The general solution of the Frenet system of differential equations for curves in the pseudo-Galilean space $\mathbf{G}_3^1$

BLAŽENKA DIVJAK*

Abstract. *In this paper the general solution of the Frenet system of differential equations for curves in the pseudo-Galilean space is given. The analogous result in the isotropic (pseudotropic), doubly isotropic and Galilean space are obtained in [3], [4], and [5]. Such result is still unknown in the Euclidean space. There is one particular solution for the Euclidean case in [1] and three particular solutions in [2].*

Key words: *pseudo-Galilean space, Frenet equations, admissible curve*

Sažetak. **Opće rješenje Frenetovog sustava diferencijalnih jednadžbi za krivulje u pseudogalilejevom prostoru $\mathbf{G}_3^1$.** *U ovom radu je dano opće rješenje Frenetovog sustava diferencijalnih jednadžbi za krivulje u pseudogalilejevom prostoru. Analogni rezultat za izotropni (i pseudoizotropni), dvostruko izotropni i Galilejev prostor dobiveni su u [3], [4] i [5], dok je takav rezultat za euklidski prostor do sada nepoznat. Jedno partikularno rješenje za euklidski slučaj dano je u [1] i još tri partikularna rješenja u [2].*

Ključne riječi: *pseudogalijev prostor, Frenetove jednadžbe, dopustiva krivulja*

AMS subject classifications: 51M30, 53A35

Received July 12, 1997 Accepted December 28, 1997

1. The pseudo-Galilean Space

The geometry of the pseudo-Galilean space is similar (but not the same) to the Galilean space which is presented in [6].

The pseudo-Galilean space G_1^3 is a three-dimensional projective space in which the absolute consists of a real plane ω (the absolute plane), a real line $f \subset \omega$ (the absolute line) and a hyperbolic involution on f.

*Faculty of Organisation and Informatics, Varaždin (University of Zagreb), Pavlinska 2, HR-42 000 Varaždin, Croatia, e-mail: `bdivjak@foi.hr`

Projective transformations which presere the absolute form of a group H_8 and are in nonhomogeneous coordinates can be written in the form

$$\begin{aligned} x &= a + \alpha x \\ y &= b + cx + r\operatorname{ch}\varphi y + r\operatorname{sh}\varphi z \\ z &= d + ex + r\operatorname{sh}\varphi y + r\operatorname{ch}\varphi z \end{aligned}$$

where $\alpha r \neq 0$ and a, b, c, d, e, r, $\varphi \in \mathbb{R}$. The group H_8 is called the similarity group of G_1^3.

If $\alpha = 1$, $r = 1$ we obtain the group $B_6 \subset H_8$ of isometries (or the group of motions) of G_1^3.

A curve $c : I \to G_1^3$ given as $\mathbf{r}(t) = (x(t), y(t), z(t))$, where $x(t)$, $y(t)$, $z(t) \in C^3$, $t \in I(\subseteq \mathbb{R})$, is said to be an admissible curve if

$$\begin{array}{ll} (i) & \dot{\mathbf{r}} \times \ddot{\mathbf{r}} = 0 \\ (ii) & \dot{x} \neq 0 \\ (iii) & \dot{y} \neq \pm \dot{z}. \end{array}$$

An admissible curve parametrized by the parametar of arc length $s = x$ (invariant of B_6) is given in the coordinate form by

$$\mathbf{r}(x) = (x, y(x), z(x)).$$

The curvature $\kappa(x)$ and the torsion $\tau(x)$ of an admissible curve are also invariants of B_6 and are given by the following formulas

$$\kappa(x) = \sqrt{|y^n(x)^2 - z''(x)^2|}$$

$$\tau(x) = \frac{1}{\kappa^2(x)} \det\left(\mathbf{r}'(x), \mathbf{r}''(x), \mathbf{r}'''(x)\right).$$

Furthermore, the associated moving trihedron is given by

$$\begin{aligned} \mathbf{t} &= \mathbf{r}'(x) = (1, y'(x), z'(x)) \\ \mathbf{n} &= \tfrac{1}{\kappa(x)}(0, y''(x), z''(x)) \\ \mathbf{b} &= \tfrac{1}{\kappa(x)}(0, z''(x), y''(x)) \end{aligned}$$

and it is called a Frenet trihedron associated to the curve c. Consequently, the following Frenet's formulas are true

$$\mathbf{t}'(x) = \kappa(x)\mathbf{n}(x), \quad \mathbf{n}'(x) = \tau(x)\mathbf{b}(x), \quad \mathbf{b}'(x) = \tau(x)\mathbf{n}(x). \tag{1}$$

2. The general solution of the Frenet system of differential equations for curves in $\mathbf{G_3^1}$

Now our goal is to find all vector fields $\mathbf{t}^*$, $\mathbf{n}^*$, $\mathbf{b}^*$ and all functions κ^*, $\tau^* : I \to \mathbb{R}$ assigned to a curve c such that the formulas analogous to Frenet's (1) are true, i.e.

