \overrightarrow{BK} \cdot \overrightarrow{AK} = 0$$

$$\begin{aligned} &\Rightarrow \left(-\overrightarrow{AB} + \lambda(\lambda_1 \overrightarrow{AB} + \lambda_2 \overrightarrow{AD})\right) \cdot \left(\lambda\lambda_1 \overrightarrow{AB} + \lambda\lambda_2 \overrightarrow{AD}\right) = 0 \\ &\Rightarrow \left((\lambda\lambda_1 - 1)\overrightarrow{AB} + \lambda\lambda_2 \overrightarrow{AD}\right) \cdot \left(\lambda\lambda_1 \overrightarrow{AB} + \lambda\lambda_2 \overrightarrow{AD}\right) = 0 \\ &\Rightarrow (\lambda\lambda_1(\lambda\lambda_1 - 1) + \lambda^2\lambda_2^2) \cdot |\overrightarrow{AB}|^2 = 0 \\ &\Rightarrow \lambda\lambda_1(\lambda\lambda_1 - 1) + \lambda^2\lambda_2^2 = 0 \\ &\Rightarrow \lambda(\lambda_1(\lambda\lambda_1 - 1) + \lambda\lambda_2^2) = 0 \end{aligned}$$

Za $\lambda = 0$ je $K \equiv A$, $N \equiv D$ pa je $\overrightarrow{MD} \perp \overrightarrow{AL}$, što je trivijalno. Za $\lambda \neq 0$ je $\lambda_1(\lambda\lambda_1 - 1) + \lambda\lambda_2^2 = 0$ odakle je $\lambda\lambda_1^2 + \lambda\lambda_2^2 = \lambda_1$, tj.

$$\lambda(\lambda_1^2 + \lambda_2^2) = \lambda_1. \quad (2)$$

$$\text{ii) } \overrightarrow{BL} \cdot \overrightarrow{AL} = 0$$

$$\begin{aligned} &\Rightarrow \left(-\overrightarrow{AB} + \mu_1 \overrightarrow{AB} + \mu_2 \overrightarrow{AD}\right) \cdot \left(\mu_1 \overrightarrow{AB} + \mu_2 \overrightarrow{AD}\right) = 0 \\ &\Rightarrow \left((\mu_1 - 1)\overrightarrow{AB} + \mu_2 \overrightarrow{AD}\right) \cdot \left(\mu_1 \overrightarrow{AB} + \mu_2 \overrightarrow{AD}\right) = 0 \\ &\Rightarrow (\mu_1(\mu_1 - 1) + \mu_2^2) \cdot |\overrightarrow{AB}|^2 = 0 \end{aligned}$$

odakle slijedi

$$\mu_1^2 + \mu_2^2 = \mu_1. \quad (3)$$

$$\text{iii) } \overrightarrow{DM} \cdot \overrightarrow{AM} = 0$$

$$\begin{aligned} &\Rightarrow \left(-\overrightarrow{AD} + \mu(\mu_1 \overrightarrow{AB} + \mu_2 \overrightarrow{AD})\right) \cdot \left(\mu\mu_1 \overrightarrow{AB} + \mu\mu_2 \overrightarrow{AD}\right) = 0 \\ &\Rightarrow \left(\mu\mu_1 \overrightarrow{AB} + (\mu\mu_2 - 1)\overrightarrow{AD}\right) \cdot \left(\mu\mu_1 \overrightarrow{AB} + \mu\mu_2 \overrightarrow{AD}\right) = 0 \\ &\Rightarrow (\mu^2\mu_1^2 + \mu\mu_2(\mu\mu_2 - 1)) \cdot |\overrightarrow{AB}|^2 = 0 \\ &\Rightarrow \mu(\mu\mu_1^2 + \mu_2(\mu\mu_2 - 1)) = 0 \end{aligned}$$

Za $\mu = 0$ je $M \equiv A$, $L \equiv B$ pa je $\overrightarrow{AN} \perp \overrightarrow{BK}$, što je trivijalno. Iz $\mu \neq 0$ slijedi

$$\mu(\mu_1^2 + \mu_2^2) = \mu_2. \quad (4)$$

$$\text{iv) } \overrightarrow{DN} \cdot \overrightarrow{AN} = 0$$

$$\begin{aligned} &\Rightarrow \left(-\overrightarrow{AD} + \lambda_1 \overrightarrow{AB} + \lambda_2 \overrightarrow{AD}\right) \cdot \left(\lambda_1 \overrightarrow{AB} + \lambda_2 \overrightarrow{AD}\right) = 0 \\ &\Rightarrow \left(\lambda_1 \overrightarrow{AB} + (\lambda_2 - 1) \overrightarrow{AD}\right) \cdot \left(\lambda_1 \overrightarrow{AB} + \lambda_2 \overrightarrow{AD}\right) = 0 \end{aligned}$$

odakle slijedi

$$\lambda_1^2 + \lambda_2^2 = \lambda_2 \quad (5)$$

Vratimo se sada jednakosti (1).

