Some Hadamard designs admitting a faithful action of Frobenius groups

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Abstract. *We have constructed three new symmetric (71,35,17) designs having $Frob_{17\cdot 8}$ as an automorphism group. Also constructed are symmetric designs with parameters (43,21,10), (67,33,16) and (71,35,17) admitting an automorphism group isomorphic to $Frob_{43\cdot 7}$, $Frob_{67\cdot 11}$ and $Frob_{71\cdot 7}$, respectively, which turn out to be isomorphic to previously known designs, constructed via cyclic difference sets. Each constructed Hadamard design leads to a series of Hadamard designs.*

Key words: *symmetric design, Hadamard design, Frobenius group, automorphism group*

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1. Introduction and preliminaries

For the definition of a symmetric (v, k, λ) design $(\mathcal{P}, \mathcal{B}, I)$, its automorphism group, and isomorphisms between such structures, we refer the reader to [5, Chapter 1, sections 1.1 and 1.2]. As usual, $\mathcal{P}$ is the point set, $\mathcal{B}$ the block set, and $I \subseteq \mathcal{P} \times \mathcal{B}$ the incidence relation. The point and the block sets are mutually disjoint.

Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ be a symmetric (v, k, λ) design and $G \leq Aut\mathcal{D}$. The group action of G produces the same number of point and block orbits. We denote that number by t, the point orbits by $\mathcal{P}_1, \dots, \mathcal{P}_t$, the block orbits by $\mathcal{B}_1, \dots, \mathcal{B}_t$, and put $|\mathcal{P}_r| = \omega_r$ and $|\mathcal{B}_i| = \Omega_i$. We shall denote the points of the orbit $\mathcal{P}_r$ by $r_0, \dots, r_{\omega_r - 1}$, (i.e. $\mathcal{P}_r = \{r_0, \dots, r_{\omega_r-1}\}$). Further, we denote by γ_{ir} the number of points of $\mathcal{P}_r$ which are incident with a representative of the block orbit $\mathcal{B}_i$. For those numbers the following equalities hold:

$$\sum_{r=1}^{t} \gamma_{ir} = k\,, \tag{1}$$

$$\sum_{r=1}^{t} \frac{\Omega_j}{\omega_r} \gamma_{ir}\gamma_{jr} = \lambda\Omega_j + \delta_{ij} \cdot (k - \lambda)\,. \tag{2}$$

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Definition 1. *A $(t \times t)$-matrix (γ_{ir}) with entries satisfying conditions* (1) *and* (2) *is called an orbit structure for the parameters (v, k, λ) and orbit lengths distributions $(\omega_1, \dots, \omega_t)$, $(\Omega_1, \dots, \Omega_t)$.*

Definition 2. *The set of those indices of points of the orbit $\mathcal{P}_r$, which are incident with a representative of the block orbit $\mathcal{B}_i$, is called the index set for the position (i, r) of the orbit structure and the given representative.*

Determining the index sets when constructing symmetric designs is called indexing.

A Hadamard matrix of order m is an $(m \times m)$-matrix $H = (h_{i,j})$, $h_{i,j} \in \{-1, 1\}$, satisfying $HH^T = H^T H = mI$, where I is the unit matrix. From each Hadamard matrix of order m one can obtain a symmetric $(m-1, \frac{1}{2}m-1, \frac{1}{4}m-1)$ design (see [5]). Also, from any symmetric $(m-1, \frac{1}{2}m-1, \frac{1}{4}m-1)$ design we can recover a Hadamard matrix. Symmetric designs with parameters $(m-1, \frac{1}{2}m-1, \frac{1}{4}m-1)$ are called Hadamard designs.

Definition 3. *Let G be an additively written group of order v. A $k-$subset D of G is a $(v, k, \lambda; n)-$difference set of order $n = k - \lambda$ if every nonzero element of G has exactly λ representations as a difference $d - d'$ with elements from D. The difference set is abelian, cyclic etc. if the group G has the respective property.*

A faithful action of a group G on a design $\mathcal{D}$ is a monomorphism from the group G into the automorphism group $Aut\mathcal{D}$.

For further basic definitions and construction procedures we refer the reader to [3], [4] and [9].

2. Hadamard (71,35,17) designs

The first symmetric $(71, 35, 17)$ design was constructed via a cyclic difference set (see [4], [6], [9]). Eight symmetric $(71, 35, 17)$ designs admitting an automorphism group isomorphic to $Frob_{21}$ were constructed recently (see [1]). None of those eight designs are isomorphic to the one constructed using a difference set.

A Frobenius group $G \cong Frob_{17 \cdot 8}$ is presented as follows:

$$G = \langle \rho, \sigma \mid \rho^{17} = 1,\, \sigma^8 = 1,\, \rho^\sigma = \rho^2 \rangle.$$

Theorem 1. *Up to isomorphism there exist three symmetric* $(71, 35, 17)$ *designs admitting an automorphism group isomorphic to $Frob_{17 \cdot 8}$ acting in such a way that the cyclic group of order* 8 *acts standardly. Full automorphism groups of these designs are all isomorphic to the group $Frob_{17 \cdot 8}$ of order* 136. *One design is self-dual, and the other two are mutually dual. The* 2*-rank of these designs equals* 36.

