Proof of the Hero's formula according to R. Boscovich*

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Abstract. *With this lecture we would like to draw your attention to a significant Croatian scientist (mathematician, physicist, astronomer and philosopher) Roger Boscovich (1711–1787). As an illustration of his mathematical work we mention his original proof of the Hero's formula. Also, a short description of his curriculum vitae is given.*

Key words: *Roger Boscovich, Hero's formula*

Sažetak. **Dokaz Heronove formule prema R. Boškoviću**. *Ovim predavanjem želimo skrenuti pažnju na istaknutog hrvatskog znanstvenika (matematičara, fizičara, astronoma i filozofa) Ruđera Boškovića (1711–1787). Kao ilustraciju njegova matematičkog rada navodimo njegov originalni dokaz Heronove formule. Također dan je kratki prikaz njegova životnog puta.*

Ključne riječi: *Ruđer Bošković, Heronova formula*

1. Introduction

With this lecture we would like to draw attention to a significant contribution of a Croatian scientist (mathematician, physicist, astronomer and philosopher) Roger Joseph Boscovich (Ruđer Josip Bošković) (1711-1787) in natural sciences specially mathematics. Let us mention that R. Boscovich was one of the initiators of the L_1 approximation theory (besides Lagrange). Even today this is a current topic in the area of approximation and statistics about which there are a lot of recent papers in famous journals from the world (see e.g. [1], [5]), as well as international conferences dedicated to the topic (e.g. [2] – see also Fig. 1).

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Figure 1. *First page of: L_1-Statistical Procedures and Related Topics* (Y. Dodge, Ed.), Lecture Notes-Monograph Series, Institute of Mathematical Statistics, Hayward, California, 1997

As an illustration of his work, in this paper we will present the original Boscovich's proof of a well known Hero's formula for the area of the triangle which was published in 1785 in the fifth volume of his work *Opera pertinentia ad opticam et astronomiam* published in Bassan in a discussion named "Demonstrations simples de quelques beaux théorèmes appartenants aux triangles". On the basis of knowing three sides of a triangle, besides calculating the area, here the angles of the triangle are defined as well as the radius of a circle inscribed in the triangle.

It is important to mention a French astronomer Joseph Jerom de Lalande (1732-1807) and his words on a great importance of R. Boscovich as a mathematician: "*Le plus grand mathematicien que j'ai connu á Rome est R. Boscovich*" (J. J. de Lalande, Voyage en Italie, 1790).

2. Proof of Hero's formula according to R. Boscovich

Although a well known formula for the area of a triangle given by three sides a, b, c:

$$P = \sqrt{s(s-a)(s-b)(s-c)}, \quad 2s = a+b+c, \qquad (H)$$

is said to be given by a Greek mathematician Hero (1. century BC), it is supposed that even an Old Greek mathematician from Siracusa Archimed (287-212 BC) was familiar with that formula. Through Arabs the formula is brought to Europe. In the 13th century it is found in the works of Fibonacci and Jordanus Nemorarius. Interesting proofs of that formula can later on be found in the work of I. Newton, Aritmetica universalis, 1707, and in the work of L. Euler, New Comments, 1748.

Here we present the proof of the Hero's formula according to R. Boscovich.

Figure 2.

In the triangle $\triangle ABC$ bisectors of the angles are drawn which bisect each other in the centre S of a circle inscribed in the triangle (see fig.). Points of contacts of the circle with triangle sides we denote by D, E, F, and introduce a mark:

$$r := |SD| = |SE| = |SF|.$$

From the vertex A (resp. B) we draw a perpendicular $\overline{AG}$ (resp. $\overline{BH}$) on the bisector of the angle $\angle ACB$.
If we denote

$$c := |AB|, \ \ a := |BC|, \ \ b := |CA|, \ \ 2s = a + b + c,$$

we can see that there holds

$$|AD| = s - a, \;\; |BD| = s - b, \;\; |CE| = s - c,$$

and also

$$\angle BSD = 180^o - \angle ASC = \angle ASG \tag{1}$$

$$\angle ASD = 180^o - \angle BSC = \angle BSH \tag{2}$$

From the right-angled triangles $\triangle BSD$ and $\triangle ASG$ we obtain

$$\sin \angle BSD = \frac{|BD|}{|BS|}, \;\; \sin \angle ASG = \frac{|AG|}{|AS|}, \tag{3}$$

hence because of (1) there holds

$$\frac{|BD|}{|BS|} = \frac{|AG|}{|AS|} \;\Rightarrow\; \frac{|BD|}{|BS|}\frac{|AS|}{|AG|} = 1. \tag{4}$$

