Thirty years of shape theory*

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Abstract. *The paper outlines the development of shape theory since its founding by K. Borsuk 30 years ago to the present days. As a motivation for introducing shape theory, some shortcomings of homotopy theory in dealing with spaces of irregular local behavior are described. Special attention is given to the contributions to shape theory made by the Zagreb topology group.*

Key words: *homotopy, shape, homotopy groups, homotopy progroups*

Sažetak. **Trideset godina teorije oblika.** *U članku je prikazan razvitak teorije oblika od njenog osnivanja od strane K. Borsuka prije 30 godina do današnjih dana. Kao motivacija za uvođenje teorije oblika opisani su neki nedostaci teorije homotopije za prostore nepravilnog lokalnog ponašanja. Posebna pažnja dana je doprinosima teoriji oblika zagrebačke topološke grupe.*

Ključne riječi: *homotopija, oblik, grupe homotopije, progrupe homotopije*

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It is generally considered that the theory of shape was founded in 1968 when the Polish topologist Karol Borsuk (1905-1982) published his well-known paper on the homotopy properties of compacta [4]. However, he submitted the paper on February 2, 1967 and spoke about his result at the Symposium on Infinite-dimensional Topology, held in Baton Rougeu, Louisiana, USA from March 27 to April 1, 1967. This means that we are only a few months away from the thirtieth birthday of shape theory. Borsuk also presented his results and ideas at the International Symposium on Topology and its Applications, held in Hercegnovi from August 25 to 31, 1968 (see [5]). This was the first time that he used the suggestive term *shape*. Several topologists from Zagreb attended that conference and thus, had the opportunity to learn about shape theory shortly after it was inaugurated.

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Borsuk's starting point was the observation that many theorems in homotopy theory are valid only for spaces with good local behavior, e.g., manifolds, CW-complexes and absolute neighborhood retracts (ANR's), but fail for spaces like metric compacta. A simple example of this phenomenon is the well-known theorem of J.H.C. Whitehead, which asserts that a mapping between connected CW-complexes $f : X \to Y$, which induces isomorphisms $\pi_n(f) : \pi_n(X) \to \pi_n(Y)$ of all homotopy groups, is a homotopy equivalence. This theorem does not hold for metric continua.

E.g., let X be the Warsaw circle, defined by the following figure.

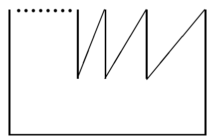

Fig. 1. *Warsaw circle*

It is readily seen that X is a pathwise connected continuum, all of whose homotopy groups vanish. Therefore, the mapping $f : X \to \{*\}$ to a single point induces isomorphisms $\pi_n(f)$, for all n. Nevertheless, f is not a homotopy equivalence, because X disconnects the plane $\mathbb{R}^2$ and $\{*\}$ does not.

To overcome this difficulty, Borsuk considered metric compacta embedded in the Hilbert cube Q (more generally, in a fixed absolute retract). Instead of considering mappings $f : X \to Y$ he considered *fundamental sequences* $(f_n) : X \to Y$, i.e., sequences of mappings $f_n : Q \to Q$, $n = 1, 2, \dots$, such that, for every neighborhood V of Y in Q, there exist a neighborhood U of X in Q and an integer m such that $f_n(U) \subseteq V$, for $n \geq m$. Moreover, the restrictions $f_n|U$ and $f_{n'}|U$ are homotopic in V, for $n, n' \geq m$. Fundamental sequences compose by composing their components, i.e., $(g_n)(f_n) = (g_n f_n)$. Two fundamental sequences $(f_n), (f'_n)$ are considered *homotopic* provided every V admits a U and an m such that $f_n|U \simeq f'_n|U$ in V, whenever $n \geq m$. Homotopy of fundamental sequences is an equivalence relation, whose classes compose by composing their representatives, i.e., $[(g_n)][(f_n)] = [(g_n)(f_n)]$. In this way one obtains a category Sh_Q. Its objects are compacta in Q, while its morphisms are homotopy classes of fundamental sequences. Since arbitrary metric compacta embed in Q, one readily extends Sh_Q to an equivalent category $Sh(\mathbf{CMet})$, whose objects are all metric compacta. This is *Borsuk's shape category.*

Every mapping $f : X \to Y$ induces a fundamental sequence, whose homotopy class depends only on the homotopy class of f. In this way one obtains a functor $S : Ho(\mathbf{CMet}) \to Sh(\mathbf{CMet})$, defined on the homotopy category of metric compacta, called the *shape functor*. Clearly, compacta of the same homotopy type have the same shape, i.e., are isomorphic objects of $Sh(\mathbf{CMet})$. It is easy to see that, for a compact ANR Y, shape morphisms $F : X \to Y$ are in one-to-one correspondence with the homotopy classes of mappings $X \to Y$. Therefore, for compact ANR's, shape coincides with the homotopy type. The Warsaw circle and the circle S^1 are examples of metric continua which have different homotopy types, but the same shape.

