On Landau's type inequalities for infinitesimal generators*

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Abstract. *We consider Landau's type inequalities of the form*

$$\|A^k u\|^n \leq C_{n,k} \|u\|^{n-k} \|A^n u\|^k, \quad u \in D(A^n), \quad 0 < k < n,$$

where A is the infinitesimal generator of either a strongly continuous semigroup or a strongly continuous cosine function of linear contractions on a Banach space X. The constants $C_{n,k}$ are computed for $n \leq 6$.

Key words: *infinitesimal generator, operator semigroup, operator cosine function, Landau's type inequalities*

Sažetak. **O nejednakosti Landauovog tipa za infinitezimalne generatore.** *Promatraju se nejednakosti Landauovog tipa*

$$\|A^k\|^n \leq C_{n,k} \|u\|^{n-k} \|A^n u\|^k, \;\; u \in D(A^n), \;\; 0 < k < n,$$

gdje je A infinitezimalni generator jako neprekidne polugrupe ili kosinusne funkcije linearnih kontrakcija na Banachovom prostoru X. Konstante $C_{n,k}$ izračunate su za $n \leq 6$.

Ključne riječi: *infinitezimalni generator, operatorska polugrupa, operatorska kosinusna funkcija, nejednakosti Landauovog tipa*

1. Introduction

In 1913 E. Landau (see [**?**]) established that if $u : R_+ \to R$ is twice differentiable and if u and u'' are bounded, then

$$\|u'\|^2 \leq 4 \, \|u\| \cdot \|u''\|,$$

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where $\|\cdot\|$ is the sup-norm. This result can be expressed as follows. Let X be the Banach space of continuous bounded functions on R_+ with the sup-norm. Then

$$[T(t)u](s) = u(s+t), \quad u \in X, \quad t, s \in R_+,$$

defines a strongly continuous semigroup $t \mapsto T(t)$ of linear operators on X (they are isometries). The infinitesimal generator of this semigroup is $A = \frac{d}{dt}$. So, the Landau's inequality reads:

$$\|Au\|^2 \leq 4\,\|u\| \cdot \|A^2 u\|, \quad u \in D(A^2). \tag{1}$$

This inequality can be proved easily by using the Taylor's formula.

In 1967 R. Kallman and G.-C. Rota have shown (see [**?**]) that essentially the same proof gives the same inequality for the infinitesimal generator A of any strongly continuous contraction semigroup $t \mapsto T(t)$ on any Banach space X.

This proof goes as follows. We have the Taylor's formula:

$$T(t)u = u + \int_0^t T(\tau)Au\,d\tau, \quad u \in D(A).$$

By iterating it we obtain

$$T(t)u = u + tAu + \int_0^t (t-\tau)T(\tau)A^2 u\,d\tau, \quad u \in D(A^2).$$

Thus,

$$Au = \frac{1}{t}u - \frac{1}{t}T(t)u - \frac{1}{t}\int_0^t (t-\tau)T(\tau)A^2 u\,d\tau, \quad u \in D(A^2), \quad t > 0.$$

Hence, we get the inequality

$$\|Au\| \leq \tfrac{2}{t}\,\|u\| + \tfrac{t}{2}\,\|A^2 u\|, \quad u \in D(A^2), \quad t > 0.$$

By minimizing over $t > 0$ we obtain (1).

Now, in [**?**] it was noticed that the same proof gives an inequality also in the case of a uniformly bounded semigroup $\|T(t)\| \leq M$:

$$\|Au\|^2 \leq 2M(M+1)\|u\| \cdot \|Au\|, \quad u \in D(A^2). \tag{2}$$

Furthermore, the same method gave an inequality for the infinitesimal generator

$$A = \left.\frac{d^2T(t)}{dt^2}\right|_{t=0}$$

of a strongly continuous uniformly bounded cosine function $t \mapsto T(t)$:

$$2\,T(t)\,T(s) = T(t+s) + T(t-s), \quad t > s \geq 0; \quad T(0) = I; \quad \|T(t)\| \leq M.$$

In this case the Taylor's formula has the form:

$$T(t)u = u + \int_0^t (t-\tau)T(\tau)Au\,d\tau, \quad u \in D(A),$$

and by iterating

$$T(t)u = u + \frac{t^2}{2}Au + \frac{1}{6}\int_0^t (t-\tau)^3 T(\tau)A^2u\,d\tau, \quad u \in D(A^2).$$

