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Constructive Procedure for Transformation of Collinear Spaces

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ABSTRACT

This paper offers the constructive solution for transformation of points between two general collinear spaces Σ and $\overline{\Sigma}$ by using Monge's projection. The constructive procedure is based on the fact that each straight line of space Σ which is parallel to vanishing plane $\mathbf{R} \in \Sigma$ and one side of autocollinear tetrahedron $D_1D_2D_3D_4$, is transformed into the straight line of space $\overline{\Sigma}$ which is parallel to vanishing plane $\overline{\mathbf{Q}} \in \overline{\Sigma}$ and the same side of autocollinear tetrahedron.

Key words: collinear spaces, autocollinear tetrahedron, vanishing plane, perspectivity of the pencils of lines

MSC 2000: 51N05

Konstruktivni postupak za transformaciju kolinearnih prostora

SAŽETAK

U radu se daje konstruktivno rješenje za preslikavanje točaka između dva opće kolinearna prostora Σ i $\overline{\Sigma}$, u Mongeovoj projekciji. Konstrukcija se temelji na činjenici da se svaki pravac prostora Σ koji je paralelan s izbježnom ravninom $\mathbf{R} \in \Sigma$ i jednom od stranica autokolinearnog tetraedra $D_1D_2D_3D_4$, preslikava u pravac prostora $\overline{\Sigma}$ koji je paralelan s izbježnom ravninom $\overline{\mathbf{Q}} \in \overline{\Sigma}$ i istom stranicom autokolinearnog tetraedra.

Ključne riječi: kolinearni prostori, autokilinearni tetraedar, izbježna ravnina, perspektivitet pramenova pravaca

1 Introduction

In the transformation of two general collinear spaces Σ and $\overline{\Sigma}$, one has to take into consideration that the composition of projections and types of projections, necessary for a space representation, are transforming into new composition of quite different types of projections in another space. Therefore, the applied method of transformation of two general collinear spaces must be based on their common invariants.

The constructive procedure in Monge's projection must have common real autocollinear tetrahedron and uses two basic invariants for the transformation of two collinear spaces.

- The first invariant consists of the following: all planes parallel to vanishing plane $\mathbf{R} \in \Sigma$ correspond to the planes parallel to vanishing plane $\overline{\mathbf{Q}} \in \overline{\Sigma}$. [2, p. 10]
- The second invariant consists of the following: each straight line which is parallel to vanishing plane $\mathbf{R} \in \Sigma$ and one side of autocollinear tetrahedron corresponds to the straight line which is parallel to the mentioned side of tetrahedron and vanishing plane $\overline{\mathbf{Q}} \in \overline{\Sigma}$. [2, p. 17]

2 Some properties of collinear spaces

Two collinear spaces Σ and $\overline{\Sigma}$, located in the same projective space, always have four fixed points which are the vertexes of one tetrahedron $D_1D_2D_3D_4$. Six edges $D_1D_2 = d_3$, $D_2D_3 = d_1$, $D_1D_3 = d_2$, $D_1D_4 = d_4$, $D_2D_4 = d_5$, $D_3D_4 = d_6$ and four sides $D_1D_2D_3 = \Delta_4$, $D_1D_2D_4 = \Delta_3$, $D_2D_3D_4 = \Delta_1$, $D_1D_3D_4 = \Delta_2$ of the tetrahedron are autocollinear, i.e. they are twofold lines and planes of the collineation with fixed points at the related vertexes D_1, D_2, D_3 and D_4. The collineation between Σ and $\overline{\Sigma}$ is uniquely determined with autocollinear tetrahedron and the pair of corresponding points or the pair of corresponding planes. [3, p. 93]

2.1 Vanishing planes, lines and points

Let collinear spaces Σ and $\overline{\Sigma}$ be located in the same real projective space. The planes, lines and points which correspond with the plane, lines and points at infinity are the *vanishing elements* of the collineation. The collineation between Σ and $\overline{\Sigma}$ can be determined with autocollinear tetrahedron $D_1D_2D_3D_4$ and vanishing plane $\mathbf{R} \in \Sigma$.