Shortly after Borsuk's seminal papers on shape theory [4, 5, 6, 7, 8], there appeared an avalanche of articles devoted to this new branch of topology. Until 1980 the literature on shape theory already consisted of about 400 papers. Around the world, groups of shape theorists were formed. Warsaw was the center of all activities and the seat of the Borsuk group of shape theorists. In USA the first contributions to shape theory were made by Ralph H. Fox (1913–1973), Jack Segal and Thomas A. Chapman. In Moscow research on shape theory was directed by Yuriĭ M. Smirnov. In Japan contributions to shape theory came from Kiiti Morita (1915-1995) and the group in Tsukuba around Yukihiro Kodama. There was a group in Zagreb, led by the author and one in Frankfurt, led by Friedrich W. Bauer. In Great Britain the first contributions to shape theory came from Timothy Porter and in France from Jean-Marc Cordier. In Spain the research on shape was initiated by José R. Sanjurjo.

Already in 1970. S. Mardešić and J. Segal generalized Borsuk's shape theory to compact Hausdorff spaces. It is well-known that every compact Hausdorff space X can be represented as the limit of a (cofinite) inverse system $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ of compact polyhedra (or compact ANR's). In [54, 55] shape morphisms $F : X \to Y$ were defined as *homotopy classes* of *homotopy mappings* $\mathbf{f} = (f, f_\mu) : \mathbf{X} \to \mathbf{Y} = (Y_\mu, p_{\mu\mu'}, \mathrm{M})$. The latter consist of an increasing function $f : \mathrm{M} \to \Lambda$ and of mappings $f_\mu : X_{f(\mu)} \to Y_\mu$ such that, for $\mu_0 \leq \mu_1$, the following diagram commutes up to homotopy

$$\begin{array}{ccc} X_{f(\mu_0)} & \xleftarrow{p_{f(\mu_0)f(\mu_1)}} & X_{f(\mu_1)} \\ \downarrow f_{\mu_0} & & \downarrow f_{\mu_1} \\ Y_{\mu_0} & \xleftarrow[q_{\mu_0\mu_1}]{} & Y_{\mu_1} \end{array} \quad . \tag{1}$$

Two homotopy mappings $(f', f'_\mu), (f'', f''_\mu)$ are considered *homotopic* if there exists an increasing function $f \geq f', f''$ such that $f'_\mu p_{f'(\mu)f(\mu)} \simeq f''_\mu p_{f''(\mu)f(\mu)}$. Equivalence with the Borsuk approach (for metric compacta) was proved using decreasing sequences of compact ANR-neighborhoods of X in Q, viewed as inclusion inverse systems.

While Borsuk's approach was rather geometric, the inverse system approach was more categorical and led quickly to further generalizations. In 1972 Fox [31] generalized Borsuk's shape theory in a different direction, i.e., to arbitrary metric spaces X. He embedded X as a closed subset in an absolute retract L and used the inclusion system of open neighborhoods of X in L to define morphisms. Both generalizations were unified by Morita [60], who gave the general description of the shape category $Sh(\mathbf{Top})$ for arbitrary topological spaces. He allowed his systems $\mathbf{X}$ to be *homotopy systems*, i.e., the usual conditions on bonding mappings were replaced by the corresponding homotopy conditions. Moreover, some theorems from [55] became conditions (M1), (M2), which are a part of the definition of a system being *associated* with a space.

(M1) For every mapping $f : X \to P$ to a polyhedron (or ANR) P, there exist a $\lambda \in \Lambda$ and a mapping $h : X_\lambda \to P$ such that $hp_\lambda \simeq f$.

(M2) For every $\lambda \in \Lambda$ and mappings $h, h' : X_\lambda \to P$ such that $hp_\lambda \simeq h'p_\lambda$, there exists an index $\lambda' \geq \lambda$ such that $hp_{\lambda\lambda'} \simeq h'p_{\lambda\lambda'}$.