Thus,

$$\|Au\| \le \frac{2}{t^2}(M+1)\|u\| + \frac{t^2}{12}M\|A^2u\|, \quad u \in D(A^2), \quad t>0,$$

and by minimizing over $t>0$ we obtain

$$\|Au\|^2 \le \tfrac{2}{3}M(M+1)\|u\| \cdot \|A^2u\|, \quad u \in D(A^2); \tag{3}$$

in the case of contractions $\|T(t)\| \le 1$

$$\|Au\|^2 \le \tfrac{4}{3}\,\|u\| \cdot \|A^2u\|, \quad u \in D(A^2). \tag{4}$$

In [?] we have considered similar inequalities for $u \in D(A^3)$. If A is the infinitesimal generator of a strongly continuous contraction semigroup we have obtained the inequalities:

$$\|Au\|^3 \le \tfrac{243}{8}\,\|u\|^2\|A^3u\|, \quad u \in D(A^3), \tag{5}$$

$$\|A^2u\|^3 \le 24\,\|u\| \cdot \|A^3u\|^2, \quad u \in D(A^3), \tag{6}$$

and in the case of a contraction cosine function:

$$\|Au\|^3 \le \tfrac{81}{40}\,\|u\|^2\|A^3u\|, \quad u \in D(A^3), \tag{7}$$

$$\|A^2u\|^3 \le \tfrac{72}{25}\,\|u\| \cdot \|A^3u\|^2, \quad u \in D(A^3). \tag{8}$$

In fact, in [?] the inequalities for the more general cases of uniformly bounded semigroups and cosine functions were established.

In this paper we consider the same method for obtaining the analogous inequalities for $u \in D(A^{n+1})$, $n \in N$, $n \ge 3$:

$$\|A^ku\|^{n+1} \le C_{n,k}\|u\|^{n-k+1}\|A^{n+1}u\|^k, \quad 1 \le k \le n.$$

2. Contraction semigroups

Let X be a Banach space and $t \mapsto T(t)$ a strongly continuous contraction semigroup ($\|T(t)\| \leq 1$) on X with the infinitesimal generator A. The iteration of the Taylor's formula

$$T(t)u = u + \int_0^t T(\tau)Au\,d\tau, \quad u \in D(A),$$

gives for $u \in D(A^{n+1})$:

$$T(t)u = u + tAu + \frac{t^2}{2}A^2u + ... + \frac{t^n}{n!}A^nu + \frac{1}{n!}\int_0^t (t-\tau)^n T(\tau)A^{n+1}u\,d\tau,$$

hence,

$$tAu + \frac{t^2}{2}A^2u + ... + \frac{t^n}{n!}A^nu = T(t)u - u - \frac{1}{n!}\int_0^t (t-\tau)^n T(\tau)A^{n+1}u\,d\tau. \quad (9)$$

Writing this equation for $t = t_1, t_2, ..., t_n$, $(0 < t_1 < t_2 < ... < t_n)$ we obtain a system of n linear inhomogeneous equations for $Au, A^2u, ..., A^nu$. The determinant of this system equals

$$D_n(t_1,...,t_n) = \begin{vmatrix} \frac{t_1}{1!} & \frac{t_1^2}{2!} & \cdots & \frac{t_1^n}{n!} \\ \vdots & \vdots & & \vdots \\ \frac{t_n}{1!} & \frac{t_n^2}{2!} & \cdots & \frac{t_n^n}{n!} \end{vmatrix} = \frac{t_1 t_2 \cdots t_n}{1!2!\cdots n!}\Delta_n(t_1,...,t_n),$$

where $\Delta_n(t_1,...,t_n) = \prod_{i<j}(t_j - t_i)$ is the Vandermonde's determinant. Thus we obtain for $1 \leq k \leq n$:

$$A^ku = \frac{1}{D_n(t_1,...,t_n)} D_{n,k}(t_1,...,t_n),$$

where (denoting by $a(t)$ the right hand side of (9))

$$D_{n,k}(t_1,...,t_n) = \begin{vmatrix} \frac{t_1}{1!} & \cdots & \frac{t_1^{k-1}}{(k-1)!} & a(t_1) & \frac{t_1^{k+1}}{(k+1)!} & \cdots & \frac{t_1^n}{n!} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \frac{t_n}{1!} & \cdots & \frac{t_n^{k-1}}{(k-1)!} & a(t_n) & \frac{t_n^{k+1}}{(k+1)!} & \cdots & \frac{t_n^n}{n!} \end{vmatrix} =$$

$$= \frac{k!}{1!2!\cdots n!}\sum_{j=1}^n (-1)^{j+k}\frac{t_1 t_2 \cdots t_n}{t_j}\Delta_{n-1,k}(t_1,...,t_{j-1},t_{j+1},...,t_n)\,a(t_j),$$

$$\Delta_{n-1,k}(t_1,...,t_{j-1},t_{j+1},...,t_n) = \begin{vmatrix} 1 & t_1 & \cdots & t_1^{k-2} & t_1^k & \cdots & t_1^{n-1} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 1 & t_{j-1} & \cdots & t_{j-1}^{k-2} & t_{j-1}^k & \cdots & t_{j-1}^{n-1} \\ 1 & t_{j+1} & \cdots & t_{j+1}^{k-2} & t_{j+1}^k & \cdots & t_{j+1}^{n-1} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 1 & t_n & \cdots & t_n^{k-2} & t_n^k & \cdots & t_n^{n-1} \end{vmatrix}.$$

