Mathematical analysis of composite structures*

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Abstract. *In this paper an exposition of the asymptotic method for periodic composite structures is given.*

Key words: *composite material, homogenization, asymptotic expansion*

Sažetak. **Matematička analiza kompozitnih struktura**. *U ovom članku dan je prikaz asimptotičke metode za kompozitne strukture.*

Ključne riječi: *kompozitni materijal, homogenizacija, asimptotički razvoj*

Examples of a composite structure are porous media, perforated materials, layered media, lattice elements in civil engineering, etc. A common property in all these cases is a rapid variability of either the geometric or the physical structure. Therefore, a boundary-value problem for a process in a composite medium cannot be solved by classical methods. In order to obtain a well posed problem, one has to average (in some sense) a microscopic model and formulate a correspondent macroscopic model, describing a process "in mean". Such macroscopic laws appeared in engineering during the last century; a good example is the well known Darcy's law, connecting the velocity and pressure of a fluid in a porous medium and taking the place instead of the Navier-Stokes equation.

A mathematical justification of macroscopic equations belongs to the homogenization theory, having the origin in last seventies. Definite results have so far been obtained for composites with a small periodicity in the structure. We shall describe the basic idea of the homogenization method on a simple example: stationary heat conduction in a bar with a periodic conduction coefficient of the material.

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Let for $\varepsilon > 0$ a sufficiently small conduction coefficient $a^\varepsilon(x)$ $(x \in (0,1))$ be defined as

$$a^\varepsilon(x) = a\left(\frac{x}{\varepsilon}\right), \tag{1}$$

where $a(y) > 0$ is a 1-periodic function on R; then a^ε is an ε-periodic function. Let $u^\varepsilon(x)$ denote the temperature of the bar. Assuming the density of an external heat flux in the form $f(x) - ku^\varepsilon(x)$, where $f(x)$ is given function and $k =$ const. ≥ 0, we have the following equation for the temperature

$$\frac{d}{dx}(a^\varepsilon(x)\frac{d}{dx}u^\varepsilon(x)) - ku^\varepsilon(x) + f(x) = 0, \quad x \in (0,1). \tag{2}$$

We take the homogeneous boundary conditions

$$u^\varepsilon(0) = (u^\varepsilon)'(1) = 0. \tag{3}$$

For each $\varepsilon > 0$ a sufficiently small problem (2), (3) has a unique solution u^ε. As the macroscopic temperature of the bar we consider the limit of u^ε, as $\varepsilon \to 0$. The purpose of the homogenization theory is to identify this limit. A formal method for it is a two-scale expansion.

One can expect that the function u^ε has a property of local ε-periodicity, i.e. that in a small neighbourhood of an interior point it holds $u^\varepsilon(x+\varepsilon) \approx u^\varepsilon(x)$. An example of such a function is the mapping $x \to u(x, x|\varepsilon)$, where $u(x,y)$ is a smooth function, 1-periodic in the variable y. A more general example is the series

$$u^0(x,y) + \varepsilon u^1(x,y) + \varepsilon^2 u^2(x,y) + \dots\ , \tag{4}$$

where $(u^k(x,y),\ k = 0,1,2,\dots)$ are a smooth functions, 1-periodic in the variable y. It is plausible to look for a solution $u^\varepsilon(x)$ of the problem (2), (3) in the form of a two-scale asymptotic expansion (4) (see [2], [3]). One can expect that (in some sense) $u^\varepsilon \to u^0$ as $\varepsilon \to 0$. Inserting (4) into (2) and taking the terms of the order ε^{-2}, we conclude that the function u^0 does not depend on y:

$$u^0 = u^0(x). \tag{5}$$

Taking the terms of the order ε^{-1}, we obtain that

$$u^1(x,y) = (u^0(x))'w(y), \tag{6}$$

where $w(y)$ is a 1-periodic solution of the equation

$$(a(y)w'(y))' + a'(y) = 0. \tag{7}$$

One easily obtains

$$w(y) = a^0 \int_0^y \frac{dy}{a(y)} - y + \text{ const.}, \qquad a^0 = \left(\int_0^1 \frac{dy}{a(y)}\right)^{-1}. \tag{8}$$

Finally, the terms of the order ε^0 give the equation

$$a(y)\frac{\partial^2}{\partial x^2}u^0(x) + a(y)\frac{\partial^2}{\partial x \partial y}u^1(x,y) + \frac{\partial}{\partial y}(a(y)\frac{\partial}{\partial x}u^1(x,y))$$

$$+\frac{\partial}{\partial y}(a(y)\frac{\partial}{\partial y}u^2(x,y)) - ku^0(x) + f(x) = 0. \qquad (9)$$

Integrating this equation due to the variable y and taking into account (6) and the periodicity conditions, we obtain the equation for the function $u^0(x)$

$$a^0(u^0(x))'' - ku^0(x) + f(x) = 0. \qquad (10)$$

According to (3) , the admissible boundary conditions for this equation are

$$u^0(0) = (u^0)'(1) = 0. \qquad (11)$$

The main mathematical problem is to establish a convergence $u^\varepsilon \to u^0, \varepsilon \to 0$. From (2) and (3) one can easily obtain the a priori estimate

$$\int_0^1 \Big(\big(\frac{d}{dx}u^\varepsilon\big)^2 + (u^\varepsilon)^2 \Big)\, dx \leq C, \qquad (12)$$

where $C > 0$ does not depend on ε. From this estimate it follows that, up to a subsequence, u^ε has a limit in $L^2(0,1)$, as $\varepsilon \to 0$. For proving the fact that the limit is the function u^0 one can use either the energy method (see [4]) or the two-scale convergence method (see [1]). Taking into account that the problem (10), (11) has a unique solution, we conclude that the limit does not depend on a choice of a subsequence. In other words, the homogenized (macroscopic) solution is identified.

References

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