Professional paper
Accepted 29. 12. 2004.

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Butterflies in the Isotropic Plane

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ABSTRACT

A real affine plane A_2 is called an isotropic plane I_2, if in A_2 a metric is induced by an absolute $\{f,F\}$, consisting of the line at infinity f of A_2 and a point $F \in f$. In this paper the well-known Butterfly theorem has been adapted for the isotropic plane. For the theorem that we will further-on call an Isotropic butterfly theorem, four proofs are given.

Key words: isotropic plane, butterfly theorem

MSC 2000: 51N025

Leptiri u izotropnoj ravnini

SAŽETAK

Realna afina ravnina A_2 se naziva izotropnom ravninom I_2 ako je metrika u A_2 inducirana apsolutnom figurom $\{f,F\}$, koja se sastoji od neizmjerno dalekog pravca f ravnine A_2 i točke $F \in f$. U ovom je radu poznati Leptirov teorem smješten u izotropnu ravninu. Za taj teorem, kojeg od sada nazivamo Izotropnim leptirovim teoremom, dana su četiri dokaza.

Ključne riječi: izotropna ravnina, leptirov teorem

1 Isotropic Plane

Let $P_2(\mathbf{R})$ be a real projective plane, f a real line in P_2, and $A_2 = P_2 \backslash f$ the associated affine plane. The *isotropic plane* $I_2(\mathbf{R})$ is a real affine plane A_2 where the metric is introduced with a real line $f \subset P_2$ and a real point F incidental with it. The ordered pair $\{f,F\}$, $F \in f$ is called *absolute figure* of the isotropic plane $I_2(\mathbf{R})$ ([3], [5]). In the affine model, where

$$x = x_1/x_0, \quad y = x_2/x_0, \tag{1}$$

the absolute figure is determined by the *absolute line* $f \equiv x_0 = 0$, and the *absolute point* F (0:0:1). All projective transformations that are keeping the absolute figure fixed form a 5-parametric group

$$G_5 \left\{ \begin{array}{l} \overline{x} = c_1 + c_4 x \\ \overline{y} = c_2 + c_3 x + c_5 y \end{array} \right. , \quad \begin{array}{c} c_1, c_2, c_3, c_4, c_5 \in \mathbf{R} \\ \& \quad c_4 c_5 \neq 0. \end{array} \tag{2}$$

We call it *the group of similarities* of isotropic plane.

Defining in I_2 the usual metric quantities such as the distance between two points, the angle between two lines etc., we look for the subgroup of G_5 for those quantities to be invariant. In such a way one obtains the fundamental group of transformations that are the mappings of the form:

$$G_3 \left\{ \begin{array}{l} \overline{x} = c_1 + x \\ \overline{y} = c_2 + c_3 x + y \end{array} \right. . \tag{3}$$

It is called *the motion group* of isotropic plane. Hence, the group of isotropic motions consists of translations and rotations, that is

$$\left\{ \begin{array}{l} \overline{x} = c_1 + x \\ \overline{y} = c_2 + y \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} \overline{x} = x \\ \overline{y} = c_3 x + y \end{array} \right. .$$

In the affine model, rotation is understood as stretching along the y-axis.

2 Terms of Elementary Geometry within I_2

We will first define some terms and point out some properties of triangles and circles in I_2 that are going to be used further on. The geometry of I_2 could be seen for example in Sachs [3], or Strubecker [5].

Isotropic straight line, parallel points, isotropic distance, isotropic span

All straight lines through the point F are called *isotropic straight lines* (isotropic lines). All the other straight lines are simply called *straight lines*. Two points A, B ($A \neq B$) are called *parallel* if they are incidental with the same isotropic line. For two no parallel points $A(a_1,a_2)$, $B(b_1,b_2)$, the *isotropic distance* is defined by $d(A,B) := b_1 - a_1$. Note that the isotropic distance is directed. For two parallel points $A(a_1,a_2)$, $B(b_1,b_2)$, $a_1 = b_1$, the quantity known as *isotropic spann* is defined by $s(A,B) := b_2 - a_2$. A straight line p through two points A and B will be denoted by $p \equiv A \vee B$, or simply $p \equiv AB$.