Therefore, denoting

$$b(t) = \int_0^t (t-\tau)^n T(\tau) A^{n+1} u \, d\tau,$$

we get

$$D_{n,k}(t_1,...,t_n) =$$

$$= \frac{k!}{1!2!\cdots n!} \sum_{j=1}^n (-1)^{j+k} \frac{t_1 t_2 \cdots t_n}{t_k} \Delta_{n-1,k}(t_1,...,t_{j-1},t_{j+1},...,t_n) T(t_j) u -$$

$$- \frac{k!}{1!2!\cdots n!} \begin{vmatrix} t_1 & \cdots & t_1^{k-1} & 1 & t_1^{k+1} & \cdots & t_1^n \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ t_n & \cdots & t_n^{k-1} & 1 & t_n^{k+1} & \cdots & t_n^n \end{vmatrix} u -$$

$$- \frac{k!}{1!2!\cdots n!} \sum_{j=1}^n (-1)^{j+k} \frac{t_1 t_2 \cdots t_n}{t_k} \Delta_{n-1,k}(t_1,...,t_{j-1},t_{j+1},...,t_n) \, b(t_j).$$

Using the facts that $\|T(t)u\| \le \|u\|$, $\|T(t)A^{n+1}u\| \le \|A^{n+1}u\|$ we obtain the inequalities of the form:

$$\|A^k u\| \le \varphi_{n,k}(t_1,...,t_n)\|u\| + \psi_{n,k}(t_1,...,t_n)\|A^{n+1}u\|, \tag{10}$$

where $\varphi_{n,k}$ is a homogeneous rational function of n variables of degree $-k$ without poles and zeroes in the domain $0 < t_1 < t_2 < ... < t_n$ and strictly positive there, and $\psi_{n,k}$ is a homogeneous polynomial of n variables of degree $n-k+1$ strictly positive in the same domain. In fact, the denominator of $\varphi_{n,k}(t_1,...,t_n)$ is

$$t_2 t_4 \cdots t_{2[\frac{n}{2}]} \prod_{\substack{i<j, \\ j-i\equiv 1 (mod\ 2)}} (t_j - t_i),$$

the nominator of $\varphi_{n,k}$ is a homogenous polynomial in $t_1,\ldots,t_n$ symmetric in $t_1, t_3, \ldots, t_{2[\frac{n-1}{2}]+1}$ and symmetric in $t_2, t_4, \ldots, t_{2[\frac{n}{2}]}$, and

$$\psi_{n,k}(t_1,...,t_n) = \frac{j!}{(n+1)!} \sigma_{n,n-k+1}(t_1,...,t_n),$$

where $\sigma_{n,j}$ is the elementary symmetric function in n variables of degree j.

The inequality (10) holds true for every choice of variables $t_1, ..., t_n$ in the domain $0 < t_1 < t_2 < ... < t_n$. Our next goal is to minimize the right-hand side of (10) over this domain. The first step is to introduce new variables $t, s_1, ..., s_{n-1}$:

$$t_1 = t,\ t_2 = ts_1,\ ...,\ t_n = ts_{n-1}, \quad t > 0, \quad 1 < s_1 < s_2 < ... < s_{n-1}.$$

Set

$$f_{n,k}(s_1, ..., s_{n-1}) = \varphi_{n,k}(1, s_1, ..., s_{n-1}),$$
$$g_{n,k}(s_1, ..., s_{n-1}) = \psi_{n,k}(1, s_1, ..., s_{n-1}).$$

By homogeneity of $\varphi_{n,k}$ and $\psi_{n,k}$ it follows from (10):

$$\|A^k u\| \leq \frac{1}{t^k} f_{n,k}(s_1, ..., s_{n-1})\|u\| + t^{n-k+1} g_{n,k}(s_1, ..., s_{n-1})\|A^{n+1}u\|. \qquad (11)$$

