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ON INTERSECTION OF SIMPLY CONNECTED SETS IN THE PLANE

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Abstract. Several authors have recently attempted to show that the intersection of three simply connected subcontinua of the plane is simply connected provided it is non-empty and the intersection of each two of the continua is path connected. In this note we give a very short complete proof of this fact. We also confirm a related conjecture of Karimov and Repovš.

1. Introduction

A homology (resp., singular) cell is a compact metric space whose Vietoris (resp., singular) homology groups are trivial. Helly [6] proved the following result which is now known as the Topological Helly Theorem:

Theorem 1.1. *Let $\mathcal{S} = \{S_0, ..., S_m\}$, $m \geq n$, be a finite family of homology cells in $\mathbb{R}^n$ such that the intersection of every subfamily $\mathcal{H}$ of $\mathcal{S}$ is nonempty if the cardinality $|\mathcal{H}| \leq n+1$ and it is a homology cell if $|\mathcal{H}| \leq n$. Then $\cap_{i=0}^{i=m} S_i$ is a homology cell.*

Versions of Theorem 1.1 for singular homology have been proved by Debrunner [5]. Alexandroff and Hopf [1, p. 295] also established a simple proof of a combinatorial version of the Helly theorem.

A topological space is said to be simply connected if it is path connected and has trivial fundamental group. It is known [4] that a compact subspace of the plane is a singular cell if and only if it is simply connected.

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In Section 2 of the paper [6] Helly proved that if S_i, $i = 1, \dots, 4$, are singular cells in $\mathbb{R}^2$ such that all intersections $S_{i_1} \cap S_{i_2} \cap S_{i_3}$ are singular cells, then $\cap_{i=1}^{i=4} S_i$ is not empty. Hence to prove the Topological Helly Theorem for singular cells in $\mathbb{R}^2$, it suffices to prove the following:

Proposition 1.2. *Let S_0, S_1 and S_2 be three simply connected compacta in the plane such that the intersection of any two of them is path connected and $\cap_{i=0}^{i=2} S_i \neq \emptyset$. Then $\cap_{i=0}^{i=2} S_i$ is simply connected.*

Bogatyi [2] has pointed out that no complete proof of this proposition can be found in the literature. He proved the proposition in the special case that S_i are Peano continua. Karimov and Repovš [7], established that, with the hypotheses of Proposition 1.2, $\cap_{i=0}^{i=2} S_i$ is cell-like connected (i.e., every two points can be connected by a cell-like continuum). We prove Proposition 1.2 by showing that $\cap_{i=0}^{i=2} S_i$ is path connected. We also give an affirmative answer to a conjecture of Karimov and Repovš [7] by proving the following proposition:

Proposition 1.3. *If X and Y are compact AR's in the plane, then so is each component of $X \cap Y$.*

2. Proof of Proposition 1.2

Since the intersection of any family of simply connected sets in the plane has a trivial fundamental group with respect to each of its points, it suffices to show that $\cap_{i=0}^{i=2} S_i$ is path connected. Let $0, 1 \in \cap_{i=0}^{i=2} S_i$ and $I \subset S_0 \cap S_1$, $J \subset S_0 \cap S_2$ and $K \subset S_1 \cap S_2$ be arcs from 0 to 1. Consider the components J_n, $n = 1, 2, .., $ of $J \backslash \big(I \cup K\big)$ which are not in S_1. If the family $\{J_1, J_2, ..\}$ is infinite, then $\lim\limits_{i \to \infty} diam J_i = 0$ because it is the family of components of an open set in the arc J. Since 0 and 1 are end-points of J, it follows that no J_i separates $I \cup J \cup K$. Suppose J_i lies in a bounded component of $\mathbb{R}^2 \backslash \big(I \cup K\big)$. Since the locally connected continuum $I \cup K$ separates J_i from ∞ in $\mathbb{R}^2$, some simple closed curve in $I \cup K \subset S_1$ does so as well [8, Chapter X, Section 61, II, Theorem 5, p. 513]. Since S_1 is simply connected, this would imply $J_i \subset S_1$, a contradiction. Thus, no J_i lies in a bounded component of $\mathbb{R}^2 \backslash \big(I \cup K\big)$.