Morita proved that the *Čech system,* formed by the nerves of all normal coverings of X, is a homotopy system associated with X [61]. In the terminology introduced in algebraic geometry by A. Grothendieck [32], shape morphisms are given by morphisms $\mathbf{X} \to \mathbf{Y}$ from the category *pro*-$Ho(\mathbf{Top})$, where $Ho(\mathbf{Top})$ denotes the homotopy category of topological spaces.

Among the first successful applications of shape theory is Fox's theory of *overlays,* a modification of covering spaces [31]. While the classical theorems on covering spaces are valid only for locally connected and semi-locally 1-connected spaces, the corresponding theorems on overlays hold without such restrictions. However, the fundamental group $\pi_1(X, *)$ has to be replaced by the *fundamental progroup* $\pi_1(\mathbf{X}, *)$.

Further significant successes of shape theory were the shape-theoretic versions of the theorems of Whitehead, Hurewicz and Smale. The statements of these results also use *progroups,* i.e., inverse systems of groups. Application of the homology functor $H_m(.; G)$ to $\mathbf{X}$ yields an inverse system of Abelian groups $H_m(\mathbf{X}; G) = (H_m(X_\lambda; G), p_{\lambda\lambda'*}, \Lambda)$, called the m^{th}-*homology progroup* of $\mathbf{X}$. Similarly, for systems of pointed spaces $(\mathbf{X}, *)$, one defines the m^{th}-*homotopy progroup* $\pi_m(\mathbf{X}, *)$. If $\mathbf{X}$ and $(\mathbf{X}, *)$ are systems of ANR's associated with the spaces X and $(X, *)$, respectively, these progroups do not depend on the choice of the associated systems. Moreover, they are shape invariants of X and $(X, *)$, respectively. The inverse limit $\check{H}_m(\mathbf{X}; G) = \lim H_m(\mathbf{X}; G)$ is the *Čech homology group*, introduced long before the advent of shape theory, by L. Vietoris [73] and E. Čech [10]. The *shape groups* $\check{\pi}_m(X, *) = \lim \pi_m(\mathbf{X}, *)$, were first defined in 1944 by D.E. Christie [18]. One should keep in mind that the Čech groups and the shape groups contain less information than the corresponding homology and homotopy progroups.

Shape-theoretic versions of the theorems of Whitehead and Hurewicz were obtained by several authors. Here we state two theorems obtained by Morita [59]. The first one asserts that a morphism of pointed shape $F : (X, *) \to (Y, *)$ between finite-dimensional topological spaces is a *shape equivalence,* i.e., an isomorphism of *pointed shape,* if and only if it induces isomorphisms of all homotopy progroups $F_\# : \pi_m(X, *) \to \pi_m(Y, *)$. In contrast to the classical Whitehead theorem, in the above theorem there are no restrictions concerning the local behavior of the spaces involved. The dimensional assumptions cannot be omitted, as it was shown by a counterexample [36], which depends on sophisticated algebraic topology [72], [1]. However, these assumptions can be weakened by requiring that the spaces involved be of finite *shape dimension* Sd. The latter notion is a numerical shape invariant, also introduced by Borsuk [5] and studied extensively by S. Nowak [62] and S. Spież [69, 70].

In the shape-theoretic Hurewicz theorem, one considers $(n-1)$-*shape connected* spaces X, $n \geq 2$, i.e., one assumes that $\pi_m(X,*) = \mathbf{0}$, for $m \leq n-1$. The theorem asserts that $H_m(X;\mathbb{Z}) = \mathbf{0}$, $m \leq n-1$, and there exists a natural isomorphism $\phi_n : \pi_n(X,*) \to H_n(X;\mathbb{Z})$.

The classical Smale theorem is the homotopy version of a theorem of Vietoris concerning *cell-like* mappings of compacta. This class of spaces is of great importance in geometric topology and was studied long before the development of shape theory. Using the shape-theoretic terminology, a mapping $f : X \to Y$ is cell-like provided all of its fibers $f^{-1}(y)$, $y \in Y$, have trivial shape, i.e., have the shape of a single point. The shape-theoretic version of Smale's theorem was obtained by J. Dydak [23, 24]. It asserts that every cell-like mapping induces isomorphisms of homotopy progroups $f_\# : \pi_n(X,*) \to \pi_n(Y,*)$, for all n and all base-points.