Now, we keep variables $s_1, ..., s_{n-1}$ fixed and minimize over $t > 0$. The right-hand side has the unique minimum point

$$t_0 = k^{1/(n+1)}\,(n-k+1)^{-1/(n+1)}\,a^{1/(n+1)}\,b^{-1/(n+1)},$$

where $a = f_{n,k}(s_1, ..., s_{n-1})\|u\|$ and $b = g_{n,k}(s_1, ..., s_{n-1})\|A^{n+1}u\|$. Note that we can suppose that $b \neq 0$, i.e. $A^{n+1}u \neq 0$. Indeed, if $A^{n+1}u = 0$, then (11) implies $A^k u = 0$. The minimum value of the right-hand side of (11) is

$$k^{-k/(n+1)}\,(n-k+1)^{k/(n+1)}\,a^{-k/(n+1)}\,b^{k/(n+1)}\,a +$$
$$+ k^{(n-k+1)/(n+1)}\,(n-k+1)^{-(n-k+1)/(n+1)}\,a^{(n-k+1)/(n+1)}\,b^{-(n-k+1)/(n+1)}\,b =$$
$$= (n+1)\,k^{-k/(n+1)}\,(n-k+1)^{-k/(n+1)}\,a^{(n-k+1)/(n+1)}\,b^{k/(n+1)}.$$

Therefore, we get from (11) the inequality of the form

$$\|A^k u\|^{n+1} \leq F_{n,k}(s_1, ..., s_{n-1})\|u\|^{n-k+1}\|A^{n+1}u\|^k \qquad (12)$$

which holds true for any $u \in D(A^{n+1})$ and for any choice of the variables $s_1, ..., s_{n-1}$ in the domain $1 < s_1 < s_2 < ... < s_{n-1}$. Here

$$F_{n,k}(s_1, ..., s_{n-1}) = \frac{(n+1)^{n+1}}{k^k(n-k+1)^{n-k+1}} f_{n,k}(s_1, ..., s_{n-1})^{n-k+1} g_{n,k}(s_1, ..., s_{n-1})^k. \qquad (13)$$

If we put

$$C_{n,k} = \inf\{F_{n,k}(s_1, ..., s_{n-1}); \quad 1 < s_1 < s_2 < ... < s_{n-1}\} \qquad (14)$$

we obtain from (12):

$$\|A^k u\|^{n+1} \leq C_{n,k}\|u\|^{n-k+1}\|A^{n+1}u\|^k, \quad u \in D(A^{n+1}), \quad 1 \leq k \leq n. \qquad (15)$$

The minimization problem (14) is very tedious (if $n \geq 3$) and we have computed the constants $C_{n,k}$ only for $n \leq 5$.

In these cases (i.e. for $1 \leq k \leq n \leq 5$) the functions $F_{n,k}$ are:

$$F_{1,1} = 4; \qquad F_{2,1}(s) = \frac{9}{2}\frac{s^3}{(s-1)^2}, \quad F_{2,2}(s) = 3\frac{(s+1)^2}{s-1};$$

$$F_{3,1}(s_1,s_2) = \frac{2^8}{3^4}\cdot\frac{s_1^4(s_1^2 - s_1s_2 + s_2 - s_1 + 1)^3}{s_2^2(s_2-s_1)^3(s_1-1)^3},$$

$$F_{3,2}(s_1,s_2) = \frac{2^6}{3^2}\cdot\frac{(s_2^2 - s_1^2 + s_2 + 1)^2(s_1s_2 + s_1 + s_2)^2}{s_2^2(s_2-s_1)^2(s_1-1)^2},$$

$$F_{3,3}(s_1,s_2) = \frac{2^5}{3^2}\cdot\frac{(s_2 - s_1 + 1)(s_2 + s_1 + 1)^3}{s_2^2(s_2-s_1)(s_1-1)};$$

$$F_{4,1}(s_1,s_2,s_3) = \frac{2^8}{15}\cdot\frac{s_1^5s_3^5[(s_1+s_3)(s_2^2+s_2+1) - s_1s_2s_3 - s_1s_3 - s_2^3 - s_2^2 - s_2 - 1]^4}{s_2^3(s_3-1)^4(s_3-s_2)^4(s_2-s_1)^4(s_1-1)^4},$$

$$F_{4,2}(s_1,s_2,s_3) = \frac{5^3}{3^5}\cdot\frac{(s_1s_2 + s_2s_3 + s_1s_3 + s_1s_2s_3)^2 g_2(s_1,s_2,s_3)^3}{s_2^3(s_3-1)^3(s_3-s_2)^3(s_2-s_1)^3(s_1-1)^3},$$