We are going to construct for every $n \geq 1$ an arc $J^n \subset S_0 \cap S_2 \cap \big(I \cup J \cup K\big) \backslash \big(J_1 \cup .. \cup J_n\big)$ from 0 to 1. Let L be a half line irreducible from J_1 to ∞ in $\mathbb{R}^2 \backslash (I \cup K)$ and $\varepsilon = \min\{d(x, y) : x \in L, y \in I \cup K\}$. Then $\varepsilon > 0$ because $I \cup K$ is compact and disjoint from the closed set L. Hence, L meets only finitely many, say $\{J_1 = J_{i_1}, .., J_{i_m}\}$, of the sets $\{J_1, J_2, ..\}$. We may suppose L meets J_{i_j} in exactly one point for each $j \in \{1, .., m\}$. We may also suppose that $J_{i_j} \cup I \cup K$ separates $J_{i_{j-1}}$ from ∞ in $\mathbb{R}^2$ for $j = 2, .., m$ if $m > 1$. Give J its natural linear order with initial point 0. If $x, y \in J$ let xy denote the arc in J irreducible from x to y. Now, let x_0 and x_1 be the end-points of J_{i_m}

with $x_0 < x_1$ and let $y_0 = \max I \cap 0x_0$, $y_1 = \min I \cap x_1 1$. Denote by M the arc in I irreducible from y_0 to y_1. Then $y_0y_1 \cup M \subset I \cup J$ is a simple closed curve containing J_{i_m}. By the Jordan curve theorem, $y_0y_1 \cup M$ bounds a disk D with boundary $y_0y_1 \cup M$. Let L^* be the unbounded component of $L \backslash J_{i_m}$ and $D_m \subset D$ be the bounded component of $\mathbb{R}^2 \backslash (I \cup J \cup K)$ whose boundary contains J_{i_m}.

Then $D_m \subset D_I \cap D_K$, where D_I (resp., D_K) is the component of $\mathbb{R}^2 \backslash (I \cup J)$ (resp., $\mathbb{R}^2 \backslash (J \cup K)$) containing D_m. Note that, as in the first paragraph of the proof, $D_I \subset S_0$ because $I \cup J \subset S_0$. Similarly, $D_K \subset S_2$. Thus, $D_m \subset S_0 \cap S_2$ and $\overline{D_m} \subset S_0 \cap S_2$ since $S_0 \cap S_2$ is closed.

Moreover, $Fr(D_m) \subset I \cup J \cup K$. It is well known [9, Theorem 2, p. 39] that each continuum contained in the union of finitely many arcs is locally connected. So $Fr(D_m)$ is locally connected. As above, let $C \subset Fr(D_m)$ be the simple closed curve that separates D_m from ∞ in $\mathbb{R}^2$. Note that $J_{i_m} \subset C$ because $L^* \subset \mathbb{R}^2 \backslash \big(I \cup J \cup K\big)$ joins $J_{i_m} \subset \overline{D_m}$ to ∞. Let $J^{1,m} \subset \big(J \cup C\big) \backslash J_{i_m} \subset S_0 \cap S_2$ be an arc from 0 to 1. If $m = 1$ let $J^1 = J^{1,m}$. If $m > 1$ repeat the above arguments with $J^{1,m}$ in place of J and $J_{i_{m-1}}$ in place of J_{i_m} to obtain an arc $J^{1,m-1}$ in $S_0 \cap S_2 \cap \big((J^{1,m} \backslash J_{i_{m-1}}) \cup (I \cup K)\big)$ from 0 to 1. After m such steps we obtain an arc $J^1 = J^{1,1} \subset (I \cup J \cup K) \cap S_0 \cap S_2 \backslash \big(J_{i_1} \cup J_{i_2} \cup .. \cup J_{i_m}\big)$ from 0 to 1.

Suppose we have already constructed an arc $J^n \subset S_0 \cap S_2$ in $I \cup J \cup K$ from 0 to 1 such that $J^n \cap \big(J_1 \cup .. \cup J_n\big) = \emptyset$. If $J^n \cap J_{n+1} = \emptyset$, let $J^{n+1} = J^n$. If $J^n \cap J_{n+1} \neq \emptyset$, we repeat the above arguments with J^n in place of J and J_{n+1} in place of J_1 to obtain an arc $J^{n+1} \subset S_0 \cap S_2 \cap \big((J^n \cup I \cup K) \backslash J_{n+1}\big)$ from 0 to 1. By induction, we construct a sequence of arcs $\{J^n\}_{n=1}^{\infty}$ from 0 to 1 with

$$J^{n+1} \subset S_0 \cap S_2 \cap \Big((I \cup J \cup K) \backslash \bigcup_{i=1}^{n+1} J_i\Big).$$

Let $J^* = \limsup J^n$. Then

$$J^* \subset \big(S_0 \cap S_2\big) \cap \Big((I \cup J \cup K) \backslash \bigcup_{i=1}^{\infty} J_i\Big) \subset S_1$$

is a continuum from 0 to 1. As above, J^* is locally connected. So, there is an arc in $J^* \subset S_0 \cap S_1 \cap S_2$ from 0 to 1.