Invariants of a pair of straight lines

Each no isotropic straight line $g \subset I_2$ can be written in the normal form $y = ux + v$, that is, in line coordinates, $g(u, v)$. For two straight lines $g_1(u_1, v_1)$, $g_2(u_2, v_2)$ the *isotropic angle*is defined by $\varphi = \angle(g_1, g_2) := u_2 - u_1$. Note that the isotropic angle is directed as well. The Euclidean meaning of the isotropic angle can be understood from the affine model that is given in figure 1.

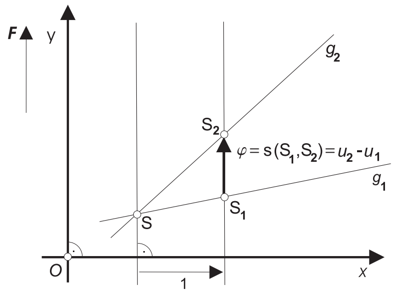

Fig. 1

For two parallel straight lines $g_1(u_1, v_1)$, $g_2(u_1, v_2)$ there exists an isotropic invariant defined by $\varphi * (g_1, g_2) := v_2 - v_1$ (see fig. 2).

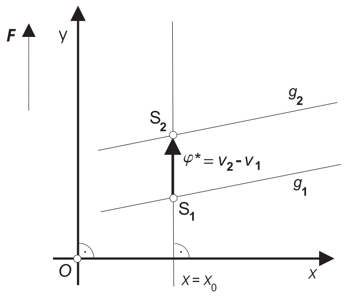

Fig. 2

Isotropic normal

An *isotropic normal* to the straight line $g(u, v)$ in the point $P(p_1, p_2)$, $P \notin g$ is an isotropic line through P. Inversely holds as well, i.e. each straight line $g \subset I_2$ is a normal for each isotropic straight line. Denoting by S the point of intersection of the isotropic normal in the point P with the straight line g, the isotropic distance of the point P from the line g is given by $d(P, g) := s(S, P) = p_2 - s_2 = p_2 - up_1 - v$ (see fig. 3).

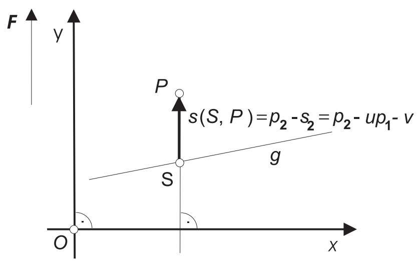

Fig. 3

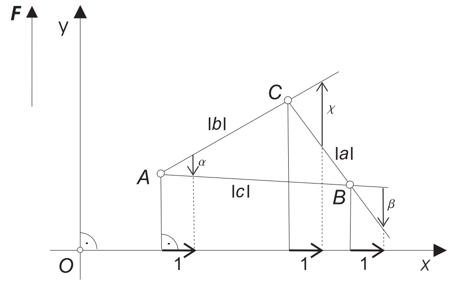

Fig. 4

Triangles and circles

Under a *triangle* in I_2 an ordered set of three no collinear points $\{A, B, C\}$ is understood. A, B, C are called *vertices*, and $a := B \vee C$, $b := C \vee A$, $c := A \vee B$ *sides* of a triangle. A triangle is called *allowable* if no one of its sides is isotropic. In a allowable triangle the *lengths* of the sides are defined by $|a| := d(B, C)$, $|b| := d(C, A)$, $|c| := d(A, B)$, with $|a| \neq 0$, $|b| \neq 0$, $|c| \neq 0$. For the directed angles we have $\alpha := \angle(b, c) \neq 0$, $\beta := \angle(c, a) \neq 0$, $\gamma := \angle(a, b) \neq 0$ (see figure 4).

Isotropic altitudes h_a, h_b, h_c associated with sides a, b, and c are isotropic straight lines passing through the vertices A, B, C, i.e. normals to the sides a, b, and c. Their lengths are defined by $|h_a| := s(L(A), A)$, where $L(A) = a \cap h_a$, etc. The Euclidian meaning is given in figure 5.