$$F_{4,3}(s_1,s_2,s_3) = \frac{25}{48}\cdot\frac{(s_1s_2 + s_2s_3 + s_1s_3 + s_1 + s_2 + s_3)^3 g_3(s_1,s_2,s_3)^2}{s_2^2(s_3-1)^2(s_3-s_2)^2(s_2-s_1)^2(s_1-1)^2},$$

$$F_{4,4}(s_1,s_2,s_3) = \frac{15}{16}\cdot\frac{(s_1+s_2+s_3+1)^4\cdot[s_1s_2 - s_1s_3 + s_2s_3 - s_2^2 - s_1 - s_2 - s_3 - 1]}{s_2(s_3-1)(s_3-s_2)(s_2-s_1)(s_1-1)};$$

where

$$g_2(s_1,s_2,s_3) = (s_1^2 + s_3^2 + s_1s_3 - s_1 - s_3)(s_2^2 + s_2 + 1) - s_1s_2^3 - s_2^3s_3 - s_1^2s_2^2,$$

$$g_3(s_1,s_2,s_3) = s_2^3 + s_2^2 - (s_1^2 + s_3^2 + s_1s_3 - 1)s_2 + s_1^2s_3 + s_1s_3^2 - s_1^2 - s_3^2 - s_1s_3 + 1,$$

$$F_{5,1}(s_1,s_2,s_3,s_4) = \frac{2^7\cdot 3^4}{5^6}\cdot\frac{s_1^6s_3^6 f_1(s_1,s_2,s_3,s_4)^5}{s_2^4s_4^4(s_4-s_1)^5(s_3-1)^5(s_4-s_3)^5(s_3-s_2)^5(s_2-s_1)^5(s_1-1)^5},$$

$$F_{5,2}(s_1,s_2,s_3,s_4) = \frac{9}{100}\cdot\frac{f_2(s_1,s_2,s_3,s_4)^4\ \sigma_4(1,s_1,s_2,s_3,s_4)^2}{s_2^4s_4^4(s_4-s_1)^4(s_3-1)^4(s_4-s_3)^4(s_3-s_2)^4(s_2-s_1)^4(s_1-1)^4},$$

$$F_{5,3}(s_1,s_2,s_3,s_4) = \left(\frac{2}{5}\right)^3\cdot\left[\frac{f_3(s_1,s_2,s_3,s_4)\sigma_3(1,s_1,s_2,s_3,s_4)}{s_2s_4(s_4-s_1)(s_3-1)(s_4-s_3)(s_3-s_2)(s_2-s_1)(s_1-1)}\right]^3,$$

$$F_{5,4}(s_1,s_2,s_3,s_4) = \left(\frac{3}{5}\right)^4\cdot\frac{f_4(s_1,s_2,s_3,s_4)^2\ \sigma_2(1,s_1,s_2,s_3,s_4)^4}{s_2^2s_4^2(s_4-s_1)^2(s_3-1)^2(s_4-s_3)^2(s_3-s_2)^2(s_2-s_1)^2(s_1-1)^2},$$

$$F_{5,5}(s_1,s_2,s_3,s_4) = \frac{2^5\cdot 3^2}{5^4}\cdot\frac{f_5(s_1,s_2,s_3,s_4)(1 + s_1 + s_2 + s_3 + s_4)^5}{s_2s_4(s_4-s_1)(s_3-1)(s_4-s_3)(s_3-s_2)(s_2-s_1)(s_1-1)},$$

where

$$f_1(s_1,s_2,s_3,s_4)=-s_1^2s_3^2(s_2+s_4+s_2s_4)+s_1s_3(s_1+s_3)[(s_4+1)s_2^2+(s_4+1)^2s_2+s_4^2+s_4]$$
$$-(s_1^2+s_3^2+s_1s_3)[(s_4^2+s_4+1)s_2^2+(s_4^2+s_4)s_2+s_4^2]$$
$$-s_1s_3[(s_4+1)s_2^3+(s_4+1)^2s_2^2+(s_4+1)(s_4^2+s_4+1)s_2^2+s_4^3+s_4^2+s_4]$$
$$+(s_1+s_2)[(s_4^2+s_4+1)s_2^3+(s_4+1)(s_4^2+s_4+1)s_2^2+s_4(s_4+1)^2s_2+s_4^3+s_4^2]$$
$$-(s_4^3+s_4^2+s_4+1)s_2^3-(s_4^3+s_4^2+s_4)s_2^2-(s_4^3+2s_4^2+s_4)s_2-s_4^3-s_4^2,$$