$$\begin{aligned} \overrightarrow{MN} \cdot \overrightarrow{LK} &= \\ &= ((\lambda_1 - \mu\mu_1)(\lambda\lambda_1 - \mu_1) + (\lambda_2 - \mu\mu_2)(\lambda\lambda_2 - \mu_2)) |\overrightarrow{AB}|^2 \\ &= (\lambda\lambda_1^2 - \lambda_1\mu_1 - \lambda\mu\lambda_1\mu_1 + \mu\mu_1^2 + \lambda\lambda_2^2 - \mu_2\lambda_2 - \lambda\mu\lambda_2\mu_2 + \mu\mu_2^2) \cdot |\overrightarrow{AB}|^2 \\ &= (\lambda(\lambda_1^2 + \lambda_2^2) + \mu(\mu_1^2 + \mu_2^2) - \lambda\mu(\lambda_1\mu_1 + \lambda_2\mu_2) - (\lambda_1\mu_1 + \mu_2\lambda_2)) \cdot |\overrightarrow{AB}|^2 \\ &= (\lambda(\lambda_1^2 + \lambda_2^2) + \mu(\mu_1^2 + \mu_2^2) - (1 + \lambda\mu)(\lambda_1\mu_1 + \lambda_2\mu_2)) \cdot |\overrightarrow{AB}|^2 \\ &= (\lambda_1 + \mu_2 - (\lambda_1\mu_1 + \lambda_2\mu_2) - \lambda\mu(\lambda_1\mu_1 + \lambda_2\mu_2)) \cdot |\overrightarrow{AB}|^2 \end{aligned} \quad (6)$$

Sada je

$$\begin{aligned} \lambda\mu &= \left| \begin{array}{l} \text{Koristimo} \\ (2) \text{ i } (4) \end{array} \right| = \frac{\lambda_1}{\lambda_1^2 + \lambda_2^2} \cdot \frac{\mu_2}{\mu_1^2 + \mu_2^2} \\ &= \frac{\lambda_1\mu_2}{(\lambda_1^2 + \lambda_2^2)(\mu_1^2 + \mu_2^2)} = \left| \begin{array}{l} \text{Koristimo} \\ (3) \text{ i } (5) \end{array} \right| \\ &= \frac{\lambda_1\mu_2}{\lambda_2\mu_1} \end{aligned} \quad (7)$$

Jednakost (6) dalje pišemo

$$\begin{aligned} \overrightarrow{MN} \cdot \overrightarrow{LK} &= \\ &= (\lambda_1 + \mu_2 - (\lambda_1\mu_1 + \lambda_2\mu_2) - \lambda\mu(\lambda_1\mu_1 + \lambda_2\mu_2)) \cdot |\overrightarrow{AB}|^2 \\ &= \frac{1}{\lambda_2\mu_1} (\lambda_1\lambda_2\mu_1 + \lambda_2\mu_1\mu_2 - \lambda_1\lambda_2\mu_1^2 - \mu_1\mu_2\lambda_2^2 - \lambda_1^2\mu_1\mu_2 - \lambda_1\lambda_2\mu_2^2) \cdot |\overrightarrow{AB}|^2 \\ &= \frac{1}{\lambda_2\mu_1} (\lambda_1\lambda_2\mu_1 + \lambda_2\mu_1\mu_2 - \mu_1\mu_2(\lambda_1^2 + \lambda_2^2) - \lambda_1\lambda_2(\mu_1^2 + \mu_2^2)) \cdot |\overrightarrow{AB}|^2 \\ &= \frac{1}{\lambda_2\mu_1} (\lambda_1\lambda_2\mu_1 + \lambda_2\mu_1\mu_2 - \mu_1\mu_2\lambda_2 - \lambda_1\lambda_2\mu_1) \cdot |\overrightarrow{AB}|^2 \\ &= \frac{1}{\lambda_2\mu_1} \cdot 0 \cdot |\overrightarrow{AB}|^2 = 0 \end{aligned}$$

Dakle, vrijedi $\overrightarrow{MN} \cdot \overrightarrow{LK} = 0$ pa je i $\overrightarrow{MN} \perp \overrightarrow{LK}$.