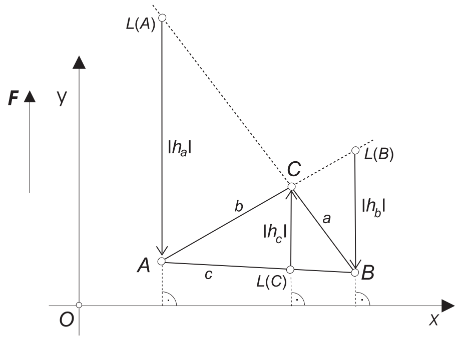

Fig. 5

An *isotropic circle* (*parabolic circle*) is a regular 2^{nd} order curve in $P_2(\mathbf{R})$ which touches the absolute line f in the absolute point F. According to the group G_3 of motions of the isotropic plane there exists in I_2 a three parametric family of isotropic circles, given by $y = Rx^2 + \alpha x + \beta$, $R \neq 0$, $\alpha, \beta \in \mathbf{R}$. Using transformations from G_3, each isotropic circle can be reduced in the normal form $y = Rx^2$, $R \neq 0$. R is a G_3 invariant and it is called the *isotropic radius* of the parabolic circle.

3 The Isotropic Butterfly Theorem

Theorem 1 (Euclidean version) *Let M be the midpoint of a chord PQ of the circle, through which two other chords AB and CD are drawn; AD cuts PQ at X and BC cuts PQ at Y. M is also the midpoint of XY.*

This theorem has been proved in a series of books and papers (e.g. [1], [2], [4]).

Theorem 2 (Isotropic version) *Let M be the midpoint of a chord $\overrightarrow{PQ}$ of the parabolic circle, through which two other chords $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are drawn; $\overrightarrow{AD}$ cuts $\overrightarrow{PQ}$ at X and $\overrightarrow{BC}$ cuts $\overrightarrow{PQ}$ at Y. M is also the midpoint of $\overrightarrow{XY}$.*

Proof 1
The point coordinates are: $P(p_1,p_2)$, $Q(q_1,q_2)$, $M(m_1,m_2)$, $X(x_1,x_2)$, $Y(y_1,y_2)$, with $p_1 \neq q_1$, since $\overrightarrow{PQ}$ is a chord and as a such a no isotropic line, wherefrom we derive that $x_1 \neq y_1 \neq m_1$ must be fulfilled as well. Let us drop perpendiculars h_1, h_2 from X, and g_1, g_2 from Y on AB and CD. Let's also denote

$$\begin{array}{l} d(P,M) = d(M,Q) = |s|, \\ d(X,M) = |x|, \quad d(M,Y) = |y|, \end{array} \tag{4}$$

$$\begin{array}{ll} H_1 = h_1 \cap AM, & H_2 = h_2 \cap DM, \\ G_1 = g_1 \cap MB, & G_2 = g_2 \cap MC. \end{array} \tag{5}$$

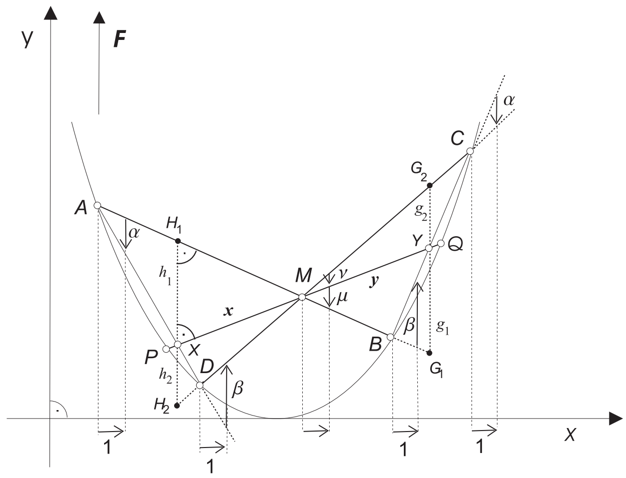

Fig. 6: The Isotropic butterfly theorem in the affine model

As first we need the following:

Lemma 1 *Let P, Q, $P \neq Q$, be two points on a parabolic circle k, and $A \neq P$, $A \neq Q$, any other point on the same circle k. The isotropic angle $\varphi = \angle\left(\overrightarrow{PA},\overrightarrow{QA}\right)$ does not depend on the position of point A.*

The proof is given in [3, p. 32].

Lemma 2 *The relations*

$$\frac{|a|}{\alpha} = \frac{|b|}{\beta} = \frac{|c|}{\chi}, \quad |h_a| = |c|\beta, \quad |h_b| = |a|\chi, \quad |h_c| = |b|\alpha$$

hold for every allowable triangle.

The proof is given in [3, p. 28].