$$f_2(s_1,s_2,s_3,s_4)=-s_1^3s_3^3(s_2+s_4+1)+s_1^2s_3^2(s_1+s_3)(s_2^2+s_4^2+s_2s_4+s_2+s_4+1)$$
$$-s_1^2s_3^2[s_2^3+(s_4+1)s_2^2+(s_4^2+s_4+1)s_2+s_4^3+s_4^2+s_4+1]$$
$$-(s_1+s_3)(s_1^2+s_3^2)[(s_4^2+s_4+1)s_2^2+(s_4^2+s_4)s_2+s_4^2]$$
$$+(s_1^2+s_3^2+s_1s_3)[(s_4^2+s_4+1)s_2^3+(s_4+1)(s_4^2+s_4+1)s_2^2+s_4(s_4+1)^2s_2+s_4^3+s_4^2]$$
$$-(s_1+s_2)[(s_4^3+s_4^2+s_4+1)s_2^3+(s_4^3+s_4^2+s_4)s_2^2+(s_4^3+s_4^2)s_2+s_4^3],$$

$$f_3(s_1,s_2,s_3,s_4)=-s_1^3s_3^3+s_1s_3(s_1^2+s_3^2+s_1s_3)(s_2^2+s_4^2+s_2s_4+s_2+s_4+1)$$
$$-(s_1+s_3)(s_1^2+s_3^2+s_1s_3)(s_4+1)(s_2^2+s_2s_4+s_2+s_4)$$
$$-s_1s_3(s_1+s_3)(s_2^3+s_4^3-s_2s_4+1)$$
$$+(s_1^2+s_3^2+s_1s_3)[(s_4+1)s_2^3+(s_4+1)^2s_2^2+(s_4+1)(s_4^2+s_4+1)s_2+s_4^3+s_4^2+s_4]$$
$$-(s_4^3+s_4^2+s_4+1)s_2^3-(s_4^3+s_4^2+s_4)s_2^2-(s_4^3+s_4^2)s_2-s_4^3,$$

$$f_4(s_1,s_2,s_3,s_4)=-s_1^2s_3^2(s_1+s_3)+s_1s_3(s_1+s_3)(s_2+s_4+1)-(s_1+s_3)(s_1^2+s_3^2)(s_2s_4+s_2+s_4)$$
$$-s_1s_3[s_2^3+(s_4+1)s_2^2+(s_4^2+s_4+1)s_2+s_4^3+s_4^2+s_4+1]$$
$$+(s_1+s_3)[(s_4+1)s_2^3+(s_4+1)^2s_2^2+(s_4+1)(s_4^2+s_4+1)s_2+s_4^3+s_4^2+s_4]$$
$$-(s_4^2+s_4+1)s_2^3-(s_4+1)(s_4^2+s_4+1)s_2^2-s_4(s_4+1)^2s_2-s_4^3-s_4^2,$$

$$f_5(s_1,s_2,s_3,s_4)=-s_1^2s_3^2+s_1s_3(s_1+s_3)(s_2+s_4+1)$$
$$-(s_1^2+s_3^2+s_1s_3)(s_2s_4+s_2+s_4)-s_1s_3[s_2^2+(s_4+1)s_2+s_4^2+s_4+1]$$
$$+(s_1+s_3)(s_4+1)(s_2^2+s_2s_4+s_2+s_4)-(s_4^2+s_4+1)s_2^2-(s_4^2+s_4)s_2-s_4^2,$$

and σ_j is the elementary symmetric function of five variables and of degree j:

$$\sigma_2(1,s_1,s_2,s_3,s_4)=s_1+s_2+s_3+s_4+s_1s_2+s_1s_3+s_1s_4+s_2s_3+s_2s_4+s_3s_4,$$
$$\sigma_3(1,s_1,s_2,s_3,s_4)=s_1s_2+s_1s_3+s_1s_4+s_2s_3+s_2s_4+s_3s_4+s_1s_2s_3+s_1s_2s_4+$$
$$+s_1s_3s_4+s_2s_3s_4,$$
$$\sigma_3(1,s_1,s_2,s_3,s_4)=s_1s_2s_3+s_1s_2s_4+s_1s_3s_4+s_2s_3s_4+s_1s_2s_3s_4.$$

It turns out that each of the functions $F_{n,k}$ ($1\le k\le n\le 5$) has exactly one critical point in the domain $1<s_1<\cdots<s_n$ and this point is the minimum point of $F_{n,k}$. It is interesting that for every $n\le 5$ all the functions $F_{n,k}$ ($1\le k\le n$) have the same minimum point! These points are:

– *for* $n = 2$: $s = 1$;
– *for* $n = 3$: $s_1 = 2 + \sqrt{2}$, $s_2 = 3 + 2\sqrt{2}$;
– *for* $n = 4$: $s_1 = \frac{1}{2}(5 + \sqrt{5})$, $s_2 = \frac{1}{2}(7 + 3\sqrt{5})$, $s_3 = 5 + 2\sqrt{5}$;
– *for* $n = 5$: $s_1 = 2 + \sqrt{3}$, $s_2 = 4 + 2\sqrt{3}$, $s_3 = 6 + 3\sqrt{3}$, $s_4 = 7 + 4\sqrt{3}$.