Among the most important contributions of Borsuk to shape theory is the introduction of two shape invariant classes of metric compacta, the *fundamental absolute neighborhood retracts* FANR's [6] and *movable compacta* [7]. FANR's generalize ANR's and are their shape-theoretic analogues. They coincide with compacta shape dominated by compact polyhedra. Borsuk introduced movability as a tool useful in detecting that some compacta, e.g., the solenoids, are not FANR's.

Further studies revealed the importance of *pointed* FANR's and *pointed movability*. The main protagonists of this research were D.A. Edwards, R. Geoghegan, H.M. Hastings and A. Heller in U.S. and J. Dydak in Poland. It was shown in [27, 29] that connected pointed FANR's coincide with continua having the shape of a polyhedron. Whether $(X,*)$ has the shape of a compact polyhedron can be detected using tools from algebraic $K-$theory. More precisely, the answer depends on the vanishing of a Wall obstruction $\sigma(X,*)$, which is an element of the reduced projective class group $\tilde{K}^0(\check{\pi}_1(X,*))$ of the first shape group $\check{\pi}_1(X,*)$ [28].

The question whether every FANR is a pointed FANR for several years eluded the efforts of shape theorists. Finally, in 1982, Hastings and Heller proved that this is always the case [37]. The proof essentially uses facts from combinatorial group theory. Whether movable continua are always pointed movable, is still an open problem.

A new direction in shape theory was inaugurated by Chapman, who applied methods of infinite-dimensional topology to the study of shape of metric compacta [15]. He considered compacta X which are Z-*embedded* in the Hilbert cube Q, i.e., have the property that there exist mappings $f : Q \to Q$, which are arbitrarily close to the identity but their image $f(Q)$ misses X. This condition, introduced by R.D. Anderson [2], implies tameness and unknottedness of compacta. It proved to be fundamental in the development of the theory of Q-manifolds [17]. Chapman's *complement theorem* asserts that two compacta X, Y, embedded in Q as Z-sets, have the same shape if and only if their complements $Q\backslash X, Q\backslash Y$ are homeomorphic. Chapman also exhibited an isomorphism of categories $T : \mathcal{WP} \to \mathcal{S}$, where $\mathcal{WP}$ is the *weak proper homotopy category* of complements $M = Q\backslash X$ of Z-sets X of Q, while $\mathcal{S}$ is the restriction of the shape category $Sh(\mathbf{CMet})$ to Z-sets X of Q. For objects $M = Q\backslash X$ of $\mathcal{WP}$, one has $T(M) = X$.

Subsequently, Chapman published a second paper, which contained a finite-dimensional complement theorem, i.e., a theorem where the ambient space was the Euclidean space [16]. This paper had a strong geometric flavor and immediately attracted attention of a number of specialists in geometric topology, in particular in PL-topology, who produced a series of finite-dimensional complement theorems. In these theorems one assumes that X and Y are "nicely" embedded in the Euclidean space $\mathbb{R}^n$ and satisfy the appropriate dimensional conditions. The conclusion is that X and Y have the same shape if and only if their complements $\mathbb{R}^n \backslash X, \mathbb{R}^n \backslash Y$ are homeomorphic. The most general of the results obtained is the complement theorem of I. Ivanšić, R.B. Sher and G.A. Venema [39].

Based on Quillen's homotopical algebra [65], Edwards and Hastings introduced a homotopy category of inverse systems, denoted by $Ho(pro\text{-}\mathbf{Top})$. It is obtained from the category *pro*-**Top** by localization at level homotopy equivalences. Using this category instead of $pro\text{-}Ho(\mathbf{Top})$, they defined a strong shape category $SSh(\mathbf{CMet})$ of compact metric spaces. Strong shape has distinct advantages over shape, e.g., the analogue of Chapman's category isomorphism theorem assumes a more natural form. It asserts the existence of an isomorphism $T : \mathcal{P} \to \mathcal{SS}$, between *proper homotopy category* $\mathcal{P}$ of complements $M = Q \backslash X$ of Z-sets X of Q and the restriction $\mathcal{SS}$ of the strong shape category $SSh(\mathbf{CMet})$ to Z-sets of Q. Through efforts of various authors over a period of several years a *strong shape category* for topological spaces $SSh(\mathbf{Top})$ was defined and so was a strong shape functor $\overline{S} : Ho(\mathbf{Top}) \to Sh(\mathbf{Top})$ [64], [3], [9], [43]-[45], [33], [25]. It is related to the shape functor S by a factorization $S = E\overline{S}$, where $E : SSh(\mathbf{Top}) \to Sh(\mathbf{Top})$ is a functor, which forgets a part of the richer structure of strong shape.