Lemma 3 *Let k be a parabolic circle in I_2, a point $P \in I_2$, $P \notin k$, and S_1, S_2 two points of intersection of a no isotropic straight line g through P with k. The product $f(P) := d(P,S_1) \cdot d(P,S_2)$ doesn't depend of the line g, but only of k and P.*

The proof is given in [3, p. 38].

Let's now continue the proof of the isotropic Butterfly theorem.
According to lemma 1,

$$\alpha = \angle\left(\overrightarrow{AB},\overrightarrow{AD}\right) = \alpha' = \angle\left(\overrightarrow{CB},\overrightarrow{CD}\right),$$

and

$$\beta = \angle\left(\overrightarrow{DA},\overrightarrow{DC}\right) = \beta' = \angle\left(\overrightarrow{BA},\overrightarrow{BC}\right). \tag{6}$$

We will also need

$$\mu = \angle\left(\overrightarrow{XM},\overrightarrow{MA}\right) = \mu' = \angle\left(\overrightarrow{YM},\overrightarrow{MB}\right),$$

and

$$\nu = \angle\left(\overrightarrow{DM},\overrightarrow{MX}\right) = \nu' = \angle\left(\overrightarrow{CM},\overrightarrow{MY}\right). \tag{7}$$

Let's apply furthermore lemma 2 on the following pairs of allowable triangles:
1st) $\triangle AXM$ & $\triangle MBY$, 2nd) $\triangle XDM$ & $\triangle MYC$,
3rd) $\triangle AXM$ & $\triangle MYC$, 4th) $\triangle XDM$ & $\triangle MBY$,
marking sides, angles and altitudes as given in figure 7.

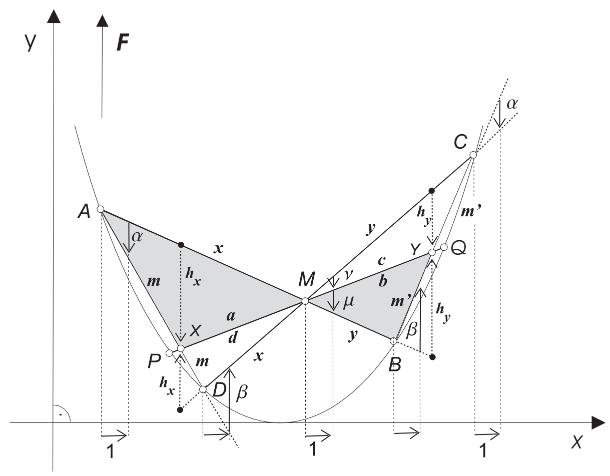

Fig. 7

1st) $\triangle AXM \quad \Rightarrow \quad \dfrac{|x|}{\angle\left(\overrightarrow{AX},\overrightarrow{XM}\right)} = \dfrac{|a|}{\alpha} = \dfrac{|m|}{\mu},$
$|h_x| = |a| \cdot \mu;$

$\triangle MBY \quad \Rightarrow \quad \dfrac{|y|}{\angle\left(\overrightarrow{BY},\overrightarrow{YM}\right)} = \dfrac{|m'|}{\mu} = \dfrac{|b|}{\beta},$
$|h_y| = |b| \cdot \mu;$

$\Rightarrow \dfrac{|h_x|}{|h_y|} = \dfrac{|a|}{|b|}$, and using marks from fig. 6 we get

$$\frac{|x|}{|y|} = \frac{|h_1|}{|g_1|}. \tag{8}$$

2nd) $\triangle XDM \quad \Rightarrow \quad \dfrac{|x|}{\angle\left(\overrightarrow{MX},\overrightarrow{XD}\right)} = \dfrac{|d|}{\beta} = \dfrac{|m|}{\nu},$
$|h_y| = |m| \cdot \beta = |d| \cdot \nu;$

$\triangle MYC \quad \Rightarrow \quad \dfrac{|y|}{\angle\left(\overrightarrow{MY},\overrightarrow{YC}\right)} = \dfrac{|c|}{\alpha} = \dfrac{|m'|}{\nu},$
$|h_y| = |m'| \cdot \alpha = |c| \cdot \nu;$