Thus, we obtain the following inequalities:

Theorem 1. *Let A be the infinitesimal generator of a strongly continuous semigroup $t \mapsto T(t)$ of linear contractions on a Banach space X. Then the following inequalities hold true:*

$$
\begin{array}{lcll}
\|Au\|^2 & \leq & 4\,\|u\| \cdot \|A^2u\|, & u \in D(A^2);\\
\|Au\|^3 & \leq & \frac{3^5}{8}\,\|u\|^2 \cdot \|A^3u\|, & u \in D(A^3);\\
\|A^2u\|^3 & \leq & 24\,\|u\| \cdot \|A^3u\|^2, & u \in D(A^3);\\
\|Au\|^4 & \leq & \frac{2^{10}}{3}\,\|u\|^3 \cdot \|A^4u\|, & u \in D(A^4);\\
\|A^2u\|^4 & \leq & \frac{10^4}{9}\,\|u\|^2 \cdot \|A^4u\|^2, & u \in D(A^4);\\
\|A^3u\|^4 & \leq & 192\,\|u\| \cdot \|A^4u\|^3, & u \in D(A^4);\\
\|Au\|^5 & \leq & \frac{5^9}{3\cdot 2^7}\,\|u\|^4 \cdot \|A^5u\|, & u \in D(A^5);\\
\|A^2u\|^5 & \leq & \frac{2\cdot 5^8}{9}\,\|u\|^3 \cdot \|A^5u\|^2, & u \in D(A^5);\\
\|A^3u\|^5 & \leq & \frac{15^2\cdot 7^5}{2^6}\,\|u\|^2 \cdot \|A^5u\|^3, & u \in D(A^5);\\
\|A^4u\|^5 & \leq & 1920\,\|u\| \cdot \|A^5u\|^4, & u \in D(A^5);\\
\|Au\|^6 & \leq & \frac{2^3\cdot 3^{10}}{5}\,\|u\|^5 \cdot \|A^6u\|, & u \in D(A^6);\\
\|A^2u\|^6 & \leq & \frac{3^2\cdot 5^4\cdot 7^6}{2^6}\,\|u\|^4 \cdot \|A^6u\|^2, & u \in D(A^6);\\
\|A^3u\|^6 & \leq & \frac{2^{15}\cdot 7^6}{5^3}\,\|u\|^3 \cdot \|A^6u\|^3, & u \in D(A^6);\\
\|A^4u\|^6 & \leq & \frac{2^6\cdot 3^{16}}{5^4}\,\|u\|^2 \cdot \|A^6u\|^4, & u \in D(A^6);\\
\|A^5u\|^6 & \leq & 2^9 \cdot 3^2 \cdot 5\,\|u\| \cdot \|A^6u\|^5, & u \in D(A^6).
\end{array}
$$

3. Contraction cosine function

Let $t \mapsto T(t)$ be a strongly continuous cosine function of linear contractions on a Banach space X:

$$2T(t)T(s) = T(t+s) + T(t-s), \quad t > s \geq 0;\; T(0) = I;\; \|T(t)\| \leq 1.$$

Let A be its infinitesimal generator:

$$A = \frac{d^2}{dt^2} T(t)\Big|_{t=0}.$$

The iteration of the Taylor's formula

$$T(t)u = u + \int_0^t (t-\tau)T(\tau)Au\,d\tau, \quad u \in D(A),$$

gives for $u \in D(A^{n+a})$:

$$T(t)u = u + \frac{t^2}{2!}Au + \cdots + \frac{t^{2n}}{(2n)!}A^n u + \frac{1}{(2n+1)!}\int_0^t (t-\tau)^{2n+1} T(t) A^{n+1} u\,d\tau.$$

By the same method as for semigroups we obtain the inequalities of the form:

$$\|A^k u\|^{n+1} \leq G_{n,k}(s_1, ..., s_{n-1})\|u\|^{n-k+1}\|A^{n+1}u\|^k$$

for every $u \in D(A^{n+1})$ and for every choice of variables $s_1, ..., s_{n-1}$ in the domain $1 < s_1 < s_2 < ... < s_{n-1}$. The functions $G_{n,k}$ are closely related to the functions $F_{n,k}$ from the previous section:

$$G_{n,k}(s_1, s_2, ..., s_{n-1}) = \left(\frac{(2k)!}{k!}\right)^{n+1} \cdot \left(\frac{(n+1)!}{(2n+2)!}\right)^k \cdot F_{n,k}(s_1^2, s_2^2, ..., s_{n-1}^2).$$