Proof. Solving equations (1) and (2) it is easy to conclude that the permutation of order 17 must act on the symmetric $(71, 35, 17)$ design fixing 3 points, with orbit structures:

OS1	1	1	1	17	17	17	17
1	1	0	0	17	17	0	0
1	0	1	0	17	0	17	0
1	0	0	1	17	0	0	17
17	1	1	1	8	8	8	8
17	1	0	0	8	8	9	9
17	0	1	0	8	9	8	9
17	0	0	1	8	9	9	8

OS2	1	1	1	17	17	17	17
1	1	0	0	17	17	0	0
1	0	1	0	17	0	17	0
1	0	0	1	17	0	0	17
17	1	1	0	8	8	8	9
17	1	0	1	8	8	9	8
17	0	1	1	8	9	8	8
17	0	0	0	8	9	9	9

Orbit structure OS1 is self-dual, and OS2 is non-self-dual.

We denote the points by $1_0, 2_0, 3_0, I_0, \ldots, I_{16},\ I = 4,5,6,7$. Generators of the Frobenius group $G = \langle \rho, \sigma \rangle$ are defined as follows:

$\rho = (1_0)(2_0)(3_0)(I_0, \ldots I_{16})\ I = 4,5,6,7,$

$\sigma = (1_0)(2_0)(3_0)(I_0)(I_1, I_2, I_4, I_8, I_{16}, I_{15}, I_{13}, I_9)(I_3, I_6, I_{12}, I_7, I_{14}, I_{11}, I_5, I_{10}),\ I = 4,5,6,7.$

Index sets which could occur in designs constructed from OS1 and OS2 are:

$$0 \ldots 1,2,4,8,9,13,15,16, \qquad 1 \ldots 3,5,6,7,10,11,12,14,$$

$$2 \ldots 0,1,2,4,8,9,13,15,16, \qquad 3 \ldots 0,3,5,6,7,10,11,12,14.$$

Indexing of the columns and rows corresponding to the fixed points and blocks is trivial. Therefore, we shall take into consideration only right-lower (4×4)-submatrices of the orbit structures. The orbit structure OS1 produces the self-dual design $\mathcal{D}_1$, and OS2 produces the non-self-dual design $\mathcal{D}_2$. Those designs are presented by (4×4)-matrices of index sets as follows:

$$\mathcal{D}_1 \qquad \mathcal{D}_2$$

$$\begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 3 & 3 \\ 0 & 3 & 1 & 2 \\ 1 & 2 & 3 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 1 & 2 \\ 1 & 1 & 3 & 0 \\ 0 & 3 & 1 & 1 \\ 1 & 2 & 3 & 3 \end{pmatrix}$$

The orbit structure dual to OS2 produces a design dual to $\mathcal{D}_2$.

A computer program by Vladimir D. Tonchev [8] computes the orders as well as the generators of the full automorphism groups of these designs. Application of another computer program by Tonchev [8] yields the assertion about the 2-ranks.□

Theorem 1 implies that there are at least 12 pairwise nonisomorphic symmetric (71,35,17) designs.

3. Series of Hadamard designs

From each Hadamard design, using the following well known theorem, one can construct a series of Hadamard designs.

Theorem 2. *Let M be an incidence matrix of a Hadamard design $\mathcal{D}$ with parameters $(m-1, \frac{1}{2}m-1, \frac{1}{4}m-1)$, and $\overline{M}$ be the incidence matrix of the complementary design $\mathcal{D}^c$. Then*

$$\left[\begin{array}{c|c|c} 0 & 1 \cdots 1 & 0 \cdots 0 \\ \hline 1 & & \\ \vdots & M & M \\ 1 & & \\ \hline 0 & & \\ \vdots & M & \overline{M} \\ 0 & & \end{array}\right]$$

is an incidence matrix of a Hadamard design $\mathcal{D}_1$ with parameters $(2m-1, m-1, \frac{1}{2}m-1)$. Each automorphism of the design $\mathcal{D}$ is an automorphism of the design $\mathcal{D}_1$. If the design $\mathcal{D}$ is self-dual, so is the design $\mathcal{D}_1$.

Proof. It is easy to verify properties 1, 2, and 3 from the definition of symmetric designs. Statements about automorphism and self-duality are clear from the construction of the incidence matrix of the design $\mathcal{D}_1$. $\square$

Applying *Theorem 2* on the result from *Theorem 1* we get the following corollary:

Corollary 1. *For each non-negative integer k there exist one dual and one self-dual Hadamard design with parameters $(72 \cdot 2^k - 1, 36 \cdot 2^k - 1, 18 \cdot 2^k - 1)$ having $Frob_{17 \cdot 8}$ as an automorphism group.*

Using the described method one can construct other Hadamard designs and the corresponding series of Hadamard designs.