$\Rightarrow \dfrac{|h_x|}{|h_y|} = \dfrac{|d|}{|c|}$, and using marks from fig. 6 we have

$$\frac{|x|}{|y|} = \frac{|h_2|}{|g_2|}. \tag{9}$$

Analogously, for the third pair of triangles we get

$$\frac{|h_1|}{|g_2|} = \frac{d\,(A,X)}{d\,(Y,C)}. \tag{10}$$

Finally, for the fourth pair of triangles we have

$$\frac{|h_2|}{|g_1|} = \frac{d\,(X,D)}{d\,(B,Y)}. \tag{11}$$

From (4), (8), (9), (10), (11), and lemma 3 one computes

$$\begin{aligned}
\frac{|x|^2}{|y|^2} &= \frac{|h_1|}{|g_1|} \cdot \frac{|h_2|}{|g_2|} = \frac{|h_1|}{|g_2|} \cdot \frac{|h_2|}{|g_1|} = \\
&= \frac{d\,(A,X)}{d\,(Y,C)} \cdot \frac{d\,(X,D)}{d\,(B,Y)} = \frac{-d\,(X,A) \cdot d\,(X,D)}{-d\,(Y,C) \cdot d\,(Y,B)} = \\
&= \frac{d\,(X,P) \cdot d\,(X,Q)}{d\,(Y,P) \cdot d\,(Y,Q)} = \frac{(p_1 - x_1)\,(q_1 - x_1)}{(p_1 - y_1)\,(q_1 - y_1)} = \\
&= \frac{(p_1 - m_1 + m_1 - x_1)\,(q_1 - m_1 + m_1 - x_1)}{(p_1 - m_1 + m_1 - y_1)\,(q_1 - m_1 + m_1 - y_1)} = \\
&= \frac{-\left(|s| - |x|\right)\ \left(|s| + |x|\right)}{-\left(|s| + |y|\right)\ \left(|s| - |y|\right)} = \frac{|s|^2 - |x|^2}{|s|^2 - |y|^2}.
\end{aligned} \tag{12}$$

$$\frac{|x|^2}{|y|^2} = \frac{|s|^2 - |x|^2}{|s|^2 - |y|^2} \quad \Rightarrow \quad |x|^2 = |y|^2 \quad \Rightarrow \quad |x| = \pm\,|y|$$

The solution $|x| = -|y| \quad \Rightarrow \quad d\,(X,M) = -d\,(M,Y) = d\,(Y,M)$, wherefrom it follows that points X and Y are parallel points, which has been excluded earlier.
So, $|x| = |y| \quad \Rightarrow \quad d\,(X,M) = d\,(M,Y)$. □

Proof 2
Let's use the notation given in (4), that is, $d\,(P,M) = d\,(M,Q) = |s|$, $d\,(X,M) = |x|$, $d\,(M,Y) = |y|$, as well as (6) and (7) for the observed angles.

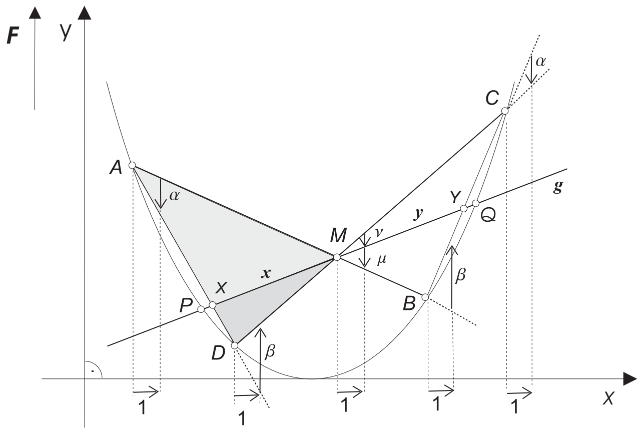

Fig. 8

From lemma 3, as shown in (12), we have

$$d\,(X,A) \cdot d\,(X,D) = d\,(X,P) \cdot d\,(X,Q)\,,$$

$$d\,(X,P) \cdot d\,(X,Q) = -\left(|s| - |x|\right)\ \left(|s| + |x|\right) = |x|^2 - |s|^2\,. \tag{13}$$