Therefore, we have:

Theorem 2. *Let A be the infinitesimal generator of a strongly continuous cosine function of linear contractions on a Banach space X. Then the following inequalities hold true:*

$$\begin{array}{llll}
\|Au\|^2 & \leq & \frac{4}{3}\,\|u\| \cdot \|A^2u\|, & u \in D(A^2); \\
\|Au\|^3 & \leq & \frac{3^4}{2^3 \cdot 5}\,\|u\|^2 \cdot \|A^3u\|, & u \in D(A^3); \\
\|A^2u\|^3 & \leq & \frac{2^3 \cdot 3^2}{5^2}\,\|u\| \cdot \|A^3u\|^2, & u \in D(A^3); \\
\|Au\|^4 & \leq & \frac{2^{10}}{3^2 \cdot 5 \cdot 7}\,\|u\|^3 \cdot \|A^4u\|, & u \in D(A^4); \\
\|A^2u\|^4 & \leq & \frac{2^4 \cdot 5^2}{7^2}\,\|u\|^2 \cdot \|A^4u\|^2, & u \in D(A^4); \\
\|A^3u\|^4 & \leq & \frac{2^6 \cdot 3^2 \cdot 5}{7^3}\,\|u\| \cdot \|A^4u\|^3, & u \in D(A^4); \\
\|Au\|^5 & \leq & \frac{5^8}{2^7 \cdot 3^4 \cdot 7}\,\|u\|^4 \cdot \|A^5u\|, & u \in D(A^5); \\
\|A^2u\|^5 & \leq & \frac{2 \cdot 5^6}{3^3 \cdot 7^2}\,\|u\|^3 \cdot \|A^5u\|^2, & u \in D(A^5);
\end{array}$$

$$\begin{array}{lll}
\|A^3u\|^5 & \le & \frac{5^4\cdot 7^4}{2^3\cdot 3^2}\,\|u\|^2\cdot\|A^5u\|^3, \quad u\in D(A^5);\\
\|A^4u\|^5 & \le & \frac{2^7\cdot 5^2\cdot 7}{3^6}\,\|u\|\cdot\|A^5u\|^4, \quad u\in D(A^5);\\
\|Au\|^6 & \le & \frac{2^3\cdot 3^7}{5^2\cdot 7\cdot 11}\,\|u\|^5\cdot\|A^6u\|, \quad u\in D(A^6);\\
\|A^2u\|^6 & \le & \frac{3^2\cdot 5^2\cdot 7^4}{2^6\cdot 11^2}\,\|u\|^4\cdot\|A^6u\|^2, \quad u\in D(A^6);\\
\|A^3u\|^6 & \le & \frac{2^{15}\cdot 7^3}{3^3\cdot 11^3}\,\|u\|^3\cdot\|A^6u\|^3, \quad u\in D(A^6);\\
\|A^4u\|^6 & \le & \frac{2^6\cdot 3^{10}\cdot 7^2}{5^2\cdot 11^4}\,\|u\|^2\cdot\|A^6u\|^4, \quad u\in D(A^6);\\
\|A^5u\|^6 & \le & \frac{2^9\cdot 3^5\cdot 5^2\cdot 7}{11^5}\,\|u\|\cdot\|A^6u\|^5, \quad u\in D(A^6).
\end{array}$$

4. Final remarks

The inequalities in *Theorem 1.* and *Theorem 2.* for $u \in D(A^3)$ were obtained in [?], for $u \in D(A^4)$ in [?], and for $u \in D(A^5)$ in [?]. Those for $u \in D(A^6)$ seem to be new.

Furthermore, these papers consider also the analogous inequalities for the infinitesimal generator of a strongly continuous group of linear isometries on a Banach space. In this case the problem is considerably more complicated because for each inequality one should minimize several different functions over several different domains.

In [?] the precise infimum is determined for $u \in D(A^3)$:

$$\|Au\|^3 \le \tfrac{9}{8}\|u\|^2\cdot\|A^3u\|, \qquad \|A^2u\|^3 \le 3\|u\|\cdot\|A^3u\|^2.$$

The inequalities obtained in [?] and [?] for $u \in D(A^4)$ and $u \in D(A^5)$ are not the best obtainable by this method.

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