In defining the strong shape category for arbitrary spaces, one needed a method of associating with any given space an inverse system of polyhedra (or ANR's) in the category **Top**. One way of doing this is provided by the Vietoris system [64], [30], [34]. Another method is provided by the notion of a *resolution* of a space X [50]. A resolution $\mathbf{p} : X \to \mathbf{X}$ is a morphism of *pro*-**Top** which satisfies a stronger version of Morita's conditions.

(R1) Given a polyhedron P and an open covering $\mathcal{V}$ of P, any mapping $f : X \to P$ admits a $\lambda \in \Lambda$ and a mapping $h : X_\lambda \to P$ such that the mappings hp_λ and f are $\mathcal{V}-$near.

(R2) There exists an open covering $\mathcal{V}'$ of P, such that whenever, for a $\lambda \in \Lambda$ and two mappings $h, h' : X_\lambda \to P$, the mappings $hp_\lambda, h'p_\lambda$ are $\mathcal{V}'-$near, then there exists a $\lambda' \geq \lambda$ such that the mappings $hp_{\lambda\lambda'}, h'p_{\lambda\lambda'}$ are $\mathcal{V}-$near.

To define a strong shape morphism $F : X \to Y$, it suffices to choose (cofinite) polyhedral resolutions $\mathbf{p} : X \to X$, $\mathbf{q} : Y \to \mathbf{Y}$ and a morphism $\mathbf{X} \to \mathbf{Y}$ of $Ho(pro\text{-}\mathbf{Top})$.

It is an important fact that the category $Ho(pro\text{-}\mathbf{Top})$ is equivalent to the coherent homotopy category $CH(\mathbf{Top})$ [52], which can be viewed as a concrete realization of the former category [44, 45]. Its morphisms are coherent homotopy classes of coherent mappings $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$. These consist of an increasing function $f : \mathrm{M} \to \Lambda$ and of mappings $f_{\mu_0} : X_{f(\mu_0)} \to Y_{\mu_0}$, which make diagram (1) commutative up to a homotopy $f_{\mu_0\mu_1} : X_{f(\mu_1)} \times I \to Y_{\mu_0}$, which is also part of the structure of $\mathbf{f}$. For three indices $\mu_0 \le \mu_1 \le \mu_2$, one has homotopies $f_{\mu_0\mu_1\mu_2} : X_{f(\mu_2)} \times \Delta^2 \to Y_{\mu_0}$, where Δ^2 is the standard 2-simplex. One requires that, on the faces of Δ^2, $f_{\mu_0\mu_1\mu_2}$ is given by the mappings $f_{\mu_1\mu_2}, f_{\mu_0\mu_2}, f_{\mu_0\mu_1}$ as indicated in *Fig. 2.*

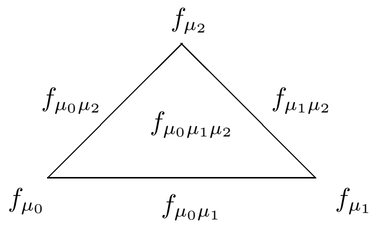

Fig. 2. *Coherence conditions for* $n = 2$.

There are other (more sophisticated) descriptions of coherent categories, but they all yield equivalent categories [19], [71], [74].

An important circle of ideas, related to strong shape, refers to *strong* or *Steenrod homology.* Its relation to singular and Čech homology is similar to the relation of strong shape to homotopy and usual shape. From the point of view of shape theory, the most important property of strong homology is its invariance with respect to strong shape [48], [53].

The approach to shape by inverse systems of polyhedra or ANR's is not the only one. Recently, a different approach was inaugurated by J. Sanjurjo [67] and further developed by Z. Čerin [13]. The basic idea consists in replacing mappings by multivalued mappings, which map points to small sets.

Generally, one expects to find applications of shape theory in problems concerning global properties of spaces having irregular local behavior. Such spaces naturally appear in many areas of mathematics. E.g., they appear as fibers of mappings, as in the case of cell-like mappings. Other examples are sets of fixed points, increments of compactifications [42], boundaries of certain geometric groups, spectra of operators [40], [22], fractal sets and attractors of dynamical systems [35], [68].