Lemma 2 applied on the allowable triangles $\triangle DMX$ and $\triangle AXM$ yields

$$\triangle DMX \Rightarrow \frac{d\,(X,D)}{\nu} = \frac{d\,(D,M)}{\angle\left(\overrightarrow{MX}\,,\,\overrightarrow{XD}\right)} = \frac{d\,(M,X)}{\beta}$$

$$\Rightarrow \frac{d\,(X,D)}{\nu} = \frac{d\,(M,X)}{\beta} \tag{14}$$

$$\triangle AXM \Rightarrow \frac{d\,(A,X)}{\mu} = \frac{d\,(X,M)}{\alpha} = \frac{d\,(M,A)}{\angle\left(\overrightarrow{AX}\,,\,\overrightarrow{XM}\right)}$$

$$\Rightarrow \frac{d\,(A,X)}{\mu} = \frac{d\,(X,M)}{\alpha}. \tag{15}$$

Lemma 4 *The sum of the directed sides of an allowable triangle in I_2 equals zero; the sum of the directed angles of an allowable triangle in I_2 equals zero as well.*

The proof is given in [3, p. 22].

For the allowable triangle $\triangle ADM$, from lemma 4,

$$\nu+\mu+\alpha+\beta=0 \quad\Rightarrow\quad \beta=-(\nu+\mu+\alpha). \tag{16}$$

Using (13)-(16) together, we obtain

$$d(X,A)\cdot d(X,D)=-d(X,M)\cdot\frac{\mu}{\alpha}\cdot d(M,X)\cdot\frac{\nu}{\beta}=$$

$$=|x|^2\frac{\nu\mu}{-\alpha(\nu+\mu+\alpha)}=|x|^2-|s|^2$$

$$\Rightarrow\ |x|^2\left(1+\frac{\nu\mu}{\alpha(\nu+\mu+\alpha)}\right)=|s|^2$$

$$\Rightarrow\ |x|^2=\frac{|s|^2\left[\alpha(\nu+\mu+\alpha)\right]}{\nu\mu+\alpha(\nu+\mu+\alpha)}. \tag{17}$$

Following the same procedure ((13)-(16)) for the segment $|y|=d(M,Y)$, due to the symmetry in ν and μ in the latter expression, we'll get exactly same result. So, $|x|^2=|y|^2$, that is $|x|=\pm|y|$, and following the conclusion from proof 1, $|x|=|y| \ \Rightarrow\ d(X,M)=d(M,Y)$. □

Proof 3

The proof is based on the following:

Lemma 5 *If in two allowable triangles in I_2 a directed angle of one is equal to a directed angle of the other, then the areas of the triangles are in the same ratio as the products of the sides composing the equal angles.*

Proof According [3, p. 26] the isotropic area of an allowable triangle ABC, $A(a_1,a_2)$, $B(b_1,b_2)$, and $C(c_1,c_2)$ is given by

$$F_{ABC}=\frac{1}{2}\begin{vmatrix}1&1&1\\a_1&b_1&c_1\\a_2&b_2&c_2\end{vmatrix}.$$

Let's mark the directed angles as given before in (6) and (7) (see figure 6), and let's observe the allowable triangles AXM and MYC (figure 9).

Lemma 1 yields that $\alpha=\angle\left(\overrightarrow{MA},\overrightarrow{AX}\right)=\alpha'=\angle\left(\overrightarrow{YC},\overrightarrow{CM}\right)$, hence, we have to proof the equality:

$$\frac{F_{AXM}}{F_{MYC}}=\frac{d(M,A)\cdot d(A,X)}{d(Y,C)\cdot d(C,M)}. \tag{18}$$

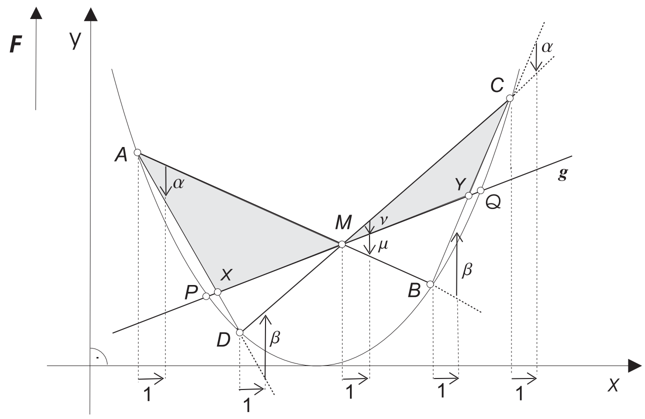

Fig. 9

For the points $A(a_1,a_2)$, $C(c_1,c_2)$, $M(m_1,m_2)$, $X(x_1,x_2)$ and $Y(y_1,y_2)$, the isotropic areas of the triangles are given by

$$F_{AXM}=\frac{1}{2}\begin{vmatrix}1&1&1\\a_1&x_1&m_1\\a_2&x_2&m_2\end{vmatrix},$$

and

$$F_{MYC}=\frac{1}{2}\begin{vmatrix}1&1&1\\m_1&y_1&c_1\\m_2&y_2&c_2\end{vmatrix}.$$