Example 1. *A Frobenius group G_1 of order 497 is presented as follows:*

$$G_1 = \langle \rho, \sigma \mid \rho^{71} = 1, \sigma^7 = 1, \rho^\sigma = \rho^{20} \rangle.$$

The group G_1 must act on a symmetric (71,35,17) design $\mathcal{D}$ standardly with orbit lengths distribution (71), (71). The only possible orbit structure for parameters (71,35,17) and the group G_1 is OS3:

OS3	*71*
71	*35*

Up to isomorphism there is only one symmetric $(71, 35, 17)$ design admitting a faithful action of a Frobenius group of order 497. The full automorphism group of this design is isomorphic to the group $Frob_{71 \cdot 35}$ of order 2485. This design is self-dual. The 2-rank of the design equals 36. This design is isomorphic to the one described in [4].

Applying Theorem 2*, we get that for each non-negative integer k there exists a self-dual Hadamard design with parameters $(72 \cdot 2^k - 1, 36 \cdot 2^k - 1, 18 \cdot 2^k - 1)$ having $Frob_{71 \cdot 35}$ as an automorphism group.*

Example 2. *According to [6], until 1996 there were two known symmetric* $(43, 21, 10)$ *designs. These designs were constructed via difference sets. Later on, S. Topalova constructed further 80 symmetric* $(43, 21, 10)$ *designs admitting an automorphism group isomorphic to* Z_{21} *(see [2] and [7]).*

Indexing of the orbit structure

OS4	*43*
43	*21*

with an assumption that a Frobenius group of order 301 *acts as an automorphism group, produces exactly two mutually nonisomorphic designs, isomorphic to the ones constructed via cyclic difference sets. So, we have proved that up to isomorphism there exist precisely two symmetric designs with parameters* $(43, 21, 10)$ *admitting a faithful action of a Frobenius group of order* 301*. The full automorphism groups of these designs are isomorphic to* $Frob_{43\cdot 7}$*, and* $Frob_{43\cdot 21}$*. Both designs are self-dual. The* 2*-rank of these two designs equals* 43*.*

Theorem 2 *yields that for each non-negative integer* k *there exists a self-dual Hadamard design with parameters* $(44\cdot 2^k - 1, 22\cdot 2^k - 1, 11\cdot 2^k - 1)$ *having* $Frob_{43\cdot 21}$ *as an automorphism group.*

Example 3. *There is only one known Hadamard* $(67, 33, 16)$ *design (see [6]). This design is constructed using a cyclic difference set (see [4]). In the same way as in the case of symmetric* $(71, 35, 17)$ *design with the group* $Frob_{71\cdot 7}$*, one can prove that up to isomorphism there exists only one symmetric design with parameters* $(67, 33, 16)$ *admitting a faithful action of a Frobenius group of order* 737*. Full automorphism group of this design is isomorphic to* $Frob_{67\cdot 33}$ *of order* 2211*. This design is self-dual. The* 2*-rank of the design equals* 67*. This design is isomorphic to the one described in [4].*

For each non-negative integer k *there exists a self-dual Hadamard design with parameters* $(68\cdot 2^k - 1, 34\cdot 2^k - 1, 17\cdot 2^k - 1)$ *having* $Frob_{67\cdot 33}$ *as an automorphism group.*

References

[1] D. Crnković, D. Held, *Some Hadamard designs with parameters (71,35,17)*, J. Combin Designs **9**(2001), 1-6.

[2] J. Dinitz, *New Results in Combinatorial Designs*, Part 1, el. document, July 2001, `http://www.emba.uvm.edu/~dinitz/newresults.part1.html`

[3] Z. Janko, Coset enumeration in groups and constructions of symmetric designs, Combinatorics **90**(1992), 275–277.

[4] D. Jungnickel, A. Pott, *Difference sets: Abelian*, in: *The CRC Handbook of Combinatorial Designs*, (C. J. Colbourn and J. H. Dinitz, Eds.), CRC press, 1996, 297-307.

[5] E. Lander, *Symmetric Designs: An Algebraic Approach*, Cambridge University Press, 11983.

[6] R. Mathon, A. Rosa, $2-(v,k,\lambda)$ *designs of small order*, in: *The CRC Handbook of Combinatorial Designs*, (C. J. Colbourn and J. H. Dinitz, Eds.), CRC press, 1996, 3-41.

[7] S. Topalova, *Hadamard 2-(43,21,10) designs with automorphisms of order 21 and binary self-dual codes of lengths 86 and 88*, Proceedings of the II Int. Workshop on Optimal Codes and Related Topics, Sozopol, Bulgaria, June, 1998.

[8] V. Tonchev, Michigan Technological University, private communication via Z. Janko of Universität Heidelberg.

[9] W. D. Wallis, A. P. Street, J. S. Wallis, *Combinatorics: Room Squares, Sum-Free Sets, Hadamard matrices*, Springer Verlag, Berlin-Heidelberg-New York, 1972.