Da bismo pokazali jednakost dužina \overline{MN} i \overline{LK} pretpostavit ćemo da je naš kvadrat upisan u koordinatni sustav, npr. tako da se vrh A nalazi u ishodištu, a

stranice \overline{AB} i \overline{AD} leže na x odnosno y osi. Bez smanjenja općenitosti možemo pretpostaviti da je kvadrat jedinični, tj. \overline{AB} možemo poistovjetiti s \vec{i} , a \overline{AD} s \vec{j} . Vratimo li se za tren na početak rješenja vidimo da je $\overline{MN} = (\lambda_1 - \mu\mu_1)\overline{AB} + (\lambda_2 - \mu\mu_2)\overline{AD}$ i $\overline{LK} = (\lambda\lambda_1 - \mu_1)\overline{AB} + (\lambda\lambda_2 - \mu_2)\overline{AD}$.

Iz $|\overline{MN}| = |\overline{LK}|$ slijedi $|\overline{MN}|^2 = |\overline{LK}|^2$, tj.

$$(\lambda_1 - \mu\mu_1)^2 + (\lambda_2 - \mu\mu_2)^2 = (\lambda\lambda_1 - \mu_1)^2 + (\lambda\lambda_2 - \mu_2)^2$$

odakle, sređivanjem, slijedi

$$\lambda_1^2 + \lambda_2^2 - \lambda^2(\lambda_1^2 + \lambda_2^2) + \mu^2(\mu_1^2 + \mu_2^2) - (\mu_1 + \mu_2) + 2(\lambda - \mu)(\lambda_1\mu_1 + \lambda_2\mu_2) = 0.$$

Zbog relacija (2) i (4) imamo

$$\lambda_1^2 + \lambda_2^2 - \frac{\lambda_1^2}{\lambda_1^2 + \lambda_2^2} + \frac{\mu_2^2}{\mu_1^2 + \mu_2^2} - (\mu_1 + \mu_2) + 2(\lambda - \mu)(\lambda_1\mu_1 + \lambda_2\mu_2) = 0.$$

Sada je zbog (3) i (4)

$$\lambda_2 - \frac{\lambda_1^2}{\lambda_2} + \frac{\mu_2^2}{\mu_1} - \mu_1 + 2(\lambda - \mu)(\lambda_1\mu_1 + \lambda_2\mu_2) = 0. \quad (8)$$

Iz (2) i (4) je

$$\lambda - \mu = \frac{\lambda_1}{\lambda_1^2 + \lambda_2^2} - \frac{\mu_2}{\mu_1^2 + \mu_2^2} = \frac{\lambda_1}{\lambda_2} - \frac{\mu_2}{\mu_1} = \frac{\lambda_1\mu_1 - \mu_2\lambda_2}{\lambda_2\mu_1}. \quad (9)$$

Koristeći (9) u (8) imamo

$$\lambda_2 - \frac{\lambda_1^2}{\lambda_2} + \frac{\mu_2^2}{\mu_1} - \mu_1 + 2 \frac{\lambda_1\mu_1 - \mu_2\lambda_2}{\lambda_2\mu_1} (\lambda_1\mu_1 + \lambda_2\mu_2) = 0,$$

odnosno,

$$\frac{1}{\lambda_2\mu_1} (\lambda_2^2\mu_1 - \lambda_1^2\mu_1 + \mu_2^2\lambda_2 - \lambda_2\mu_1^2 + 2\lambda_1^2\mu_1^2 - 2\mu_2^2\lambda_2^2) = 0.$$

Konačno, koristeći (3) i (5) dobivamo

$$\begin{aligned} \frac{1}{\lambda_2\mu_1} (\lambda_2^2(\mu_1^2 + \mu_2^2) - \lambda_1^2(\mu_1^2 + \mu_2^2) + \mu_2^2(\lambda_1^2 + \lambda_2^2) - \\ - \mu_1^2(\lambda_1^2 + \lambda_2^2) + 2\lambda_1^2\mu_1^2 - 2\mu_2^2\lambda_2^2) = 0. \end{aligned}$$

Dakle, pretpostavka $|\overline{MN}| = |\overline{LK}|$ je točna i time smo u potpunosti riješili zadatak.

Teško je vjerovati da bi bilo kome, u prvu ruku, palo na um rješavati ovaj zadatak na ovaj način. Upravo ta netipičnost rješenja me ponukala da ga objavim u našem časopisu.