$$\frac{d\mathbf{t}^*}{dx} = \kappa^*\mathbf{n}^*, \quad \frac{d\mathbf{n}^*}{dx} = \tau^*\mathbf{b}^*, \quad \frac{d\mathbf{b}^*}{dx} = \tau^*\mathbf{n}^*. \tag{2}$$

We first write

$$
\begin{aligned}
\mathbf{t}^* &= a_{11}\mathbf{t} + a_{12}\mathbf{n} + a_{13}\mathbf{b}\\
\mathbf{n}^* &= a_{21}\mathbf{t} + a_{22}\mathbf{n} + a_{23}\mathbf{b}\\
\mathbf{b}^* &= a_{31}\mathbf{t} + a_{32}\mathbf{n} + a_{33}\mathbf{b}
\end{aligned}
\tag{3}
$$

where $a_{ij} : I \to \mathbb{R}$, $i, j = 1, 2, 3$, are yet unknown coefficients.

By differentiating (3) and using (1) we get

$$
\begin{aligned}
\tfrac{d\mathbf{t}^*}{dx} &= a'_{11}\mathbf{t} + (a'_{12} + a_{11}\kappa + a_{13}\tau)\mathbf{n} + (a'_{13} + a_{12}\tau)\mathbf{b}\\
\tfrac{d\mathbf{n}^*}{dx} &= a'_{21}\mathbf{t} + (a'_{22} + a_{21}\kappa + a_{23}\tau)\mathbf{n} + (a'_{23} + a_{22}\tau)\mathbf{b}\\
\tfrac{d\mathbf{b}^*}{dx} &= a'_{11}\mathbf{t} + (a'_{12} + a_{11}\kappa + a_{13}\tau)\mathbf{n} + (a'_{13} + a_{12}\tau)\mathbf{b}\,.
\end{aligned}
\tag{4}
$$

By substituting (3) into the right-hand side of (1) we obtain

$$
\begin{aligned}
\tfrac{d\mathbf{t}^*}{dx} &= \kappa^*(a_{21}\mathbf{t} + a_{22}\mathbf{n} + a_{23}\mathbf{b})\\
\tfrac{d\mathbf{n}^*}{dx} &= \tau^*(a_{31}\mathbf{t} + a_{32}\mathbf{n} + a_{33}\mathbf{b})\\
\tfrac{d\mathbf{b}^*}{dx} &= \tau^*(a_{21}\mathbf{t} + a_{22}\mathbf{n} + a_{23}\mathbf{b})\,.
\end{aligned}
\tag{5}
$$

By comparing (4) and (5) we get the following differential equations for unknown functions

$$
\begin{aligned}
&a'_{11} = \kappa^* a_{21}\\
&a'_{12} + a_{11}\kappa + a_{13}\tau = a_2 2\kappa^*\\
&a'_{13} + a_{12}\tau = a_{23}\kappa^*\\
&a'_{21} = \kappa^* a_{31}\\
&a'_{22} + a_{21}\kappa + a_{23}\tau = a_{32}\kappa^*\\
&a'_{23} + a_{22}\tau = a_{33}\kappa^*\\
&a'_{31} = a_{21}\tau^*\\
&a'_{32} + a_{31}\kappa + a_{33}\tau = a_{22}\tau^*\\
&a'_{33} + a_{32}\tau = a_{23}\tau^*\,.
\end{aligned}
\tag{6}
$$

Now, we will concentrate on finding solutions of the system (6).

Since the vector $\mathbf{t}^*$, $\mathbf{n}^*$, and $\mathbf{b}^*$ are orthonormal vectors in G_3^1, they have to be of the following form

$$
\begin{aligned}
\mathbf{t}^* &= \mathbf{t} + f\mathbf{n} + g\mathbf{b}\\
\mathbf{n}^* &= \operatorname{ch}\varphi\mathbf{n} + \operatorname{sh}\varphi\mathbf{b}\\
\mathbf{b}^* &= \operatorname{sh}\varphi\mathbf{n} + \operatorname{ch}\varphi\mathbf{b},
\end{aligned}
\tag{7}
$$

where f, g, φ are certain functions of x and

$$|f' + \kappa + g\tau| > |g' + f\tau|.$$

By comparing (7) with (3) we can conclude

$$
\begin{array}{lll}
a_{11} = 1 & a_{12} = f & a_{13} = g\\
a_{21} = 0 & a_{22} = \operatorname{ch}\varphi & a_{23} = \operatorname{sh}\varphi\\
a_{31} = 0 & a_{32} = \operatorname{sh}\varphi & a_{33} = \operatorname{ch}\varphi.
\end{array}
\tag{8}
$$