Let side $\Delta_4 = D_1D_2D_3$ of autocollinear tetrahedron lie in the horizontal plane and (r_1, r_2) are the traces of vanishing plane $\mathbf{R} \in \Sigma$ (Fig1).

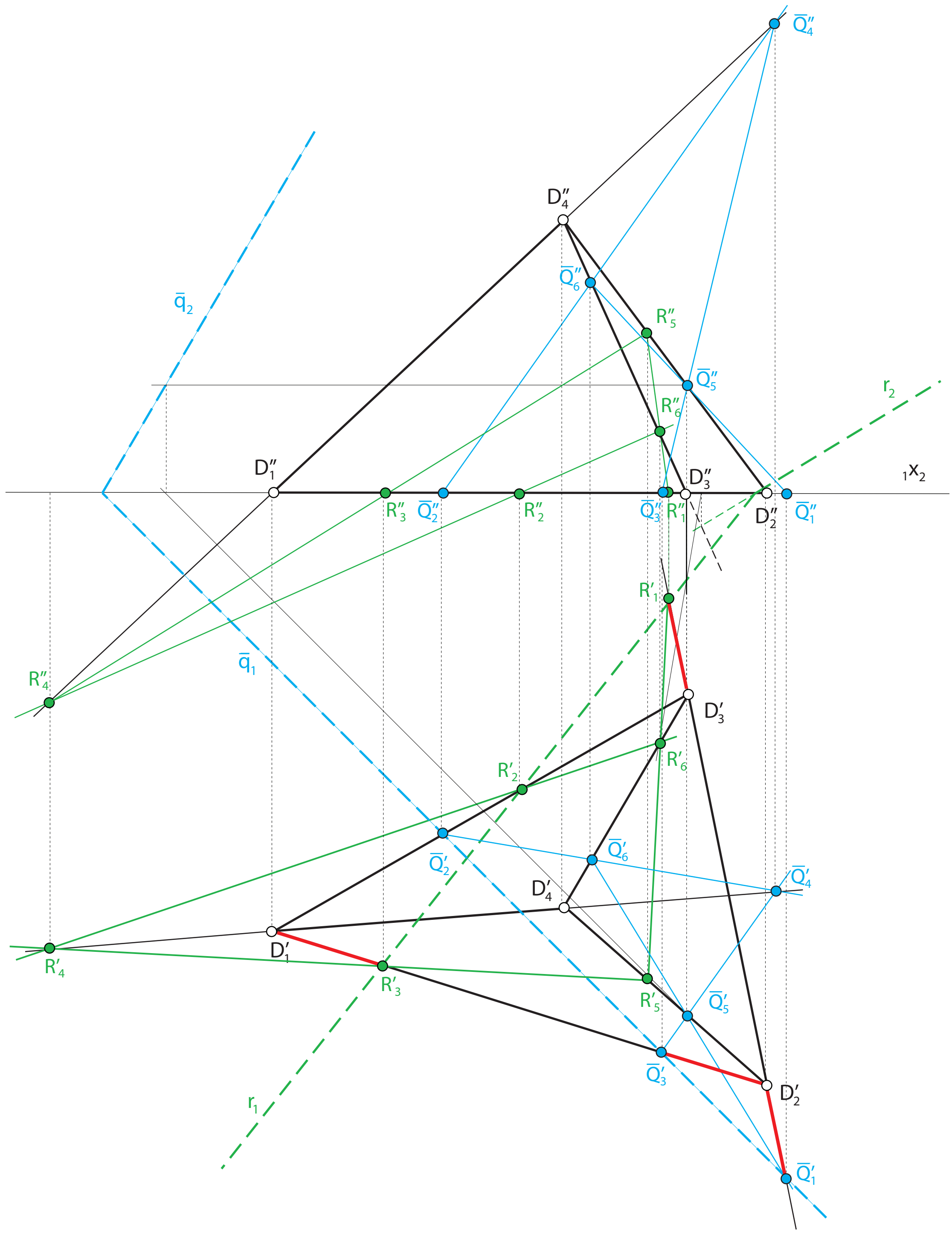

Figure 1

The points of intersection of six lines ($d_1 - d_6$) and vanishing plane $\mathbf{R}$ defines six vanishing points ($R_1 - R_6$) on autocollinear edges. The lines of intersection of four sides ($\Delta_1 - \Delta_4$) and vanishing plane $\mathbf{R}$ are four vanishing lines in autocollinear planes. Each of them joins three vanishing points: $\Delta_1 \cap \mathbf{R}$ joins R_1, R_5, R_6; $\Delta_2 \cap \mathbf{R}$ joins R_2, R_4, R_6; $\Delta_3 \cap \mathbf{R}$ joins R_3, R_4, R_5; $\Delta_4 \cap \mathbf{R}$ joins R_1, R_2, R_3. The construction is simple because three vanishing points (R_1, R_2, R_3) lie on the first trace r_1 and it is neccessary to construct only one other vanishing point as the intersection of autocollinear edge and vanishing plane (in Fig 1 it is point R_6).

The construction of vanishing plane $\overline{\mathbf{Q}} \in \overline{\Sigma}$, six vanishing points ($\overline{Q}_1 - \overline{Q}_6$) on autocollinear edges ($d_1 - d_6$) and four related vanishing lines in autocollinear sides ($\Delta_1 - \Delta_4$) is based on the following statement: *The vanishing points of two projective ranges of points which are located on the same line, are always equidistanced from the fixed points of the projectivity*. [3, p. 17]