There are many situations, where shape itself does not apply, but its methods are applicable. A typical example are properties at infinity of locally compact spaces, in particular the theory of ends. Ideas of shape theory stated in an abstract way, as in *categorical shape theory* [21], yield further possibilities of application, e.g., in pattern recognition [63], [20].

We conclude this short survey by recalling some events related to the development of shape theory in the former Yugoslavia and Croatia.

My first encounter with shape theory occurred during an international conference on topology, held in Hercegnovi in 1968, where Borsuk delivered a talk on the shape of compacta [5]. This was the first time he used the suggestive term *shape.* My first paper on shape (written jointly with Jack Segal) appeared already in 1970 [54]. Since that time members of the Zagreb topology group published more than 100 papers on the subject.

In addition to the work discussed above, we here mention a few of the topics considered. Mardešić defined the shape category for arbitrary topological spaces. Mardešić and Š. Ungar [57] proved a relative shape-theoretical Hurewicz theorem. Mardešić generalized the existing shape-theoretical Whitehead theorems. Ungar proved the analogue of the Blakers-Massey excision theorem and the analogue of the van Kampen theorem. Čerin obtained a series of results on cell-like mappings and homotopy properties of locally compact spaces at infinity. He introduced some new properties like calmness and smoothness. Particularly interesting was his class of calm compacta, which lies between the class of FANR's and the class of movable compacta [12]. Mardešić, jointly with T.B. Rushing, introduced and studied shape fibrations. Ivanšić studied compacta X, which embed in $\mathbb{R}^n$ up to shape, i.e., they admit a compactum $X' \subseteq \mathbb{R}^n$ such that $sh(X) = sh(X')$. Jointly with L.S. Husch, he found sufficient conditions for the existence of embeddings up to shape and sufficient conditions in order that a shape equivalence be a shape concordance. Čerin wrote a series of papers on regular convergence [11], an area where ideas of shape theory raised new problems and provided new techniques. He also studied notions obtained from shape-theoretical notions by imposing the additional requirement that all homotopies involved be dominated by given coverings. In such controlled shape theory, he was able to characterize various old and new classes of compacta and spaces. Ivanšić and N. Uglešić worked on shape improved compacta and weak fibrant compacta. Q. Haxhibeqiri worked in shape fibrations and N. Šekutkovski in coherent homotopy. Jointly with S.A. Antonian, Mardešić studied equivariant shape theory for compact groups. Using the method of multivalued mappings Čerin defined equivariant shape for arbitrary groups [14]. Mardešić proved the existence of paracompact spaces X, whose homology progroups have nonvanishing derived limits $\lim^n H_m(X;\mathbb{Z})$ of arbitrarily high order n [51]. If X is compact, $\lim^n H_m(X;\mathbb{Z}) = 0$, for $n \geq 2$ [53].

Over a number of years many topologists from abroad, interested in shape theory, visited Zagreb. Stays lasting one academic year or at least one semester were realized by J. Segal (U. of Washington), T.B. Rushing (U. of Utah), L.S. Husch (U. of Tennessee), D.S. Coram (State U. of Oklahoma), T. Watanabe (U. of Yamaguchi), A.P. Šostak (U. of Riga), Yu.T. Lisitsa (Moscow Power Institute), S.A. Antonian (U. of Erevan), L.R. Rubin (U. of Oklahoma) and A.V. Prasolov (U. of Minsk). In 1976, 1981 and 1986 at the Interuniversity Centre of Postgraduate Studies in Dubrovnik the Zagreb group organized three international topology conferences, devoted to shape theory and geometric topology. Proceedings of the two latter conferences were published as volumes 870 and 1283 of the Springer Lecture Notes in Mathematics (editors, S. Mardešić and J. Segal). In 1976 the University of Zagreb conferred to Professor Borsuk a doctorate honoris causa in recognition of his scientific contributions and influence he had on the development of the Zagreb topology group. In 1978 at the International congress of mathematicians in Helsinki, in an invited address, Mardešić reported on the development of shape theory during its first ten years [49]. In 1982, North Holland published the Mardešić-Segal monograph on shape theory as volume 26 of their prestigious collection *North-Holland Mathematical Library* [56].

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