In addition, we put (8) in (6) and get

$$
\begin{aligned}
f' + \kappa + g\tau &= \operatorname{ch}\varphi\kappa^*\\
g' + f\tau &= \operatorname{sh}\varphi\kappa^*
\end{aligned}
\tag{9}
$$

and as a result of (9) we have

$$\varphi = \operatorname{arth} \frac{g' + f\tau}{f' + \kappa + g\tau}, \tag{10}$$

$$\kappa = \sqrt{(f' + \kappa + g\tau)^2 - (g' + f\tau)^2}. \tag{11}$$

Finally, if we set $a_{23} = \operatorname{sh}\varphi$, $a_{22} = \operatorname{ch}\varphi$ and $a_{33} = \operatorname{ch}\varphi$ in $a'_{23} + a_{22}\tau = a_{33}\tau^*$ we obtain

$$\tau^* = \tau + \varphi'. \tag{12}$$

Now, the following theorem is proven.

Theorem 1. *Let $c : I \to G_3^1$, $I \subseteq \mathbb{R}$ be an admissible C^4 curve, κ and τ its curvature and torsion, respectively, and $f, g : I \to \mathbb{R}$ C^2 functions. Then, the general solution of the Frenet system is given by (8), (11) and (12), where $\varphi : I \to \mathbb{R}$ is a differentiable function defined by (10).*

This theorem can be generalized as follows. Let $c : I \to G_3^1$, $I \subseteq \mathbb{R}$ be an admissible curve of the class C^4 and $\kappa_1 = \kappa(x)$, $\tau_1 = \tau(x)$ its curvature and torsion, respectively. Now, we define a sequence of functions $\kappa_i, \tau_i : I \to \mathbb{R}$ in the following way

$$\kappa_{i+1} = \sqrt{(f_i' + \kappa_i + g_i\tau_i)^2 + (g_i' + f_i\tau_i)^2},$$

$$\tau_{i+1} = \tau_i + \varphi_i'$$

where

$$\varphi_i = \operatorname{arth} \frac{g_i' + f_i\tau_i}{f_i' + \kappa_i + g_i\tau_i}, \quad i = 1, 2, 3, ...,$$

$f_i, g_i : I \to \mathbb{R}$ are arbitrary functions of the class C^1 and $f_1 = f$, $g_1 = g$. It has to be

$$|f_i' + \kappa_i + g_i\tau_i| > |g_i' + f_i\tau_i|, \quad i = 1, 2, 3....$$

Moreover, let $F_i = \{\mathbf{t}_i, \mathbf{n}_i, \mathbf{b}_i\}$ be a sequence of the orthogonal trihedra in G_3^1 defined by

$$\begin{aligned} \mathbf{t}_{i+1} &= \mathbf{t}_i + f_i\mathbf{n}_i + g_i\mathbf{b}_i \\ \mathbf{n}_{i+1} &= \operatorname{ch}\varphi_i\mathbf{n}_i + \operatorname{sh}\varphi_i\mathbf{b}_i \\ \mathbf{b}_{i+1} &= \operatorname{sh}\varphi_i\mathbf{n}_i + \operatorname{ch}\varphi_i\mathbf{b}_i. \end{aligned}$$

We set $\mathbf{t}_1 = \mathbf{t}$, $\mathbf{n}_1 = \mathbf{n}$, $\mathbf{b}_1 = \mathbf{b}$.

Then it is easy to prove by induction the following

Theorem 2. *For derivatives of the vector fields of the trihedra F_i and the functions κ_i, τ_i the following Frenet type formulas hold*

$$\frac{d\mathbf{t}_i}{dx} = \kappa_i\mathbf{n}_i, \quad \frac{d\mathbf{n}_i}{dx} = \tau_i\mathbf{b}_i, \quad \frac{d\mathbf{b}_i}{dx} = \tau_i\mathbf{n}_i$$

References

[1] S. Bilinski, *Eine Verallgemeinerung der Formeln von Frenet und eine Isomorphie gewisser Teile der Differentialgeometrie der Raumkurven*, Glasnik Mat.-Fiz.-Astr. **10**(1955), 175–180.

[2] Z. Kurnik, V. Volenec, *Über die begleitenden Dreibene der Raumkurve*, Glasnik Mat. **6**(26)(1971), 129–142.

[3] B. Pavković, *Allgemeine Lösung des Frenetschen Systems von Differentialgleichungen in isotropen and pseudotropen dreidimensionalen Raum*, Glasnik Mat. **10**(30)(1975), 207–218.

[4] B. Pavković, I. Kamenarović, *The general solution of the Frenet system in the doubly isotropic space* $I_3^{(2)}$, Rad JAZU **428**(1987), 17–24.

[5] B. Pavković, *The general solution of the Frenet system of differential equations for curves in the Galilean space* G_3, Rad JAZU **450**(1990), 123–128.

[6] O. Röschl, *Die Geometrie des Galileischen Raumes*, Habilitationsschrift, Leoben, 1984.