The sides composing the equal angles are $d(M,A)=(a_1-m_1)$, $d(A,X)=(x_1-a_1)$, $d(Y,C)=(c_1-y_1)$, and $d(C,M)=(m_1-c_1)$. For the directed angles α and α' we have

$$\alpha=\angle\left(\overrightarrow{MA},\overrightarrow{AX}\right)=\frac{x_2-a_2}{x_1-a_1}-\frac{a_2-m_2}{a_1-m_1}$$

$$\alpha'=\angle\left(\overrightarrow{YC},\overrightarrow{CM}\right)=\frac{m_2-c_2}{m_1-c_1}-\frac{c_2-y_2}{c_1-y_1}$$

$$\alpha=\alpha' \ \Rightarrow\ \frac{x_2-a_2}{x_1-a_1}-\frac{a_2-m_2}{a_1-m_1}=\frac{m_2-c_2}{m_1-c_1}-\frac{c_2-y_2}{c_1-y_1}$$

$$\Rightarrow\ \frac{x_1m_2-x_2m_1-a_1m_2+a_2m_1+a_1x_2-a_2x_1}{y_1c_2-y_2c_1-m_1c_2+m_2c_1+m_1y_2-m_2y_1}=$$

$$=\frac{a_1x_1-x_1m_1+m_1a_1-a_1^2}{m_1c_1-m_1y_1+c_1y_1-c_1^2}.$$

The latter equation can be reach writing extensively equation (18). □

Let's apply now lemma 5 on the following pairs of allowable triangles:

$\triangle MAX$ and $\triangle YCM \Rightarrow$

$$\frac{F_{MAX}}{F_{YCM}} = \frac{d(M,A) \cdot d(A,X)}{d(Y,C) \cdot d(C,M)}, \tag{19}$$

$\triangle CMY$ and $\triangle DMX \Rightarrow$

$$\frac{F_{CMY}}{F_{DMX}} = \frac{d(C,M) \cdot d(M,Y)}{d(D,M) \cdot d(M,X)}, \tag{20}$$

$\triangle XDM$ and $\triangle MBY \Rightarrow$

$$\frac{F_{XDM}}{F_{MBY}} = \frac{d(X,D) \cdot d(D,M)}{d(M,B) \cdot d(B,Y)}, \tag{21}$$

$\triangle YMB$ and $\triangle XMA \Rightarrow$

$$\frac{F_{YMB}}{F_{XMA}} = \frac{d(Y,M) \cdot d(M,B)}{d(X,M) \cdot d(M,A)}. \tag{22}$$

$$(19) \cdot (20) \cdot (21) \cdot (22) = \frac{F_{MAX}}{F_{YCM}} \cdot \frac{F_{CMY}}{F_{DMX}} \cdot \frac{F_{XDM}}{F_{MBY}} \cdot \frac{F_{YMB}}{F_{XMA}} = 1$$

$$\Rightarrow \frac{d(A,X) \cdot d(M,Y)}{d(Y,C) \cdot d(M,X)} \cdot \frac{d(X,D) \cdot d(Y,M)}{d(B,Y) \cdot d(X,M)} = 1$$

$$\Rightarrow \frac{d(A,X) \cdot d(X,D)}{d(B,Y) \cdot d(Y,C)} = \frac{d(M,X) \cdot d(X,M)}{d(M,Y) \cdot d(Y,M)}. \tag{23}$$

According lemma 3, and using the notation given in (4), we have

$$d(A,X) \cdot d(X,D) = d(P,X) \cdot d(X,Q) = |s|^2 - |x|^2, \tag{24}$$

and

$$d(B,Y) \cdot d(Y,C) = d(P,Y) \cdot d(Y,Q) = |s|^2 - |y|^2. \tag{25}$$

Inserting (24) and (25) in (23) we obtain

$$\frac{|s|^2 - |x|^2}{|s|^2 - |y|^2} = \frac{-|x|^2}{-|y|^2} \Rightarrow |x|^2 = |y|^2 \Rightarrow |x| = \pm|y|,$$

and finally, as it has been shown before,

$$|x| = |y| \Rightarrow d(X,M) = d(M,Y).\square$$

Proof 4

Let k be a parabolic circle in I_2, and let M be the midpoint of the chord $\overrightarrow{PQ}$ of k. Let's choose the coordinate system as shown (in the affine model) in figure 10, i.e, the tangent on the circle k parallel to the chord $\overrightarrow{PQ}$ as the x-axis, and the isotropic straight line through M as the y-axis.