It enables very simple construction of vanishing points $\overline{Q}_1 - \overline{Q}_6$ on lines $d_1 - d_6$, related vanishing lines and the traces ($\overline{q}_1, \overline{q}_2$) of vanishing plane $\overline{\mathbf{Q}} \in \overline{\Sigma}$. For example, in Fig 1 trace $\overline{q}_1$ joins points $\overline{Q}_1$ and $\overline{Q}_2$, where $d(\overline{Q}_1, D_2) = d(\overline{R}_1, D_3)$ and $d(\overline{Q}_3, D_2) = d(\overline{R}_3, D_1)$.

2.2 Perspectivities of pencils of lines in horizontal plane

For two colliner fields, located in the same real projective plane, the pencils of corresponding lines with vertexes in the pair of corresponding points on the edges of autocollinear triange $D_1D_2D_3$, are perspectivelly ajusted. The axes of that perspectivities always pass through the opposite vertexes of autocollinear triangle. If one of the vertexes of pencils of lines is the point at infinity, the axis of perspectivity is parallel to a vanishing line. [1, p. 2].

In Fig 2 this property is shown for pencils of lines (S) and $(\overline{S}^{\infty})$, where S and $\overline{S}^{\infty}$ are the pair of corresponding points on edge D_2D_3. The axis of perspectivity $(S) \overline{\overline{\wedge}} (\overline{S}^{\infty})$ is parallel to vanishing line v and passes through point D_1.

This statement enables the construction of corresponding points in all autocolinear planes of collinear spaces Σ and $\overline{\Sigma}$. In Fig 3 this construction is shown in the horizontal plane from Fig 1. Perspectivities $(R_1) \overline{\overline{\wedge}} (\overline{R}_1^{\infty})$ and $(R_2) \overline{\overline{\wedge}} (\overline{R}_2^{\infty})$ are used for the construction of corresponding points X and $\overline{X}$ which are in general position to twofold elements.