Analogously, from the right-angled triangles $\triangle ASD$ and $\triangle BSH$, and in accordance with (2), we obtain

$$\frac{|AD|}{|AS|}\frac{|BS|}{|BH|} = 1. \tag{5}$$

By multiplying the equations (4) and (5) we obtain:

$$\frac{|BD|}{|AG|}\frac{|AD|}{|BH|} = 1,$$

resp.

$$|BD| \cdot |AD| = |AG| \cdot |BH| \;\Rightarrow\; |AG| \cdot |BH| = (s-a)(s-b) \tag{6}$$

From the right-angled triangles $\triangle AGC$ and $\triangle CHB$ we obtain

$$\sin \frac{\angle ACB}{2} = \frac{|AG|}{|AC|}, \;\; \sin \frac{\angle ACB}{2} = \frac{|BH|}{|BC|},$$

hence it follows:

$$\sin^2 \frac{\angle ACB}{2} = \frac{|AG|}{|AC|}\frac{|BH|}{|BC|} = \frac{(s-a)(s-b)}{b \cdot a}. \tag{7}$$

Since $a = (s-b)+(s-c)$ i $b = (s-a)+(s-c)$, there holds $ab = s(s-c)+(s-a)(s-b)$, and therefore (7) can be written in the following form:

$$\sin^2 \frac{\angle ACB}{2} = \frac{(s-a)(s-b)}{s(s-c)+(s-a)(s-b)}. \tag{8}$$

From the right-angled triangle $\triangle CSE$ we obtain

$$\sin \frac{\angle ACB}{2} = \frac{|SE|}{|SC|} \;\Rightarrow\; \sin^2 \frac{\angle ACB}{2} = \frac{|SE|^2}{|SC|^2} = \frac{|SE|^2}{|SE|^2 + |CE|^2}. \tag{9}$$

Now from (9) and (8) we obtain

$$\frac{|SE|^2 + |CE|^2}{|SE|^2} = \frac{s(s-c) + (s-a)(s-b)}{(s-a)(s-b)},$$

resp.

$$\frac{|CE|^2}{|SE|^2} = \frac{s(s-c)}{(s-a)(s-b)}. \tag{10}$$

Since $|SE| = r$ and $|CE| = s - c$ from (10) we obtain

$$s^2 r^2 = s(s-a)(s-b)(s-c) \tag{11}$$

By using a well known formula for the area of the triangle

$$P = r \cdot s$$

from (11) we obtain the Hero's formula (H). $\square$

3. Roger Boscovich's curriculum vitae

R. Boscovich was born on May 18, 1711 in Dubrovnik (Croatia). He started his education in Collegium Ragusinum (a Jesuitical College) in Dubrovnik. In 1725 he continues his education in Rome where he receives his PhD in theology in 1740. As a student of divinity he becomes a mathematics professor in Rome, and then from 1764 at the university in Pavia.

In the period 1759-1763 he was in Paris. He was recognized there as a scientist, but he also faced resistance from Jesuits Jansenists and encyclopedists headed by Voltaire and D'alambert. Therefore, in 1760 he left for England where he was welcomed and accepted as a member of "Royal Society", the biggest English scientific society.

From 1763-1773 he is again in Rome where he gives himself up to teaching and establishing and decorating the Brer astronomical observatory. After having completed this task, another manager is appointed, which is taken by R. Boscovich as one of the greatest grievances in his life.

Since the Jesuit order is cancelled in 1773 (by the Pope Klement XIV bull "*Dominus ac redemptor noster*"), R. Boscovich goes to Paris where he becomes a manager of optics for maritime affairs and takes the French citizenship. Due to health problems, he returns to Italy in 1782. He dies in Milano on 13 February 1787, sick, abandoned, mentally deranged. That is how one of the most brilliant scientists of the 18th century died.

R. Boscovich's first work "*De solis ac lunae defectibus*" was published in 1735. Since that time, at least one discussion in mathematics, physics, astronomy, meteorology or natural sciences in general was published every year. He sprang into fame with the work published in Vienna in 1758 "*Philosophiae naturalis theoria*" and a reviewed version under a different title "*Theorie philosophiae naturalis*" published in Venecie in 1763. While in Bassan (1783-1785) he published his work in five volumes which was mentioned in the introductory part and which also contains the mentioned discussion.

His most important mathematical work "*Elementa matheseos universae*" in three volumes was published in Rome in 1754. The first volume contains geometry and trigonometry, the second algebra and the third cone cross-sections. His work "*De continuitatis lege*" was published in Rome in the same year and there he talked about his point of view regarding continuity of the set of real numbers. He proposes the use of the limit in his work "*De natura et usu infinitorum et infinite parvorum*" which was published in Rome in 1741.

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