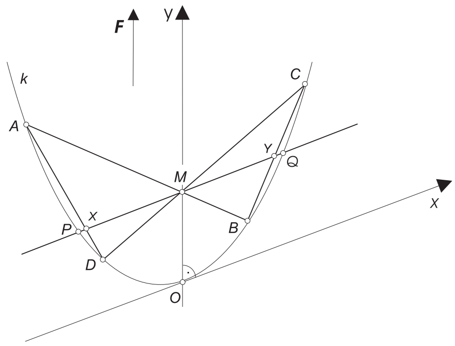

Fig. 10

Let $A(a_1, Ra_1^2)$, $B(b_1, Rb_1^2)$,$A \neq B \Rightarrow a_1 \neq b_1$, and $C(c_1, Rc_1^2)$, $D(d_1, Rd_1^2)$, $C \neq D \Rightarrow c_1 \neq d_1$, be four points on the parabolic circle k. Choosing $M(0,m)$, for the chord $\overrightarrow{PQ}$ we have $\overrightarrow{PQ} \equiv y = m$. Besides, for $\overrightarrow{AB}$ being a chord through M, the following relations are obtained:

M, A, B collinear points $\Leftrightarrow$

$$\begin{vmatrix} 0 & m & 1 \\ a_1 & Ra_1^2 & 1 \\ b_1 & Rb_1^2 & 1 \end{vmatrix} = 0 \Leftrightarrow a_1 b_1 = -\frac{m}{R}. \tag{26}$$

Analogously, for $\overrightarrow{CD}$ being a chord through M, we have:

M, C, D collinear points $\Leftrightarrow$

$$\begin{vmatrix} 0 & m & 1 \\ c_1 & Rc_1^2 & 1 \\ d_1 & Rd_1^2 & 1 \end{vmatrix} = 0 \Leftrightarrow c_1 d_1 = -\frac{m}{R}. \tag{27}$$

Let's denote further on $X(x_1, m)$ and $Y(y_1, m)$.

One obtains the following:

A, D, X collinear points $\Leftrightarrow$

$$\begin{vmatrix} x_1 & m & 1 \\ a_1 & Ra_1^2 & 1 \\ d_1 & Rd_1^2 & 1 \end{vmatrix} = 0 \Leftrightarrow Rx_1(a_1 + d_1) = m + Ra_1 d_1. \tag{28}$$

C, B, Y collinear points $\Leftrightarrow$

$$\begin{vmatrix} y_1 & m & 1 \\ b_1 & Rb_1^2 & 1 \\ c_1 & Rc_1^2 & 1 \end{vmatrix} = 0 \Leftrightarrow Ry_1(b_1 + c_1) = m + Rb_1 c_1. \tag{29}$$

Finally, using (26), (27), (28), and (29) it follows:

$$x_1+y_1=\frac{m+Ra_1d_1}{R(a_1+d_1)}+\frac{m+Rb_1c_1}{R(b_1+c_1)}=$$

$$=\frac{(m+Ra_1d_1)(b_1+c_1)+(m+Rb_1c_1)(a_1+d_1)}{R(a_1+d_1)(b_1+c_1)}=$$

$$=\frac{R(a_1b_1d_1+a_1c_1d_1+a_1b_1c_1+b_1c_1d_1)+m(a_1+b_1+c_1+d_1)}{R(a_1+d_1)(b_1+c_1)}=$$

$$=\frac{R\left(-\frac{m}{R}d_1-\frac{m}{R}a_1-\frac{m}{R}c_1-\frac{m}{R}b_1\right)+m(a_1+b_1+c_1+d_1)}{R(a_1+d_1)(b_1+c_1)}=0$$

$\Rightarrow M$ is the midpoint of $\overrightarrow{XY}$. $\square$

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