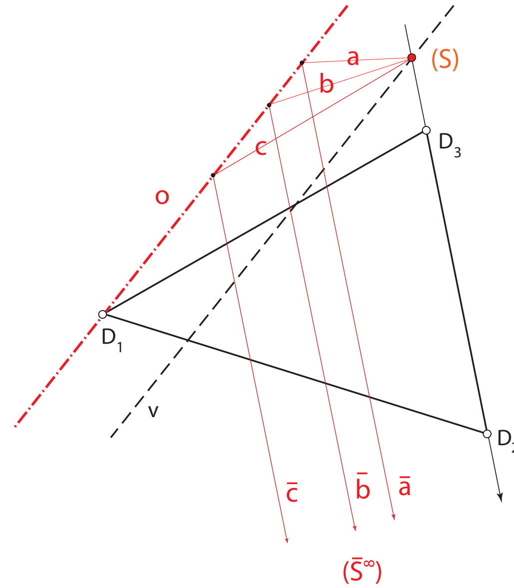

Figure 2

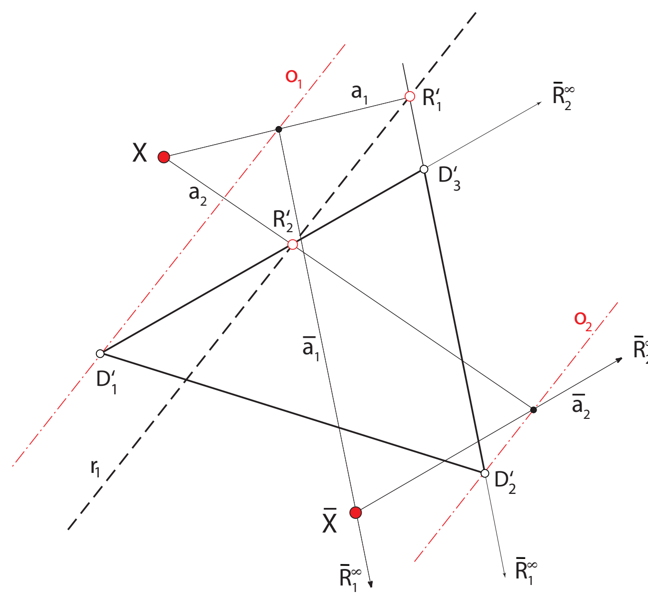

Figure 3

3 Construction of corresponding points for collinear spaces in Monge's projection

The constructive procedure is based on the fact that each straight line of space Σ, which is parallel to vanishing plane $\mathbf{R} \in \Sigma$ and one side of autocollinear tetrahedron $D_1D_2D_3D_4$, is transformed into the straight line of space $\overline{\Sigma}$ which is parallel to vanishing plane $\overline{\mathbf{Q}} \in \overline{\Sigma}$ and the same side of autocollinear tetrahedron.

Let the collineation between Σ and $\overline{\Sigma}$ be defined with autocollinear tetrahedron $D_1D_2D_3D_4$ and vanishing plane $\mathbf{R} \in \Sigma$, and let them be positioned as in Fig 1. Point A (see Fig 4) is in general position to autocollinear elements and point $\overline{A}$ is constructed in the following way:

1. Traces $(\overline{q}_1, \overline{q}_2)$ of vanishing plane $\overline{\mathbf{Q}}$ and vanishing lines R_1R_6, R_2R_6, $\overline{Q}_1\overline{Q}_6$, $\overline{Q}_2\overline{Q}_6$ are constructed as in Fig 1.
2. Lines a_1, a_2 which pass through point A and are parallel to R_1R_6 and R_2R_6, respectively, are constructed. They lie in plane $\mathbf{T} \parallel \mathbf{R}$. Points A_1, A_2 are the intersections of lines a_1, a_2 and horizontal plane.
3. Since points A_1 and A_2 lie in autocollinear plane (horizontal plane) we can construct corresponding points $\overline{A}_1$ and $\overline{A}_2$ as in Fig 3, by using perspectivities $(R_1)\,\overline{\overline{\wedge}}\,(\overline{R}_1^{\infty})$ and $(R_2)\,\overline{\overline{\wedge}}\,(\overline{R}_2^{\infty})$. For the construction of point $\overline{A}_2$, line $\overline{t}_1$ (the first trace of plane $\overline{\mathbf{T}} \parallel \overline{\mathbf{Q}}$) is also used.
4. According to the statement mentioned in the introduction: the lines through points $\overline{A}_1$ and $\overline{A}_2$, parallel to $\overline{Q}_1\overline{Q}_6$ and $\overline{Q}_2\overline{Q}_6$, respectively, are corresponding lines of a_1 and a_2, i.e. they are $\overline{a}_1$ and $\overline{a}_2$.

Point $\overline{A}$ is the intersection of lines $\overline{a}_1$ and $\overline{a}_2$.