3. Proof of Proposition 1.3

Let C be a component of $X \cap Y$. If K is the topological hull of C, then $K \subset X$ and $K \subset Y$ since neither X nor Y separates $\mathbb{R}^2$. So, $K = C$. By unicoherence of $\mathbb{R}^2$ it follows that $Fr(C)$, the boundary of C in $\mathbb{R}^2$, is connected.

By the well-known result of Borsuk [3] (that every locally connected plane continuum not separating the plane is an AR), it remains to prove that C is locally connected. Since C is a continuum in the plane, it suffices to prove that $Fr(C)$ is locally connected. To prove this it suffices to show that every pair of points of $Fr(C)$ is separated by a finite set (see [10, p. 99]).

Since X is simply connected, locally connected subcontinuum in the plane, by [10, Chapter IV], all true cyclic elements of X are topological disks D_i such that the cardinality of $D_i \cap D_j$ is at most 1 for $i \neq j$ and, if the sequence $\{D_i\}$ is infinite, then $\lim diam D_i = 0$. Hence, each $Fr(D_i)$ is a simple closed curve and $Fr(X) = X \backslash \bigcup int(D_i)$ is a locally connected continuum with a particularly simple structure. Let x and y be distinct points in $Fr(C) \subset Fr(X) \cup Fr(Y)$. If x and y do not both lie in any one cyclic element of X, then an one point set separates x and y in X and, hence, in C. Thus, we may suppose that there are cyclic elements D in X and E in Y with $x, y \in D \cap E$. Now x in $int(D)$ implies there is a neighborhood W of x in $Fr(X) \cup Fr(Y)$ with $\overline{W} \subset int(D)$. Then a finite set P separates $Fr(Y) \backslash W$ from x in $Fr(Y)$ since $Fr(Y)$ is rim-finite. Hence, P separates x from $Fr(X) \cup Fr(Y) \backslash W$. So we may suppose $x, y \in Fr(D) \cap Fr(E)$ (see [8, 49.V, Theorem 3, p. 244]).

Let F be a two-point set in $Fr(E)$ which separates x and y in $Fr(E)$. Then F separates x and y in $Fr(Y)$ [10, IV.3.1, p. 67]. So $F \cup (D \backslash Fr(Y))$ separates x and y in D. Since D is hereditarily normal, a closed subset A of $F \cup (D \backslash Fr(Y))$ separates x and y in D. Since D is unicoherent, a component A' of A separates x and y in D. We can construct an arc $A'' \subset D$ such that A'' separates x and y in D and $A'' \cap Y \subset F$. Indeed, let N be an open neighbourhood of $A' \backslash F$. So N is the union of a null collection, locally finite except at points of F, of open balls B_i each centered at a point of $A' \backslash F$ and having closure in $\mathbb{R}^2 \backslash Fr(Y)$. Then the closure of N and, hence, $Fr(N)$ separates x and y in D. Hence, a component G of $Fr(N)$ separates x and y in D. As above, $G \subset Fr(N) \subset \cup Fr(B_i) \cup F$ is locally connected at each point of $G \backslash F$. Since F is finite and a continuum cannot fail to be locally connected only at points of a zero dimensional set, G is locally connected. Since x and y are on the boundary of D, there is an arc A'' in G which separates x and y in D.

If we also take A'' to be irreducible with respect to separating x and y in D (see [8, V.49, Theorem 3, p. 244]), then $A'' \cap Fr(D)$ will contain just two points c and d. As above, A'' separates x and y in X because D is a cyclic element of X. So $A'' \cap \big(Fr(X) \cup Fr(Y)\big) \subset F \cup \{c, d\}$ separates x and y in $Fr(C) \subset \big(Fr(X) \cup Fr(Y)\big) \subset X$. So, $Fr(C)$ is rim-finite, hence, locally connected.

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