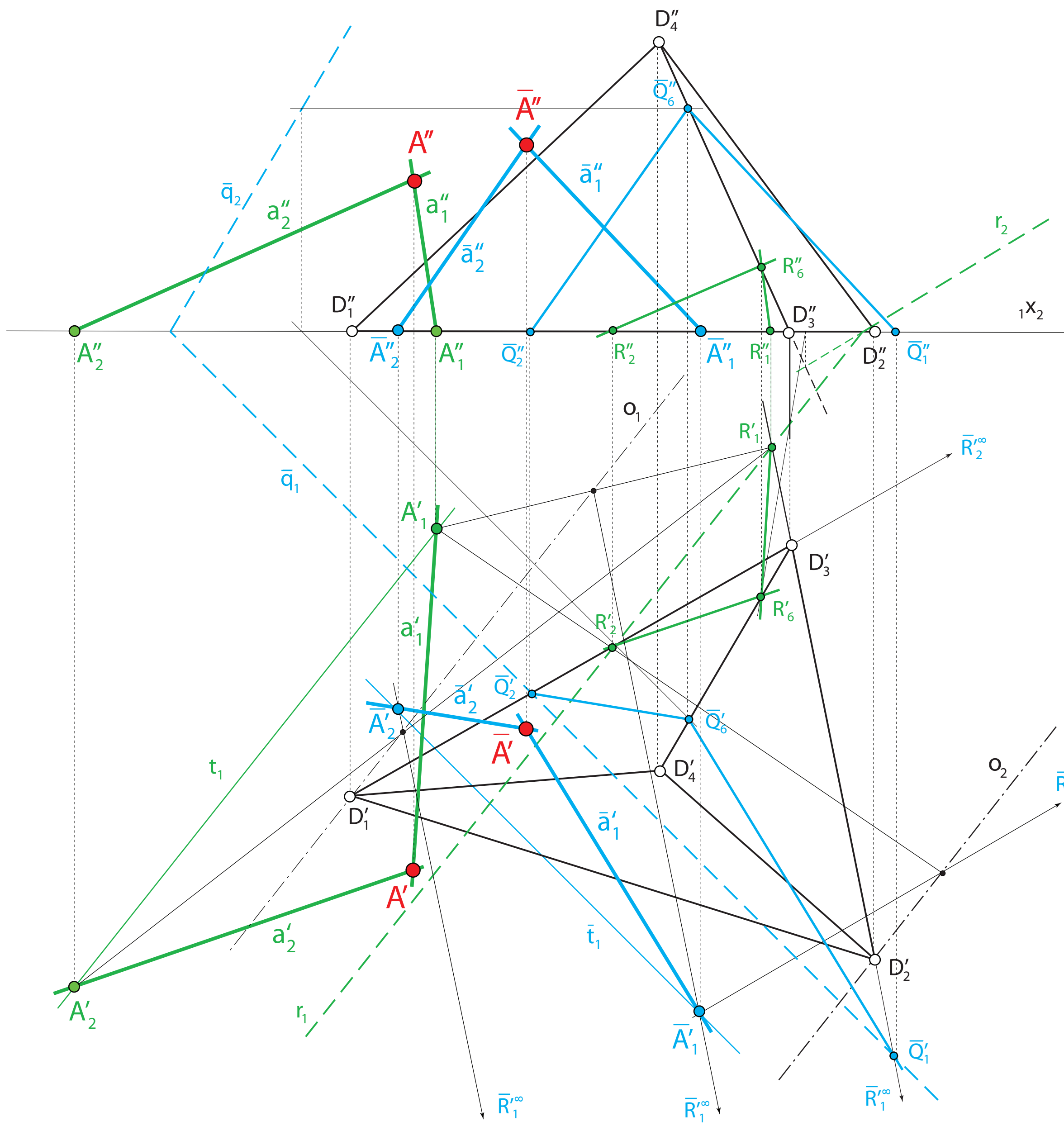

Figure 4

References

[1] ČUČAKOVIĆ, A., *Invariables of conformity and coordinatization of general collinear and of general affine planes and spaces*, Ph.D. thesis, Belgrade 1992.

[2] GAGIĆ, LJ., *Graphic Transformation of Collinear Spaces*, Ph.D. thesis, Belgrade 1977.

[3] NIČE, V., *Deskriptivna geometrija I*, Školska knjiga, Zagreb, 1979.

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